Preservation Properties for Some Discrete Mean Residual Life Ordering

M. A. Ibrahim and A. N. Ahmed

Department of Mathematical Statistics, Institute of Statistical Studies and Research

Cairo University, Cairo, Egypt

abushouck@gmail.com.

Abstract

The random variable $X_{(t)} = X - t | X \leq t$, which is called the residual life random variable, has gathered the attention of most researchers in reliability. The properties of mean residual life in continuous case have been studied by several authors. But in discrete case, only in recent years, some studies have been done. In this paper, we present the main results concerning the statistical properties of the discrete mean residual life ordering, such as, convolution, mixtures and convergence in distributions. Furthermore, two results concerning discrete renewal process in connection with the orders in the paper are obtained.

1 Introduction

Discrete lifetimes usually arise through grouping or finite-precision measurement of continuous time phenomena. They may also be found in natural choice where failure may occur only due to incoming shocks. Parametric models for discrete life distributions may be found in Bain (1991), Adams and Watson (1989) and Xekalaki (1983). Nonparametric families of discrete life distributions have been considered in the reliability literature mainly in connection with shock models leading to various continuous-time ageing families, see, e.g., Barlow and Proschan (1975), have studied inter-relations and closure properties of some non-parametric ageing families of distributions having a finite support. Related partial orders have been considered by Abouammoh (1990). Recently, Nanda and Sengupta (2005), have discussed reversed hazard rate in discrete setup and obtained several interesting results.

Let X be a non-negative random variable with probability mass function (p.m.f) given by:

$$f(x) = P(X = x), \qquad x \in N = 0, 1, ,$$

the cumulative distribution function:

$$F(x) = \sum_{i=1}^{x} f(i), \qquad \forall x \in N,$$

and the survival function:

$$\bar{F}(x) = 1 - F(x) = \sum_{i=x+1}^{\infty} f(i), \qquad \forall x \in N.$$

In particular, if $f(0) = Pr(X = 0)$, or accounting random variable X has a support on $N_+ = 0, 1, \ldots,$ we may say that the discrete distribution is zero-truncated. Recently, Pavlova et al. (2006) defined discrete hazard rate and discrete mean residual lifetime of F by:

$$h_F(x) = \frac{f(x)}{f(x) + \bar{F}(x)},$$

$\forall x \in N, and\, P(X \geq x) > 0$, and

$$M_F(x) = E[X - x | X > x] = \frac{\sum_{i=x}^{\infty} \bar{F}(i)}{\bar{F}(i)},$$

$\forall x \in N, and\, \bar{F}(x) > 0$.

There is an abundance of literature on continuous life distributions used in modeling failure data. However, very little has appeared in the literature for discrete failure models.

Discrete failure data arise in various common situations. Consider the following examples:

1. A device can be monitored only once per time period and the observations is taken as the number of time periods successfully completed prior to the failure of the device.

2. A piece of equipment operates in cycles. In this case the random variable of interest is the successful number of cycles before the failure. For instance, the number of flashes in a car flasher prior to failure of the device.

3. In some situations the experimenter groups or finite precision measurement of continuous time phenomena.

Shaked et al. (1995) stated that, discrete failure rates arise in several common situations in reliability theory, where clock time is not the best scale on which to describe lifetime, For example, in weapons reliability, the number of rounds fixed until failure is more important than the age in failure. They also showed the usefulness of these functions for modeling imperfect of discrete models repair and for characterizing ageing in the discrete setting. For more applications in reliability and survival analysis, see,Ebrahimi (1986), and Padgett and Spurrier (1985). More precise concepts of discrete reliability theory have been discussed by Salvia and Bollinger (1982). Roy and Gupta (1992) examined classification of discrete life distributions and they introduced the concepts of second rate of finite failure to maintain analog with the continuous ageing class. Salvia (1996) presented some results on discrete mean residual life.

