Identifying Mixtures of Some Probability Distributions

By

Ali A. A. Rahman(*)

Abstract. In this paper, we identify mixtures of some probability distributions through a differential equation of the second order, as well as a recurrence relation between three consecutive conditional moments of some function $h^k(X)$, k = 1, 2, ... given X <y. Some well known results follow as special cases from our results.

Keywords. Characterization, Mixture of Probability Distributions, Power, Exponential, Weibull, Burr, Pareto Distributions.

1- Introduction.

Mixtures of probability distributions play an important role in statistics and reliability studies. Suppose a manufacturer produces α_i fraction of a certain product in assembly line i and the life length of a unit produced in assembly line i has a distribution F_i. Now if the outputs of the assembly lines are merged, then a randomly chosen unit from the merged stream will possess the life length distribution $F = \sum_i F_i$. This has motivated several authors to deal with their characterizations (see, e.g., Fakhry [7], Gharib [9], Holzmann et al. [10], Ismail and El Kodary [11], Nassar [13] and Nassar and Mahmoud [14]). Also, some authors have been interested in inferences on mixtures of some distributions, among them, Ahmed et al [3], Abu-Zinadh [2], Bartoszewiez [4], El Sherpieny [5], Everitt and Hand [6], Maclachlan and Peel [12] and Zakerzadeh and Dolati [18].

Let X be a mixture of two continuous random variables with distribution function F(x) defined by:

$$F(x) = \sum_{i=1}^{2} \lambda_i F_i(x) \tag{1.1}$$

where

$$F_i(x) = (d - h(x))^{c_i}, \qquad x \in (\alpha, \beta)$$

Such that:

$$0 < \lambda_i < 1, i = 1, 2 \text{ and } \sum_{i=1}^{2} \lambda_i = 1 .$$

1) $c_i \in \{-1, 0\}, i = 1, 2$ and d are constants.

2) h(x) is a real valued differentiable function on (α, β) with $\lim_{x \to \alpha^+} h(x) = d$ and $\lim_{x \to \beta^-} h(x) = d - 1$

(*) Associate Professor, Institute of Statistical Studies and Research, Cairo University.

In this paper, we are interested in identifying the distribution (1.1) through a differential equation of second order, a recurrence relation between conditional moments of $h^k(X)$, k = 1, 2, ….. given X <y and the first conditional moment of h(X) given X <y.

2- The Main Results

The following Theorem identifies the distribution (1.1) using a differential equation of the second order.

Theorem 2.1 Let X be a continuous random variable with cdf F(.),and density function f(.) such that $F(\alpha)=0$ and $F(\beta)=1$ with $F'(x)>0$ for all x > α (so that F(x) < 1 for all x). Let h(x) be a real valued continuous function defined on (α, β) possessing continuous derivatives on$(\alpha, \beta)$with $\lim_{x\to\alpha^+} h(x)=d$ and $\lim_{x\to\beta^-} h(x)=d-1$. Then X has the distribution defined by (1.1) iff

(2.1) $\frac{(d-h(x))^2}{h'(x)}\left(\frac{F'(x)}{h'(x)}\right)' + (c_1+c_2-1)(d-h(x))\frac{F'(x)}{h'(x)} + c_1c_2F(x)=0$, *where the symbole* $' \equiv \frac{d}{dx}$

Proof. The necessity of this theorem can be verified directly. To prove sufficiency, set

$$z=\ln(d-h(x))$$

Then $$\dot{z}=\frac{-\dot{h}(x)}{d-h(x)},$$

$$F'(x)=\frac{dF(x)}{dx}=\frac{dF(z)}{dz}\frac{dz}{dx}=\frac{-h'(x)}{d-h(x)}F'(z)$$

$$\left(\frac{F'(x)}{h'(x)}\right)'=\frac{d}{dx}\left(\frac{F'(x)}{h'(x)}\right)=-\frac{d}{dx}\left(\frac{F'(z)}{d-h(x)}\right)$$

