Computation of the moments and product moments of order statistics for skewed continuous distribution.

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ABSTRACT

In this article, we present a new algorithm to compute the moments, product moments, variances and covariances of order statistics in skewed continuous distributions for any sample size. The algorithm is written using the Gauss-Legender quadrature algorithm of GAUSS mathematical and statistical system matrix programming language. The accuracy of the calculations was tested for the Gamma, Weibull, and the extreme value distributions. A nonlinear least squares approximation for the first moments of order statistics for the gamma, Weibull and extreme value distributions is also presented.

Key words: *Order statistics, Simpson's algorithm, Gauss-Legender quadrature algorithm, series approximations, Nonlinear least squares approximations, Recurrence relations.*

AMS: 62G30, 62Q05, 60G70.

1. INTRODUCTION

Let $X_1, X_2, \ldots, X_n$ be a random sample from an absolutely continuous distribution with cumulative distribution function (cdf) F(x) and probability density function (pdf) f(x), and let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics obtained from this sample. The k^{th} moments of the order statistic $X_{(r)}$ is given by

$$E(X_{(r)}^k) = \frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} y^k\,[F_X(y)]^{r-1}[1-F_X(y)]^{n-r}f_X(y)dy \quad (1)$$

Sen (1959) showed that, if $E|X|^\delta$ exists for some $\delta > 0$, then $E(X_{(r)}^k)$ exists for all r satisfying $r_o < r < n - r_o + 1$, where $r_o\delta = k$. The product moments between any two order statistics $X_{(r)}$ and $X_{(s)}$, $r < s$; r, s=1, 2,..., n, are defined by

$$E(X_{(r)}X_{(s)}) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\,\cdot$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{y} xy\,[F_X(x)]^{r-1}[F_X(y)-F_X(x)]^{s-r-1}[1-F_X(y)]^{n-s}f_X(x)f_X(y)dxdy \quad (2)$$

The moments of order statistics in random samples of small size n can be obtained explicitly only for a few populations, such as the uniform and the exponential. Tables of means, variances, and covariances are available in the literature for some standard distributions, Harter and Balakrishnan (1996).The tables use different computation methods and some series approximations and they are available only for small sample sizes. Some statisticians still believe on what Hirakawa claims in 1973 "moments of order statistics can be evaluated by numerical integration, but straightforward integration has shortcomings in view of quantity of computation and the accuracy".

Recurrence relations between the moments of order statistics have been studied extensively with the principal aim of reducing the number of independent calculations required for the evaluation of the moments. Such relations may also be used as partial checks on direct calculations of the moments.

Algorithms to compute the moments and product moments of order statistics for small samples from the normal distribution had been presented by Davis and Stephens (1978) and Royston (1982). The Maple procedures presented by Childs and Balakrishnan (2002) utilizes the series approximations presented by David and Johnson (1956) to approximate the moments and product moments of order statistics from any continuous distributions. Hussien (2009) proposed a new algorithm for the computation of the moments, product moments, variances and covariances of order statistics from a continuous distribution and for any sample size. The accuracy of the calculations was tested for the normal, the uniform and the exponential distributions.

In this paper this algorithm will be extended to compute the moments, product moments, variances and covariances of order statistics for the gamma, Weibull and extreme value distributions. Moreover a simple but accurate nonlinear least squares approximations for the first moments of order statistics from the three distributions will be presented. In Section 2 we describe the proposed algorithm and some formulae that can be used for checking its accuracy with a simple and accurate approximation to the first moment. Sections 3, 4 and 5 compare the results of the proposed algorithm with the available table-values and approximations for the moments of order statistics from the Gamma, Weibull and extreme value distributions. Section 6 presents nonlinear least squares approximation for the first moments of order statistics for the gamma, Weibull and extreme value distributions.