Similar to continuous distributions, discrete distributions can also be classified by the properties of the failure rates, the mean residual lifetimes, and survival functions of discrete distributions. Some commonly used classes of discrete distributions include the classes of discrete decreasing failure rate (d-DFR), discrete decreasing failure rate on average (d-DFRA), discrete new worse than used (d-NWU), discrete increasing mean residual lifetime (d-IMRL), discrete harmonic new worse than used in expectation (d-HNWUE), and their dual ones including the classes of discrete increasing failure rate (d-IFR), discrete increasing failure rate on average (d-IFRA), discrete new better than used (d-NBU), discrete decreasing mean residual lifetime (d-DMRL) and discrete harmonic new better than used in expectation (d-HNBUE). These classes of discrete distributions have been used extensively in different fields of statistics and probability such as insurance, finance, reliability, survival analysis, and others. See, for example, Cai and Willmot (2005), Willmot et al. (2005), Johnston et al. (2005), Hu et al. (2003), Kijima (2003), Willmot and Cai (2001), Cai and Kalashnikov (2000), Willmot and Lin (2000), Shaked et al. (1995), Shaked and Shanthikumar (1994), Fagiuoli and Pellerey (1994), Barlow and Proschan (1975), and references therein.

2 Preliminaries

In this section, we present definitions, notations and basic facts used throughout the paper. We use "increasing" in place of "non-decreasing" and "decreasing" in place of "non-increasing". Let X and Y be two non-negative random variables with F and G as their respective distribution functions. Let $\bar{F}(t) = 1 - F(t)$ and $\bar{G}(t) = 1 - G(t)$. We will assume that $\bar{F}(0) = \bar{G}(0) = 1$ in all cases.

Before we introduce our contributed discrete definitions of partial ordering, we note that, the continuous versions of these definitions have appeared in, e.g., Ahmed (1988), and Shaked and Shanthikumar (2007).

Definition 1. The random variable X is said to have a smaller discrete mean residual lifetime than that of Y, written $X \leq_{d-MRL} Y$, if

$$\frac{\sum_{i=x}^{\infty} \bar{F}(i)}{\bar{F}(i)} \leq \frac{\sum_{i=x}^{\infty} \bar{G}(i)}{\bar{G}(i)}, \qquad \forall x \in N. \tag{1}$$

Note that, 1 is equivalent to saying

$$\frac{\sum_{i=x}^{\infty} \bar{F}(i)}{\sum_{i=x}^{\infty} \bar{G}(i)},$$

is increasing in x, $\forall x \in N$.

Definition 2. The random variable X has a smaller discrete hazard rate than that of Y, written $X_{d-HR}Y$, if, $\frac{\bar{F}(x)}{\bar{G}(i)}$, is increasing in x, $\forall x \in N$.

Definition 3. The distribution F is called discrete decreasing failure rate (discrete increasing failure rate) or d-DFR (d-IFR), if it's failure rate, $h_F(x) = \frac{f(x)}{f(x)+\bar{F}(x)}$ is non-increasing (non-decreasing) for $x \in N_+$ and $f(x) + \bar{F}(x) > 0$. We notice that, the discrete decreasing failure rate life distribution govern,

1. In the grouped data case, the number of periods until failure of a device governed by a DFR life distribution.

2. The number of seasons a TV show is run before being canceled.

Thus the d-DFR life distributions are of great significance in spite of their relative neglect in the reliability literature.

Definition 4. The distribution F is called discrete new better than used in expectation (discrete new worse than used in expectation) or d-NBUE (d-NWUE), if

$$\sum_{i=x}^{\infty} \bar{F}(i) \leq (\geq) \bar{F}(x) \sum_{j=0}^{\infty} \bar{F}(j), \qquad \forall x \in N.$$

Definition 5. The distribution F is called discrete decreasing mean residual lifetime or d-DMRL if, it's mean residual lifetime $M_F(x) = \frac{\sum_{i=x}^{\infty} \bar{F}(i)}{\bar{F}(i)}$, is increasing in x, $\forall x \in N$.