$$=-\frac{\left[(d-h(x))\frac{dF'(z)}{dx}+F'(z)h'(x)\right]}{(d-h(x))^2}=$$

$$=-\left[\frac{(d-h(x))\frac{dF'(z)}{dz}\frac{dz}{dx}+F'(z)h'(x)}{(d-h(x))^2}\right]=$$

$$=-\left[\frac{-h'(x)F''(z)+F'(z)h'(x)}{(d-h(x))^2}\right]$$

Substituting these results in equation (2.1) one gets:

$$F''(z)-(c_1+c_2)F'(z)+c_1c_2F(z)=0$$

The solution of this differential equation (see, e.g., Ross

$$F(z) = Ae^{c_1 z} + Be^{c_2 z}$$

,i.e.,

$$F(x) = A\left(d - h(x)\right)^{c_1} + B\left(d - h(x)\right)^{c_2}$$

The assumption that $F(\beta) = 1$ gives A +B = 1. Also, the fact that

$0 < F(x) < 1$ for all $x > \alpha$ implies that A > 0 and B > 0.

The proof is complete.

Remarks 2.1.

(1) Theorem (2.1) can be used to characterize a mixture of two power distributions. To this end, set h(x) = -x, d = 0 α = 0 , β = 1.

Therefore,

$$F(x) = \sum_{i=1}^{2} \lambda_i x^{c_i}$$

iff

$$x^2 F''(x) - (c_1 + c_2 - 1)xF'(x) + c_1 c_2 F(x) = 0$$

which is the result of Abdel-Rahman [1].

(2) Ferguson [8] was interested in characterizing the following distributions:

a. $F(x) = \left(\frac{x-b}{r-b}\right)^{\theta}$, $b < x < r, \theta > 0$

b. $F(x) = \left(\frac{r-x}{r-b}\right)^{-\theta}$, $x < b, \theta > 0$

c. $F(x) = \exp[(x-b) | \theta]$, $x < b, \theta > 0$

Theorem (2.1.) can be used to characterize mixtures of the above distributions as follows:

a. Set $h(x) = \frac{-x}{x-b}$, $d = \frac{-b}{r-b}$, α = b, β = r, $c_i = \theta_i$, i = 1, 2 , one gets:

$$F(x) = \sum_{i=1}^{2} \lambda_i \left(\frac{x-b}{r-b}\right)^{\theta_i}, \qquad b < x < r$$

Iff $$(x-b)^2 F''(x) - (\theta_1 + \theta_2 - 1)(x-b)F'(x) + \theta_1\theta_2 F(x) = 0$$

b. Set $h(x) = \frac{x}{r-b}$, $d = \frac{r}{r-b}$, α = -∞, β = b, $c_i = -\theta_i$, i = 1, 2 , one gets :

$$F(x) = \sum_{i=1}^{2} \lambda_i \left(\frac{r-x}{r-b}\right)^{-\theta_i}, \qquad x < b$$

Iff $(r-x)^2 F''(x) - (\theta_1 + \theta_2 + 1)(r-x)F'(x) + \theta_1\theta_2 F(x) = 0$

c. Set $h(x) = \exp(x-b)$, $d = 0$, $\alpha = -\infty$, $\beta = b$, $c_i = \frac{1}{\theta_i}$, $i = 1, 2,$

one gets :

$$F(x) = \sum_{i=1}^{2} \lambda_i exp \frac{(x-b)}{\theta_i}, \qquad x < b$$

Iff $\quad \ddot{F}(x) - \left[\frac{1}{\theta_1} + \frac{1}{\theta_2}\right] \dot{F}(x) + \frac{F(x)}{\theta_1\theta_2} = 0$

Now, we identify the distribution defined by (1.1) by a recurrence relation between consecutive conditional moments of $h^k(X)$ given X < y.