2. THE PROPOSED MOMENTS EVALUATION ALGORITHM

The proposed algorithm computes the kth moment, k=1, 2, the product moments, variances and covariances of order statistics from a skewed continuous distribution with a very high accuracy. This level of accuracy will be at least as good as the available tabulated values. The computation remains accurate for sample size as large as 150. One cannot obtain the required level of accuracy in computation of (1) or (2) unless F(x) and f(x) are evaluated with a high level of accuracy. The evaluation of F(x) for many distributions depends on some mathematical approximation. The major mathematical programming languages provide mathematical approximation procedures to compute F(x) for many distributions. Such procedure will provide accuracy essentially to "machine precision".

Any numerical integration procedure approximates $\int_a^b g(x)dx$ where the limits of integration a and b are finite. This is not true for (1) and (2) for the distributions we study. So we set $a = F^{-1}(p_1)$ and $b=F^{-1}(1-p_2)$. Typically, $p_1 = p2 = 0.10^{-6}$. One can change those limits to get different level of accuracy.

To compute $E(X_{(r)})$ and $var(X_{(r)})$, r= 1, 2, ..., n, the adaptive Simpson's algorithm for numerical integration (*intsimp* procedure of Gauss 9.0) with tolerance limits 10^{-8} will be used. In the adaptive process, the interval [a,b] will be divided into two sub-intervals and then decide whether each of them can be divided into more subintervals. The procedure is continued until some specified accuracy is obtained throughout the entire interval [a,b]. The Gauss-Legender quadrature algorithm (*intquad1* procedure of Gauss

9.0) allows accurate and fast integration for many functions, but it is not adaptive to yield any level of accuracy as the *intsimp* procedure, (see Gauss 9.0 Language Reference, (2007)). For the product moments, the double integration limits will be

$$\int_a^b \int_a^y g(x,y)\, dx dy$$

with the same definition of a and b as above. The Gauss-Legendre quadrature algorithm for double numerical integration (*intgrat2* procedure of Gauss 9.0) will be used. This procedure allows the integration over a region which is bounded by functions, rather than just scalars as the *intquad2* procedure.

The accuracy of the proposed numerical integration procedure will be checked by the following general relations, given by David and Nagaraja (2003)

$$\sum_{r=1}^{n} E(X_{(r)}) = n\,\mu \qquad (3)$$

$$\sum_{r=1}^{n} E(X_{(r)}^2) = n\,E(X^2) \qquad (4)$$

$$\sum_{r=1}^{n}\sum_{s=1}^{n} cov(X_{(r)}, X_{(s)}) = n\,\sigma^2 \qquad (5)$$

Thus, the following formulae will be used to check the accuracy of the computation for each distribution studied, for $2 \le n \le 150$,

$$max_n |\Sigma E(X_{(r)}) - n\mu|, \qquad (6)$$

$$max_n |\Sigma E(X^2_{(r)}) - nE(X^2)| \qquad (7)$$

$$max_n |\Sigma\Sigma cov(x_{(r)}, x_{(s)}) - n\sigma^2| \qquad (8)$$

3. THE MOMENTS AND PRODUCT MOMENTS OF GAMMA DISTRIBUTION

Let $X_1, X_2, \ldots, X_n$ be a random sample from the gamma distribution given by the pdf

$$f(x) = \frac{x^{\theta-1} e^{-x}}{\Gamma(\theta)}, \; x > 0 \;\text{ and }\; \theta > 0$$

and the corresponding cdf given by

$$F(y) = \frac{\gamma(\theta, y)}{\Gamma(\theta)}, \; y > 0 \text{ and } \theta > 0$$

where $\gamma(\theta,y)$ is the incomplete gamma function defined by

$$\gamma(\theta,y) = \int_0^x t^{\theta-1} e^{-t} dt, \; y > 0$$

The k<u>th</u> moment of $X_{(r)}$ is given by

$$E(X^k_{(r)}) = \frac{n!}{(r-1)!(n-r)!\{\Gamma(\alpha)\}^n}\int_0^\infty y^{k-\theta-1}e^{-y}\{\gamma(\theta,y)\}^{r-1}\{\Gamma(\theta)-\gamma(\theta,y)\}^{n-r}dy$$

and the product moment of $X_{(r)}$ and $X_{(s)}$ is given by

$$E(X_{(r)},X_{(s)}) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!\{\Gamma(\alpha)\}^n}\int_{-\infty}^{\infty}\int_{-\infty}^{y} x^{\theta}y^{\theta}[\gamma(\theta,x)]^{r-1}[\gamma(\theta,y) - \gamma(\theta,x)]^{s-r-1}[\Gamma(\theta) - \gamma(\theta,y)]^{n-s}\, e^{-x-y}dxdy$$