The following two definitions will be used in sequel:

Definition 6. A probability vector $\underline{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ is said to be smaller than the probability vector $\underline{\beta} = (\beta_1, \beta_2, ..., \beta_n)$, in the sense of the discrete likelihood ratio order, denoted by $\alpha \leq_{d-LR} \beta$, if

$$\frac{\beta_i}{\alpha_i} \leq \frac{\beta_j}{\alpha_j}.$$

Definition 7. A function $g : R^2 \to [0, \infty)$ is said to be log-concave, if

$$g(x_1, y_1)g(x_2, y_2) - g(x_1, y_2)g(x_2, y_1) \geq 0,$$

wherever $x_1 < x_2, y_1 < y_2$.

The remainder of this paper is organized as follows. In Section 3, we present the main results concerning the statistical properties of the discrete mean residual life ordering, such as convolution, mixtures and convergence in distributions. Section 4, contains two results concerning discrete renewal process in connection with the orders in the paper.

3 The Main Results

In this section, we present preservation results for the discrete mean residual life ordering. We point out that similar results hold for both the hazard rate ordering and the likelihood ratio ordering. We begin by showing that the discrete mean residual life ordering is preserved under weak limits in distributions.

Theorem 3.1. The discrete mean residual life ordering ($\leq_{d-MRL}$) preserves the weak convergence property.

Proof. Suppose F_n and G_n converge weakly to F and G and that $F_n \leq_{d-MRL} G_n$. Then, if y is a continuity point of both F and G, it follows that $\mu_F(y) \leq \mu_G(y)$. Thus, $\mu_F(y) > \mu_G(y)$ is possible only if y is a discontinuity point of either F and G. Such discontinuity points are the most countable, so there exist continuity points x_n of F and G for which $x_n \downarrow y \, as \, n \to \infty$. Consequently, appealing to the right-continuity property of distribution function

$$\mu_F(y) = \lim_{n\to\infty} \mu_F(x_n) \leq \mu_G(y) = \lim_{n\to\infty} \mu_G(x_n),$$

whence a contradiction.

The following results, shows that the mean residual life ordering is preserved under the convolution.

For next results, we shall use the notation a_i and b_i to replace $\bar{F}(i)$ and $\bar{G}(i)$, respectively.

Theorem 3.2. Let X_1, X_2 and Y be three non-negative discrete random variables, where Y is independent of both X_1, X_2, also, let Y have a probability mass function g. Then $X_1 \leq_{d-MRL} X_2$. X_2 and g are log-concave imply that $X_1 + Y \leq_{d-MRL} X_2 + Y$.

Proof. We have to show that

$$\frac{\sum_u^\infty \sum_0^\infty g(t-u)a_{x+u}}{\sum_u^\infty \sum_0^\infty g(t-u)b_{x+u}} \geq \frac{\sum_u^\infty \sum_0^\infty g(s-u)a_{x+u}}{\sum_u^\infty \sum_0^\infty g(s-u)b_{x+u}}, \tag{2}$$

or equivalently,

$$\begin{vmatrix} \sum_{u=0}^\infty \sum_{x=0}^\infty g(s-u)b_{x+u} & \sum_{u=0}^\infty \sum_{x=0}^\infty g(s-u)a_{x+u} \\ \sum_{u=0}^\infty \sum_{x=0}^\infty g(t-u)b_{x+u} & \sum_{u=0}^\infty \sum_{x=0}^\infty g(t-u)a_{x+u} \end{vmatrix} \geq 0. \tag{3}$$

Next, by the well known basic composition formula (karlin, 1986, p. 17), the left side of (4) is equal to

$$\begin{vmatrix} g(x_1-y_1) & g(x_1-y_2) \\ g(x_2-y_1) & g(x_2-y_2) \end{vmatrix} \begin{vmatrix} \sum_{x=0}^\infty b_{x+u_1} & \sum_{x=0}^\infty a_{x+u_1} \\ \sum_{x=0}^\infty b_{x+u_2} & \sum_{x=0}^\infty a_{x+u_2} \end{vmatrix} \tag{4}$$

The conclusion now follows if, we note that, the first determinant is non-negative since g is log-concave, and that the second determinant is non-negative since $X_1 \leq_{d-MRL} X_2$.