Theorem 2.2 . Let X be a continuous random variable with cdf F(.), density function f(.) and reversed failure rate g(.) such that $0 \le F(x) \le 1$ for all $x > \alpha$. Let h(.) be a real valued differentiable function defined on (α, β) with $\lim_{x\to\beta^-} h(x) = d - 1$ and $\lim_{x\to\alpha^+} h(x) = d$. Then

$$F(x) = \sum_{i=1}^{2} \lambda_i (d - h(x))^{c_i}, \qquad x \,\varepsilon\, (\alpha, \beta)$$

$c_i \notin \{-1, 0\}$, $i = 1, 2$

iff (2.2) $u_k = E(h^k(X) \mid X < y) =$

$$\theta\,[c_1\ c_2\, h^k(y) + k\,(d - h(y))^2\, h^{k-1}(y)\frac{g(y)}{h(x)} +$$
$$dk(c_1 + c_2 + 2k - 1)\, u_{k-1} - k\,(k-1)\, d^2 u_{k-2}\,],$$

$$k = 1,2,3,\ldots\ldots, \quad \theta = [c_1c_2 + k(c_1 + c_2 + k)]^{-1}$$

Proof. Necessity

Let

$$F(x) = \sum_{i=1}^{2} \lambda_i (d - h(x))^{c_i}, \qquad x \,\varepsilon\, (\alpha, \beta),$$

$c_i \notin \{-1,0\}$

By definition:

$$u_k = E[h^k(X) \mid X < y] = \frac{\int_\alpha^y h^k(x)dF(x)}{F(x)}$$

Now, using integration by parts, we get:

$$I = \int_\alpha^y h^k(x)dF(x) = h^k(y)F(y) - k\int_\alpha^y h^{k-1}(x)h'(x)F(x)dx$$

Using Theorem 2.1, we observe that

$$F(x) = \frac{-(d - h(x))^2}{c_1c_2h'(x)}\left(\frac{F'(x)}{h'(x)}\right)' - (c_1 + c_2 - 1)(d - h(x))\frac{F'(x)}{c_1c_2h'(x)}$$

Substituting this result in the 2[nd] term of I we get:

$$J = \int_{\alpha}^{y} h^{k-1}(x)h^{`}(x)F(x)dx =$$

$$= \frac{-1}{c_1 c_2}\int_{\alpha}^{y} h^{k-1}(x)(d-h(x))^2\left(\frac{F^{`}(x)}{h^{`}(x)}\right)^{\backslash} dx$$

$$-\frac{c_1+c_2-1}{c_1 c_2}\int_{\alpha}^{y} h^{k-1}(x)(d-h(x))F^{`}(x)dx$$

Integrating by parts, we obtain :

$$J = \frac{1}{c_1 c_2}h^{k-1}(y)(d-h(y))^2\frac{F^{`}(y)}{h^{`}(y)}+$$

$$\frac{k-1}{c_1 c_2}\int_{\alpha}^{y}\left(d^2h^{k-2}(x)-2dh^{k-1}(x)+h^k(x)\right)F^{`}(x)dx$$

$$-\frac{(c_1+c_2+1)}{c_1 c_2}\int_{\alpha}^{y}\left(dh^{k-1}(x)-h^k(x)\right)F^{`}(x)dx$$

Therefore

$$I = h^k(y)F(y)+\frac{k}{c_1 c_2}h^{k-1}(y)(d-h(y))^2\frac{F^{`}(y)}{h^{`}(y)}-$$

$$-\frac{k(k-1)}{c_1 c_2}\int_{\alpha}^{y} d^2h^{k-2}(x)F^{`}(x)dx+$$

$$+\frac{kd}{c_1 c_2}(2k+c_1+c_2-1)\int_{\alpha}^{y} h^{k-1}(x)F^{`}(x)dx-\frac{k(c_1+c_2+k)}{c_1 c_2}\int_{\alpha}^{y} h^k(x)F^{`}(x)dx$$