Most of the work related to moments of gamma order statistics expresses $E(X^k_{(r)})$ in terms of recurrence relations (Gupta (1960), Krishnaiah and Rizvi (1967), Breiter and Krishnaiah (1968), Tadikamalla (1978) and Walter and Stitt (1988)). Only Gupta (1960) and Prescott (1974) presented tables for the covariances of order statistics from the gamma distribution. However, it is an approximate calculation depending on Gupta's recurrence relations and it is available for small sample sizes only. Sobel and Wells (1990) express $E(X^k_{(r)})$in terms of Dirichlet integrals which is not available as a standard routine. Nadarajah and Pal (2008) derived an explicit expression for $E(X^k_{(r)})$ as a finite sum of a special function called the Lauricella function. Numerical routines for the direct computation of the Lauricella function are available (Trott (2006)).

The main advantages of the numerical integration procedure that it gets accurate calculations for the moments for any sample size without using numerically computed special functions, used by Nadarajah and Pal (2008). Moreover, it accurately calculates the product moments and covariances for any value of the shape parameter θ, and for any sample size. Thus, accurate and complete tables of the covariances can be produced using the numerical integration procedure.

The computation of the first two moments and the covariances of order statistics were checked for their accuracy by the identities (3), (4), and (5). Then, $max_n|\Sigma E(X_{(i)}) - n\mu$, $max_n|\Sigma E(X^2_{(i)}) - nE(X^2)|$ and $max_n|\Sigma\Sigma cov(x_{(i)},x_{(j)}) - n\sigma^2|$ for n=2(1)150 were computed. The results for specific values of θ are given in Table 1. This shows that the numerical integration procedure computations of the moments and covariances are accurate to more than 3 decimals for all sample sizes and all values of θ considered. Moreover, it is much faster than Nadarjan and Pal (2008) procedure.

To compare the numerical integration procedure results with the tabulated values we define

MAXDm (k)=$\max_n|E^k_r - \mu^k_r|$ $\quad k = 1,2,3,4;\ r = 1,2,\dots,n$

and

AVRDm (k)=$\text{average}_n|E^k_r - \mu^k_r|$ $\quad k = 1,2,3,4;$ r=1,2,...,n

where, $E^k_r = E(X^k_{(r)})$ calculated from Breiter and Krishnaiah (1969) tables r=1,2,...,n, and $\mu^k_r = E(X^k_{(r)})$ calculated by the proposed numerical integration algorithm, r = 1,2,...n . Values of MAXDm (k) and AVRDm (k) for selected values of θ are given in Table 2. Note that Breiter and Krishnaiah (1969) tables are restricted for n<10 and θ=0.5(1)8.5.

Also, we define

$$\text{MAXDv} = \max_{rs}|C_{rs} - \mu_{rs}| \qquad \text{and}$$

$$\text{AVRDv} = \text{average}_{rs}|C_{rs} - \mu_{rs}|$$

where $C_{rs} = E(X_{(r)}X_{(s)})$ $r < s$ calculated from Prescott (1974) tables r=1,2,…,n and

$\mu_{rs} = E(X_{(r)}X_{(s)})$ $r < s$ calculated by the proposed numerical integration algorithm, r=1,2,…n . Values of MAXDv and AVRDv for the sample sizes tabulated in Prescott (1974) are given in Table 3.