Corollary 3.3. If $X_1 \leq_{d-MRL} Y_1$ and $X_2 \leq_{d-MRL} Y_2$, where X_1 is independent of Y_1, X_2 is independent of Y_2, then the following statements holds:

1. If X_1 and Y_2 have log-concave probability mass functions, then $X_1+X_2 \leq_{d-MRL} Y_1 + Y_2$.

2. If X_2 and Y_1 have log-concave probability mass functions, then $X_1+X_2 \leq_{d-MRL}$ $Y_1 + Y_2$.

Proof. The following chain of inequalities establish (1):

$$X_1 + X_2 \leq_{d-MRL} X_1 + Y_2 \leq_{d-MRL} Y_1 + Y_2.$$

The proof of (2), is similar.

Let $X(\theta)$ be a non-negative discrete random variable having distribution function F_θ and let θ_i, be a random variable having distribution function $G_i(i = 1,2)$ and support R^+.

The following theorem shows that the d-MRL ordering is preserved under mixtures.

Theorem 3.4. Let $\{X(\theta), \theta \in R^+\}$ be a family of random variables independent of Θ_1 and Θ_2. If $\Theta_1 \leq_{LR} \Theta_2$ and $X(\theta_1) \leq_{d-MRL} X(\theta_2)$ wherever $\theta_1 \leq \theta_2$ then $X(\Theta_1) \leq_{d-MRL} X(\Theta_2)$.

Proof. Let F_i be the distribution function of $X(\Theta_i)$ with $i = 1,2$, we know that

$$\bar{F}_i(x) = \int_0^\infty \bar{F}_\theta(x) dG_i(\theta) \tag{5}$$

In the right of (6), we shall prove that

$$\Phi(i,k) = \sum_{u=0}^{k} F_i(k-u)$$

is TP_2 in (i, k), we notice that,

$$\begin{aligned}\Phi(i,k) &= \sum_{u=0}^{k} F_i(k-u) \\ &= \sum_{u=0}^{k} \int_0^\infty \bar{F}_\theta(k-u) dG_i(\theta) \\ &= \int_0^\infty \sum_{u=0}^{k} \bar{F}_\theta(k-u) g_i(\theta) d\theta \\ &= \int_0^\infty g_i(\theta) \Psi(\theta,k) d\theta.\end{aligned}$$

By assumption, $X(\theta_1) \leq_{d-MRL} X(\theta_2)$ wherever $\theta_1 \leq \theta_2$; we have $\Psi(\theta, k)$ is TP_2 in (θ, k), while the assumption $\Theta_1 \leq_{LR} \Theta_2$ implies that $g_i(\theta)$ is TP_2 in (i, θ). Thus the assertion follows from the basic composition formula (see, Karlin, 1968). Let $\underline{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ be less ordered than $\underline{\beta} = (\beta_1, \beta_2, , \beta_n)$, in the sense of the discrete likelihood ratio ordering. We shall compare the distribution function of

$$F(x) = \alpha_1 F_1(x) + ... + \alpha_n F_n(x)$$

for a random variable X, and

$$G(x) = \beta_1 F_1(x) + ... + \beta_n F_n(x)$$

for a random variable Y.

Theorem 3.5. Let $X_1, , X_n$ be a collection of discrete random variables with corresponding distribution functions $F_1, , F_n$ such that,

$$X_1 \leq_{d-MRL} X_2 \leq_{d-MRL} \cdots \leq_{d-MRL} X_n.$$

Also, let $\underline{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\underline{\beta} = (\beta_1. \beta_2, , \beta_n)$ be two probability vectors with $\alpha \leq_{d-MRL} \beta$ then, $X \leq_{d-MRL} Y$.

Proof We need to establish

$$\frac{\sum_{x=0}^{\infty}\sum_{i=0}^{n}\beta_i a_{t+x}^i}{\sum_{x=0}^{\infty}\sum_{i=0}^{n}\alpha_i a_{t+x}^i} \leq \frac{\sum_{t=0}^{\infty}\sum_{i=0}^{n}\beta_i a_{t+y}^i}{\sum_{t=0}^{\infty}\sum_{i=0}^{n}\alpha_i a_{t+y}^i}, \tag{6}$$

$\forall 0 \leq x \leq y$.