Recalling that $u_k = \frac{1}{F(y)}\int_{\alpha}^{y} h^k(x)F^{`}(x)dx$, and the reversed failure rate $g(y) = \frac{f(y)}{F(y)}$, we get: $u_k = h^k(y)+\frac{k}{c_1 c_2}h^{k-1}(y)(d-h(y))^2\frac{g(y)}{h^{`}(y)}-$

$$-\frac{k(k-1)}{c_1 c_2}d^2u_{k-2}+\frac{k}{c_1 c_2}d(c_1+c_2+2k-1)u_{k-1}$$

$$-\frac{k(c_1+c_2+k)}{c_1 c_2}u_k$$

Solving this equation for u_k, one gets:

$$u_k = \theta\left[c_1c_2h^k(y) + k(d-h(y))^2h^{k-1}(y)\frac{g(y)}{h'(y)} + kd(c_1+c_2+2k-1)u_{k-1} - k(k-1)d^2u_{k-2}\right],$$

where $\theta = [c_1c_2 + k(c_1+c_2+k)]^{-1}$.

Sufficiency

Equation (2.2) can be written in integral form as follows:

$$\frac{1}{F(y)}\int_\alpha^y h^k(x)F'(x)dx = \theta[c_1c_2h^k(y) + k(d-h(y))^2h^{k-1}(y)\frac{F'(y)}{h'(y)F(y)} +$$

$$+\frac{kd}{F(y)}(c_1+c_2+2k-1)\int_\alpha^y h^{k-1}(x)F'(x)dx - \frac{k(k-1)d^2}{F(y)}\int_\alpha^y h^{k-2}(x)F'(x)dx]$$

Multiplying both sides by F(y) θ^{-1} and differentiating both sides with respect to y, one gets:

$$[c_1c_2 + k(c_1+c_2+k)]h^k(y)F'(y) = c_1c_2h^k(y)F'(y) + c_1c_2kh^{k-1}(y)h'(y)F(y)$$

$$+kh^{k-1}(y)(d-h(y))^2\left(\frac{F'(y)}{h'(y)}\right)' + k(k-1)h^{k-2}(y)(d-h(y))^2F'(y) -$$

$$-2kh^{k-1}(y)(d-h(y))F'(y) + dk(c_1+c_2+2k-1)h^{k-1}(y)F'(y)$$

$$-k(k-1)d^2h^{k-2}(y)F'(y)$$

Canceling out $c_1c_2h^k(y)F'(y)$ from both sides, dividing both sides by $kh^{k-1}(y)$, rearranging the terms and dividing the results by $h'(y)$, one gets:

$$\frac{(d-h(y))^2}{h'(y)}\left(\frac{F'(y)}{h'(y)}\right)' + (c_1+c_2-1)\frac{(d-h(y))}{h'(y)}F'(y) + c_1c_2F(y) = 0$$

Using Theorem (2.1), it follows that

$$F(y) = \sum_{i=1}^{2}\lambda_i(d-h(y))^{c_i}$$

Our proof is complete.

Remarks 2.2

(1) Set $h(y) = -y$, $d = 0$, $\alpha = 0$, $\beta = 1$. Therefore, $h(y)$ satisfies the assumptions of Theorem (2.2). Hence we conclude that:

$$u_k = \theta\left[c_1c_2(-y)^k - k(-y)^{k+1}g(y)\right]. \quad k = 1, 2, \ldots$$

iff

$$F(y) = \sum_{i=1}^{2}\lambda_i y^{c_i}, \qquad 0 < y < 1$$

(2) Set $h(y) = \exp(y-b)$, $d = 0$, $\alpha = -\infty$, $\beta = b$, $c_i = \dfrac{1}{\theta_i}$, $i = 1, 2$. Then $h(y)$ satisfies the assumptions of Theorem (2.2). Hence we conclude that:

$$u_k = \theta \exp k(y-b)[\theta_1\theta_2 + kg(y)]$$

where

$$\theta = [\theta_1 + \theta_2 + k(\theta_1 + \theta_2 + k)]^{-1}, \quad k = 1, 2, \ldots$$

iff $$F(y) = \sum_{i=1}^{2} \lambda_i \exp\frac{(y-b)}{\theta_i}, \quad y < b$$

Corollary 2.1. A continuous random variable X follows the distribution defined by (1.1) iff

$$E(h(X)|X < y) = \theta\left[c_1c_2h(y) + (d - h(y))^2\frac{g(y)}{h(y)} + d(c_1 + c_2 + 1)\right]$$

where $$\theta = [(c_1+1)(c_2+1)]^{-1}$$

Proof. Set k = 1 in Theorem (2.2) and noting that $u_0 = 1$, we obtain the result.