4. THE MOMENTS AND PRODUCT MOMENTS OF THE WEIBULL DISTRIBUTION

Let $X_1, X_2, \ldots, X_n$ be a random sample from a standardized Weibull distribution with a probability density function

$$f(x) = \alpha x^{\alpha-1} e^{-x^{\alpha}} \qquad x>0,$$

where $\alpha > 0$ is the shape parameter. Explicit formulas for the moments and product moments of order statistics for the standardized Weibull distribution were given by Lieblein (1955) as follows:

$$E(X_{(i)}^k) = \frac{n!}{(i-1)!(n-i)!}\Gamma\left(1+\frac{k}{\alpha}\right)\sum_{x=0}^{i-1}(-1)^x\binom{i-1}{x}(n+x-i+1)^{-1-\frac{k}{\alpha}}$$

, i=1,2,3,…,n and k=1,2,3,…..

$$E(X_{(i)}X_{(j)}) = \alpha^2 K\sum_{x=0}^{i-1}\sum_{y=0}^{j-i-1}(-1)^{x+y}\binom{i-1}{x}\binom{j-i-1}{y}\psi(j-i+x-y, n-j+y+1)$$

i<j , i,j = 1,2,3,...,n,

$$K = \frac{n!}{(i-1)!(j-i-1)!(n-j)!},$$

$$\psi(t,u) = \alpha^{-2}(tu)^{-r}\Gamma(2r)B_p(r,r),$$

$$r = 1+\frac{1}{\alpha}, \quad p = \frac{t}{t+u}, \quad t>0, \; u>0, \; t+u \le n$$

$B_p(a,b) = \int_0^p x^{a-1}(1-x)^{b-1}dx$,the incomplete beta function.

Harter (1970b) tabulated the means for $n \le 40$ and $\alpha = 0.5(.5)4(1)8$. Harter and Balakrishnan (1996) tabulated the variances and covariances for $n \le 10$ and $\alpha = 0.5(.5)4$.

For n>20, their computed values for $E(X^k_{(i)})$ became extremely large. For n>30, Lesch et al. (2009) proposed a simple technique that can be used to calculate accurate approximations to the covariances of Weibull (and Gamma) order statistics., They used David and Johnson (1954) quantile based approximations to the mean and variance of order statistics.

The computation of the first two moments and the covariances of order statistics by the new numerical integration procedure were checked for their accuracy by

$$max_n|\Sigma X_{(i)} - n\mu|\ ,\ max_n|\Sigma X^2_{(i)} - nE(X^2)| \text{ and } max_n|\Sigma\Sigma cov(x_{(i)}, x_{(j)}) - n\sigma^2|$$

for n=2(1)150. The results for some values of α are given in Table 4. The entries of this table show that the numerical integration procedure for computing the moments and covariances is accurate to more than 3 decimals for all sample sizes and all values of α considered. Tables of the variances and covariances of order statistics for Weibull distribution with α =2 and n=5, 10, 15, 20 and 30, are available from the author.

Define $E^k_r = E(X^k_{(r)})$ as the k[th] moment calculated from Lieblein explicit formula r=1,2,...,n. Define $\mu^k_r = E(X^k_{(r)})$ as the k[th] moment calculated from the suggested numerical integration algorithm, r=1,2,...n. For a specific sample size n, define

$$\text{MAXm(k)}=\max_r|E^k_r - \mu^k_r| \qquad k = 1,2,3,4$$

Values of MAXm(1) and MAXm(2) for selected sample sizes are given in Table 5. The table shows that both Lieblein and numerical integration methods give similar results for n <30 and the loss in accuracy of Lieblein explicit formula increases rapidly as the sample size increases while the numerical integration values do not affect by the increase in the sample size. The stability of calculated values by the numerical integration for large samples holds also for all higher moments.