Multiplying by the denominators and canceling out equal terms shows that (7) is equivalent to

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\beta_i\alpha_j\sum_{u=0}^{\infty}a_{u+x}^i\sum_{v=0}^{\infty}a_{v+y}^j \leq \sum_{i=0}^{n}\sum_{j=0}^{n}\beta_i\alpha_j\sum_{u=0}^{\infty}a_{u+y}^i\sum_{v=0}^{\infty}a_{v+x}^j, \tag{7}$$

or, when $j > i$ we get

$$\sum_{i=0}^{n}\sum_{j=0}^{n}[\beta_i\alpha_j\sum_{u=0}^{\infty}a^i_{u+x}\sum_{v=0}^{\infty}a^j_{v+y}] + [\beta_i\alpha_j\sum_{u=0}^{\infty}a^j_{u+x}$$
$$\sum_{v=0}^{\infty}a^i_{v+y}] \le \sum_{i=0}^{n}\sum_{j=0}^{n}[\beta_i\alpha_j\sum_{u=0}^{\infty}a^i_{u+x}\sum_{v=0}^{\infty}a^j_{v+y}]$$
$$+ [\beta_i\alpha_j\sum_{u=0}^{\infty}a^j_{u+y}\sum_{v=0}^{\infty}a^i_{v+x}] \quad (8)$$

now, for each fixed pair (i, j) with $i < j$, we have

$$\beta_i\alpha_j\sum_{v=0}^{\infty}a^i_{v+y}\sum_{u=0}^{\infty}a^j_{u+x} + \beta_j\alpha_i\sum_{v=0}^{\infty}a^j_{v+x}\sum_{u=0}^{\infty}a^i_{u+y}$$
$$- \beta_i\alpha_j\sum_{u=0}^{\infty}a^i_{u+x}\sum_{v=0}^{\infty}a^j_{v+y} - \beta_j\alpha_i\sum_{u=0}^{\infty}a^j_{u+x}\sum_{v=0}^{\infty}a^i_{v+y}(\beta_i\alpha_j$$
$$- \beta_j\alpha_i)[\sum_{v=0}^{\infty}a^i_{v+y}\sum_{u=0}^{\infty}a^j_{u+x} - \sum_{u=0}^{\infty}a^i_{u+x}\sum_{u=0}^{\infty}a^j_{v+y}]. \quad (9)$$

which is non-negative, because both terms are non-negative by assumption. This completes the proof.

In any attempt to construct new discrete mean residual life ordered random variables from known ones, the following theorem might be used:

Theorem 3.6. If $X_1, X_2,$ and $Y_1, Y_2,$ are two sequences of independent random variables with $X_i \le_{d-MRL} Y_i$ and X_i, Y_i have log-concave probability mass functions for all i, then

$$\sum_{i=1}^{n}X_i \le_{d-MRL} \sum_{i=1}^{n}Y_i, \qquad n = 1, 2, \ldots .$$

Proof. We shall prove the theorem by induction.

Clearly, the result is true for $n = 1$.

Assume that the result is true for $p = n - 1$, this means that

$$\sum_{i=1}^{n-1}X_i \le_{d-MRL} \sum_{i=1}^{n-1}Y_i, \qquad n = 1, 2, \ldots . \quad (10)$$

Notice that each of the two sides of (11) has log-concave probability (see, e.g., Karlin, 1968, p. 128). Appearing to Corollary 3.3, the result follows.

Remark. Similar results hold if, the discrete mean residual life ordering is replaced by the discrete hazard rate ordering in Theorem 3.2, Corollary 3.3, Theorem 3.4 and Theorem 3.5.

To demonstrate the usefulness of the above results in recognizing discrete mean residual life ordered random variables, we consider the following:

Example 1. Let X_p denote the convolution of n geometric distributions with parameters $p_1, p_2, , p_n$, respectively. Assume without less of generality that $p_1 > p_2 > ... > p_n$. Since geometric probability mass functions are log-concave, Theorem 3.5 implies that, $X_p \leq_{d-MRL} Y_q$ wherever $p_i \geq q_i, i = 1, , n$.