Remarks 2.3

(1) Set $c_1 = c_2 = c$, one gets:

$$E(h(X)|X < y) = \frac{ch(y) + d}{c+1}$$

iff $$F(x) = (d - h(x))^c, \quad x\epsilon(\alpha, \beta)$$

which is the result of Ouyang [15].

(2) Set $h(x) = \frac{-u(x)}{u(\beta) - z(k)/(1-f(k))}$, $d = \frac{-z(k)/(1-f(k))}{u(\beta) - z(k)/(1-f(k))}$, and $c_1 = c_2 = \frac{f(k)}{1-f(k)}$, where u(x) is a differentiable function such that $\lim_{x\to\alpha^+} u(x) = \frac{z(k)}{1-f(k)}$ and $\lim_{x\to\beta^-} u(x) = u(\beta)$ then Corollary (2.1) reduces to the result of Talwalker [17]

(3) Set $h(x) = e^{x-b}$, $d = 0$, $\alpha = -\infty$, $\beta = b$, $c_i = \dfrac{1}{\theta_i}$, $i = 1, 2$. Then h(x) satisfies the assumptions of Theorem (2.1). Hence we can conclude that:

$$E(e^{X-b}|X < y) = \frac{e^{y-b}}{(\theta_1+1)(\theta_2+1)}[1 + \theta_1\theta_2 g(y)]$$

iff

$$F(x) = \sum_{i=1}^{2} \lambda_i \exp(x-b)|\theta_i, \quad x < b$$

(4) Set $h(x) = -x^a$ for $x \in (0, 1)$, $d = 0$, $c_1 = c_2 = 1$. then h(x) satisfies the assumptions of Theorem (2.2). hence we conclude that:

$$E(X^a|X < y) = \frac{1}{2}y^a, \quad y\epsilon(0, 1)$$

iff $$F(x) = x^a, \quad x\epsilon(0, 1)$$

which is the result of Ouyang [15].

(5) Set $h(x)=e^{-bx^{a}}$ for $x \in (0, \infty)$, $d=1$, $c_1=c_2=1$. Then h(x) satisfies the result of Theorem (2.2) and we conclude that:

$$E(e^{-bX^{a}}|X<y)=\frac{1}{2}(e^{-by^{a}}+1), \quad y\epsilon(0,\infty)$$

iff $$F(x)=1-e^{-bx^{a}}, \quad x\epsilon(0,\infty)$$

which is the result of Ouyang [15].

(6) Set $h(x)=x^{-a}$, $d=1$, $\alpha=1$, $\beta=\infty$, $c_1=c_2=1$. Then h(x) satisfies the assumptions of Theorem (2.2) and we conclude that

$$E(X^{-a}|X<y)=\frac{1}{2}(y^{-a}+1), \quad \text{for } a>0, y\epsilon(1,\infty)$$

iff $$F(x)=1-x^{-a}, \quad x\epsilon(1,\infty)$$

which is the result of Ouyang [15].

(7) Set $h(x)=(1+x^{a})^{-b}, d=1, \alpha=0, \beta=\infty, c_1=c_2=1$. Then h(x) satisfies the assumptions of Theorem (2.2) and we conclude that:

$$E((1+X^{a})^{-b}|X<y)=\frac{1}{2}((1+y^{a})^{-b}+1)$$

$$\text{for } a>0,\ b>0, \text{ and } y\in(0,\infty)$$

iff $$F(x)=1-(1+x^{a})^{-b}, \quad x\in(0,\infty)$$

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