5. THE MOMENTS AND PRODUCT MOMENTS OF TYPE I EXTREME VALUE DISTRIBUTION

Let $X_{(1)}, X_{(2)},\ldots,X_{(n)}$ be the order statistics corresponding to a simple random sample from a standard type 1 extreme value distribution with probability density function

$$f(x) = \exp(x-e^x) \qquad -\infty < x < \infty.$$

The moments of the extreme value order statistics can be written as (Lieblein (1953))

$$E(X^k_{(i)}) = n\binom{n-1}{i-1}\sum_{r=0}^{n-i}\binom{n-i}{r}(-1)^{-r}g_k(i+r)$$

where

$$g_k(c) = \int_{-\infty}^{\infty} x^k e^{x-ce^x}dx \qquad c > 0$$

In particular, the first two moments are

$$E(X_{(i)}) = -n \binom{n-1}{i-1} \sum_{k=0}^{i-1} \binom{i-1}{k} (-1)^{-k} \frac{\gamma + \ln(n-i+k+1)}{n-i+k+1}$$

and

$$E(X_{(i)}^2) = n \binom{n-1}{i-1} \sum_{k=0}^{i-1} \binom{i-1}{k} (-1)^{-k} \frac{\frac{\pi^2}{6} + (\gamma + \ln(n-i+k+1))^2}{n-i+k+1}$$

where $\gamma=-\Gamma'(1)$ is the Euler's constant, 0.5772156649…

Lieblein (1953) derived an expression for the product moments as

$$E(X_{(i)}X_{(j)}) = \binom{n}{j}\binom{j}{i-1} \sum_{r=0}^{j-i-1} \sum_{s=0}^{n-j} \binom{j-i-1}{r} \binom{n-i}{s} (-1)^{-r-s} \xi(i+r, j-i-r+s)$$

where the function

$$\xi(t,u) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} xy e^{(x-te^{x})} e^{(y-ue^{y})} dx dy \qquad t, u > 0$$

is expressed in terms of Spence's function which have been extensively tabulated by Abramowitz and Stegun (1965).

White (1969) presented a table for the means of order statistics for n=1(1)50(5) 100. Balakrishnan and Chan (1992) presented tables of means, variances and covariances of all order statistics for n=1(1)15(5)30. For larger sample sizes the accuracy in their numerical computations reduces rapidly as n increases. Childs and Balakrishnan (2002) presented a Maple procedure that approximate the means, variances, and covariances of order statistics from the extreme value distribution using series approximations in the form of David and Johnson's approximation.

Fard and Holmquist (2007) presented an approximation of the first moment of the order statistics by finding α and β that minimizing

$$Q(\alpha, \beta) = \sum_{i=2}^{n} (M_i - g_i(\alpha, \beta))^2$$

where M_i is the expected values in White (1969) tables, and

$$g_i(\alpha, \beta) = F^{-1}\left(\frac{i-\alpha}{n-\alpha-\beta+1}\right),$$

Then approximate $E(X_{(i)})$ by the Blom (1958) approximation with best fitted $\hat{\alpha}$ and $\hat{\beta}$ as

$$E(X_{(i)}) \approx F^{-1}((i-\hat{\alpha})/(n-\hat{\alpha}-\hat{\beta}+1)).$$

Accordingly, they concluded that differences between this approximation and the exact values are less than two decimal places.

Using Taylor's series expansion, Fard and Holmquist (2008) suggested an approximation for the variances and covariances for order statistics from the extreme value distribution. The approximation was constructed using exact values of the covariances from the tables of Balakrishnan and Chan (1992) for n < 30. For n > 30, the exact covariances were replaced by the empirical covariances based on Monte Carlo simulations. They presented a similar approximation procedure for the variances depend on the tabulated values of White (1969) for n<100 and replaced by the empirical variances based on Monte Carlo simulations for n>100. Their maximum absolute deviation between the exact and the approximate covariances is less than two decimal places for all sample sizes.

The proposed numerical integration procedure gives accurate results for the moments and product moments for all sample sizes. The computation of the first two moments and the covariances of order statistics were checked for their accuracy and $max_n|\Sigma X_{(i)} - n\mu|$, $max_n|\Sigma X^2_{(i)} - nE(X^2)|$ and $max_n|\Sigma\Sigma cov(x_{(i)}, x_{(j)}) - n\sigma^2$ for n=2(1)150 were computed. The results are summarized in table 6. This shows that the numerical integration procedure for computing the moments and covariances are accurate to more than 2 decimals for all sample sizes <100. Moreover, the procedure is much simpler than Fard and Holmquist (2007, 2008) approximations.