Example 2. Let X_p be as described in Example 1. An application of Theorem 3.4 immediately yields

$$\sum_{i=1}^{\infty} \alpha_i X_{p_i} \leq_{d-MRL} \sum_{i=1}^{\infty} \beta_i X_{p_i}$$

for every two probability vector $\underline{\alpha}$ and $\underline{\beta}$ such that $\alpha \leq_{d-MRL} \beta$,

Another application of Theorem 3.4 is contained in:

Example 3. Let X_p and X_q-as given in Example 1. For $0 \leq \theta_1 \leq \theta_2 \leq 1$ and $\theta_1 + \theta_2 = 1$, we have

$$\theta_1 X_p + \theta_2 X_q \leq_{d-MRL} \theta_2 X_p + \theta_1 X_q,$$

It is remarkable that, the above example can be generalized to higher dimensions, with obvious modifications in $\underline{\alpha}$ and $\underline{\beta}$. Let

$$F_t(x) = \frac{\bar{F}(x+t)}{\bar{F}(t)}.$$

be the conditional reliability of a unit of age t, then we have the following characterization of d-MRL distributions:

Theorem 3.7. $F \leq_{d-MRL} F_t$ for all $t \geq 0$ if, and only if, F is d-DMRL.

Proof. Observe that $F \leq_{d-MRL} F_t$ for all $t \geq 0$ if, and only if

$$\frac{\sum_{k=x}^{\infty} \bar{F}_t(x)}{\bar{F}_t(x)} \leq \frac{\sum_{k=x}^{\infty} \bar{F}(x)}{\bar{F}(x)},$$

but the latter is equivalent to $\epsilon(t+x) \leq \epsilon(x)$ that is $\epsilon(t)$ is decreasing. This completes the proof.

4 Discrete Renewal Process Applications

Let $(N_F(t), t \geq 0)$ and $(N_G(t), t \geq 0)$ denote two renewal processes having inter-arrival distributions F and G, respectively.

Theorem 4.1. If $F \leq_{d-MRL} G$ then $N_F(t) \leq_{d-V} N_G(t)$.

Proof. The Theorem follows by mincing the elegant proofs of and Theorem 9.6.4 and Lemma 9.6.5 of Ross (1983), and the fact that:

$$\begin{aligned} E[\sum_{i=1}^{N_F(t)+1} X_i] &= E[X_1|X_1 > t] \\ &\geq E[Y_1|Y_1 > t] = E[\sum_{i=1}^{N_F(t)+1} Y_i], \end{aligned}$$

where X_i and Y_i are two sequences of independent identically distributed random variables having F and G as Their respective distributions.

A version of the arguments used to prove Corollary 3.16 and Theorem 3.17 in Chapter 6 of Barlow and Proschan (1975) can be used to show that the following are valid:

Corollary 4.2. Let $F \leq_{d-MRL} G$ and $0 \leq h(1) \leq h(2) \leq ...$, then

$$\sum_{n=1}^{\infty} h(n)\, F_{(n)}(t) \leq \sum_{n=1}^{\infty} h(n)\, G_{(n)}(t).$$

Theorem 4.3. If $F \leq_{d-MRL} G$, $c(k)$ is convex increasing and $c(0) = 0$ then

$$\sum_{k=1}^{\infty} c(k)\, p(N_F(t) = k) \leq \sum_{k=1}^{\infty} c(k)\, p(N_G(t) = k).$$

For an application of Theorem 4.3 in minimizing the expected shortage in spare part one may consult Barlow and Proschan (1975).

Acknowledgement

The authors would like to thank the two referees for careful reading and for their comments which improved the paper.

References

[1] Abouammoh, A. M. (1990), Partial Ordering of Discrete Life Distribution Based on Ageing, Pakistan Journal of Statistics, 6, 25-45.

[2] Adams, G. and Watson, R. (1989), A discrete Time Parametric Model for the Analysis of Failure Time Data, Australian Journal of Statistics, 31, 365-384.