6. NONLINEAR LEAST SQUARES APPROXIMATION TO THE FIRST MOMENT

Another use of the numerical integration algorithm is finding a simple approximation to the first moment $E(X_{(r)})$ as a function of r, r=1,2,...,n. This simple approximation is not only highly accurate but it also provides a simple rule to put a bound on the first moments of extreme order statistics. These bounds are much easier to compute than the bounds presented by Joshi (1969) and Joshi and Balakrishnan (1983) especially for large samples.

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution F_0. Let $C_r(F_0)$ be the approximated $E(X_{(r)})$ by the numerical integration procedure, then

$$C_r(F_\theta) \approx \eta(r,\beta) + \varepsilon_r \qquad r = 1,2,\ldots,n \qquad (9)$$

where, $\eta(r, \beta)$ is a linear or nonlinear function in r, and ε_r is the error of the approximation. In particular, the linear model will be

$$\eta(r,\beta) = \textstyle\sum_{j=0}^{k} g_j(r)\beta_j \qquad 2 \le k < n \qquad (10)$$

The nonlinear least squares approximation for (9) is a kx1 vector $\hat{\beta}$ that minimizes

$$Q(\beta) - \textstyle\sum_{r=1}^{n}((C_r - \eta(r,\beta))/w_r)^2$$

$$= (C - \eta(\beta))^T W(C - \eta(\beta)) \qquad (11)$$

The value w_r is a measure of the error in approximation in C_r. If the function η is nonlinear the minimization of Q(.) with respect to β will be done iteratively using the Gauss-Newton method or the Levenberg-Marquardt method, (Nocedal and Wright , 2006). The discrepancy between the model and the calculated C is measured by the mean squared error

$$MSE = (e^T We)/(n - k - 1).$$

where the residuals

$$e = C - \eta(\hat{\beta})$$

is an estimate of the approximation errors $\varepsilon = C - \eta(\beta)$.

To find the "best approximation" one needs to consider many forms of the function η and choose the one that minimizes the MSE (or maximizes the coefficient of determination

$$R^2=1-(e^TWe)/C^T[I-(1/n)J]C,$$

where J is an nxn matrix of ones.

For all distributions considered in this study, the relationship between the estimated C_r and r is the same for all sample sizes considered. So one can search for one form for the approximating function η that best fit C. Using the statistical package DataFit one can evaluate more than 300 linear and nonlinear forms of the function η and choose the one with the highest R^2 (≈1.00).

For this approximation to be easy to use it requires that the number of parameters in the fitted model to be at most 4 parameters. To achieve this we have to allow a model with less accuracy ($R^2 < 0.999$). Joshi and Balakrishnan (1983), and David and Nagaraja (2003) showed that the approximations methods for the moments of order statistics do not give satisfactory results for the extreme order statistics. The residuals e are the difference between the calculated expected value by the numerical integration algorithm and the corresponding value estimated by the nonlinear least squares. The corresponding standardized residuals will be $e^*=e/\sqrt{MSE}$. Thus, one can increase R^2 again by "trimming" the values with $|e^*|>2.5$, i.e. values with less accurate approximation by the numerical integration procedure. Then fit a trimmed nonlinear least squares model.

$$Q_T(\beta) = (C_T - \eta(\beta))^T W_T (C_T - \eta(\beta)) \qquad (12)$$

where C_T is the calculated C after trimming certain proportion of the extremes. That is, replace the less accurately fitted extremes by its predicted value from the model with very high accuracy (small MSE and nearly perfect R^2). The fitting of this model can be described by the following steps.

Step 1: Given a random sample of size n > 5 from a distribution F_{θ^*}, use the numerical integration procedure to calculate

$$C = (E(X_{1:n}), (E(X_{2:n}), \ldots, (E(X_{n:n}))^T.$$

Step 2: Find the best fitting models $\eta(r,\beta) = \sum_{j=0}^{k} g_j(r)\beta_j$ such that $2 \leq k \leq 4$

Step 3: From the standardized residual plot of the chosen model $\eta^*(r,\beta)$ consider an observation to be an "outlier" (or less acuratly approximated values) if the absolute standardized residuals $e^*=|e/\sqrt{MSE}|> 2.5$

Step 4: If no outliers and $R^2 > 0.999$, fit the same model for different sample sizes n=10(5)100 and plot the standardized residual for each sample size.

If no outliers and $R^2 > 0.999$ for all sample sizes, report the model

$C(F_{\theta^*}) \approx \eta^*(r.\widehat{\beta})$.

Step 5: If the standardized residual plot reveals outliers, $|e^*| > 2.5$, trim the outlying observations and recompute R^2. (A further study is needed to determine the optimal trimming proportions as a function the skewness of F_{θ^*}).

Use the trimmed nonlinear least squares in (12) to fit the model

$$C(F_{\theta^*}) \approx \eta^*(r.\widehat{\beta}_T)$$

Fit the same model for different sample sizes n=10(5)100.

Step 6: Report the trimmed model if R^2 improved over the untrimmed model. Otherwise, use the untrimmed model.■

Applying this algorithm, the "best" nonlinear least squares fit for $E(X_{(r)})$ for the gamma distribution is given by

$$E(X_{(r)}|\theta) \approx r/(a + b\,r + c\,r^2). \qquad (13)$$

The estimated coefficients a, b and c for θ =1, 2, and 4 with n=10(10)100 are given in Table 7. A tenth degree polynomial gives the "almost perfect" fit ($R^2 \approx 100$) for all sample sizes, but it is not reported here. The trimmed least squares approximation does not substantially increase R^2 so it does not improve the accuracy of the results.

The "best" nonlinear least square fit for $E(X_{(r)})$ for the Weibull distribution is given by

$$E(X_{(r)}|\alpha) \approx b_0 + b_1 r + b_2 r^2 + b_1 r^3 \qquad (14)$$

where the error of approximation is very small for all sample sizes considered, MSE <0.0001 , $R^2 > 0.99$ for all $n \le 100$. For nearly symmetric shape (α=4) almost perfect fit were achieved by right and left trimming a fixed proportion for all sample sizes. Table 8.a gives the best fit for α=4 and proportion of trimming [0.05n] from both sides for different sample sizes. Table 8.b gives the best fit for α=2 and proportion of trimming [0.06n] from the right side for different sample sizes. Table 8.c gives the best fit for α=1 and proportion of trimming [0.15n] from the right side for different sample sizes. The accuracy of thee fitted values in tables 8a, 8b, and 8c are computed to 15 decimal places but truncated for the space. Graphs 1 and 2 illustrate the effect of "outliers" of the above fit for α=4. Graphs 3 and 4 illustrate the effect of "outliers" for α=2. For positively skewed shape (α<3.5) nearly perfect fit is achieved by right trimming a fixed proportion for all sample sizes. The proportion of trimming increases as the distribution became more skewed. For nearly symmetric shape a better fit will be achieved by trimming same proportion from both sides.

The "best" nonlinear least square fit for $E(X_{(r)})$ for the extreme value distribution to be

$$E(X_r) \approx b_0 + b_1 r^3 + b_2 ln(r) \quad (15)$$

The error of approximation is very small for all sample sizes considered, MSE <0.0001 and $R^2 > 0.99$ for all $n \leq 100$. One can get better fit (R^2>0.999) by trimming a small proportion of extreme values from each tail, [0.01n] from the left and [0.07n] from the right. Note that the coefficient of skewhess for the extreme value distribution is 1.139547. Table 9a gives the best fit without trimming. Table 9b gives the best fit after trimming [0.01n] from the left and [0.07n] from the right. Graphs 5 and 6 illustrate the effect of "outliers". The mean square are and R will always increase for any sample size by trimming 7% observations from the right.