[3] Ahmed, A. N. (1988), Preservation Properties for the Mean Residual Life Ordering. Statistical Papers, 29, 143-150.

[4] Bain, L. J. (1991), *Statistical Analysis of Reliability and Life Testing Models*, Marcel Dekker, New York.

[5] Barlow, R. E. and Proschan, F. (1975), *Statistical Theory of Reliability and Life Testing. Probability Models*, Holt, Rinehart and Winston, New York.

[6] Cai, J. and Kalashnikov, V. (2000), NWU Property of a Class of Random Sums, Journal of Applied Probability, 37, 283-289.

[7] Cai, J. and Willmot, G. (2005), Monotonicity and Aging Properties of Random Sums, Statistics and Probability Letters, 73, 381-392.

[8] Ebrahimi, N. (1986), Classes of Discrete Decreasing and Increasing Mean Residual Life Distributions, IEEE Transactions on Reliability, 35, 403-405.

[9] Fagiuoli, E. and Pellerey, F. (1994), Preservation of Certain Classes of Life Distribution Under Poisson Shock Models, Journal of applied Probability, 31, 458-465.

[10] Hu, T., Ma, M. and Nanda, A. K. (2003), Moment Inequality for Discrete Aging Families, Communication in Statistics, Part A Theory and Methods, 32, 61-91.

[11] Johnson, M. L., Kotz, S. and Kemp, A, W, (2005), *Univariate Discrete Distributions*, Second ed. John Willey and Sons, INC, New York.

[12] Karlin, S. (1968), Total Positivity, Vol. 1, Stanford University Press, Stanford C A.

[13] Kijima, M. (2003), *Stochastic Process with Applications of Finance*, Chapman and Hall/ CRC, Boca Raton.

[14] Nanda, A. K. and Sengupta, D. (2005), Discrete Life Distribution with Decreasing Reversed Hazard, Sankhya-A, 55, 164-168.

[15] Padgett, W. J. and John D. Spurrier, J. D. (1985), On Discrete Failure Models, IEEE Transactions of Reliability, R-34, 253-256.

[16] Pavlova, K. P., Cai, J. and Willmot, G. E. (2006), The Preservation of Classes of Discrete Distributions Under Convolution and Mixing, Insurance Mathematics and Economics, 38, issue 2, 391-405.

[17] Ross, S. M. (1983), *Stochastic Processes*, second edition, John Wiley and Son, New York, 1983.

[18] Roy. D. and Gupta, R. P. (1992), Classification of Discrete Lives, Microelectronic Reliability, 32, 1459-1473.

[19] Salvia, A. A. (1996), Some Results on Discrete Mean Residual Life, IEEE Transaction on Reliability, 45, 359-361.

[20] Salvia, A. A. and Bollinger, R. C. (1982), On Discrete Hazard Functions, IEEE Transaction on Reliability, R-31, 5, 458-459.

[21] Shaked, M. and Shanthikumar, J. G. (2007), *Stochastic Order*, Springer, New York.

[22] Shaked, M. and Shanthikumar, J. G. (1994), *Stochastic Order and their Applications*, Academic Press, New York.

[23] Shaked, M., Shanthikumar, J. G. and Valdez-Torres, J. B. (1995). Discrete hazard rate functions, Computers and Operation Research, 22, 391-402.

[24] Willmot, G. E. and Cai, J. (2001), Aging and other Distributional Properties of Discrete Compound Geometric Distributions, Insurance: Mathematics and Economics, 28, 361-379.

[25] Willmot, G. E., Drekic, S. and Cai, J. (2005), Equilibrium Compound Distributions and Stop-loss Moments, Scandinavian Actuarial Journal, 1, 6-24.

[26] Willmot, G. E. and Lin, X. S. (2000), *Lundberg Approximations for Compound Distributions with Insurance Applications*, Springer, New York.

[27] Xekalaki, E. (1983), Hazard Functions and Life Distributions in Discrete Time, Commons Statistics, Theory and Methods, 12, 2503-2509.