7-COMMENTS AND CONCLUTIONS

The proposed algorithm facilitates "nearly exact" computation of the moments of order statistics for any sample size and for a continuous distribution. In particular, for the gamma distribution its calculation is accurate and does not need numerically computed special functions as the algorithms of Nadarajah and Pal (2008) and Sobel and Wells (1990). While the available tables for the covariances of order statistics is restricted to small samples only, Prescott (1974), the numerical integration procedure accurately calculates the product moments and covariances for any value of the shape parameter θ, and for any sample size.

For the Weibull distribution, Harter (1970b) and Harter and Balakrishnan (1996) tabulated the variances and covariances for small sample sizes only. Both Lieblein (1955) explicit formula and numerical integration procedure give similar results for n <30 and the loss in accuracy of Lieblein explicit formula increases rapidly as the sample size increases while the numerical integration values do not affect by the increase in the sample size. For the extreme-value distribution, numerical integration procedure for computing the moments and covariances are accurate to more than 2 decimals for all sample sizes <100. Moreover, the procedure is much simpler than Fard and Holmquist (2007, 2008) approximations.

The accuracy of the calculation for the logistic, , beta, and the student's t distributions will be presented in a forthcoming article, The existing tables for the above distributions either do not exist or incomplete (for the expected values only or for small sample sizes only).

The nonlinear least squares approximation presents an accurate and simple calculation for the first moments of order statistics. However, the optimal trimming proportions as a function the skewness of F_θ. need to be investigated.

A further study is needed to approximate higher moments by nonlinear least squares procedure. The exact calculation of the moments of order statistics can improve the statistical analysis in many applications: Lloyd's weighted least squares estimators for the location scale family, Linear estimators for censored samples from any continuous distribution and regression based goodness of fit tests.

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Appendix A

Table 1 Accuracy of the computation for gamma moments

α		$max_n \lvert \Sigma EX_{(i)} - n\mu \rvert$	$max_n \lvert \Sigma EX^2_{(i)} - nE(X^2) \rvert$	$max_n \lvert \Sigma\Sigma cov(x_{(i)}, x_{(j)}) - n\sigma^2 \rvert$
1	all n	0.002803	0.058238825	0.304403019
	n≤30	0.0008744	0.0056303	0.0237102
	30<n≤77	0.001698072	0.015949996	0.22411007
	77<n≤100	0.002578389	0.058238825	0.304403019
2	all n	0.001345491	0.024375959	0.018989477
	n≤30	4.04E-04	7.31E-03	5.69E-03
	30<n≤77	0.001036056	0.018769483	0.014614096
	77<n≤100	0.001345491	0.024375959	0.018989477
3	all n	0.000170687	0.020333693	0.0145676
	n<=30	0.000051215	0.00610703	0.00437103
	30<n≤77	0.000131429	0.015656944	0.01121694
	77<n≤100	0.000170687	0.020333693	0.0145676
4	all n	٠,٠٠٠٦٧٧٧٨٩٨	0.030543148	0.020009765
	n≤30	0.00203103	0.00916503	00600103
	30<n≤77	٠,٠٠٠٥٢١٨٩٨٢	0.023518224	0.015407533
	77<n≤100	٠,٠٠٠٦٧٧٧٨٩٨	0.030543148	0.020009765

Table 2 Maximum and average absolute differences of moments with Breiter and Krishnaiah tables

θ		$E(X_{(r)})$	$E(X^2_{(r)})$	$E(X^3_{(r)})$	$E(X^4_{(r)})$
1.5	MAXDm	0.0000500814	0.000410127	0.00046387	0.004776459
	AVRDm	0.0000152055	0.0000444338	0.0000928751	0.000695378
5.5	MAXDm	0.00040222	0.00055033	0.005015948	0.096961253
	AVRDm	0.0000359135	0.000215485	0.002005333	0.016873319
7.5	MAXDm	0.009546677	0.00456351	0.048151864	0.465129394
	AVRDm	0.000271027	0.000701033	0.006339337	0.073916505

Table 3 Maximum and average absolute differences of covariances with Prescott tables

	θ=2		θ=3		θ=4	
n	MAXDv	AVRDv	MAXDv	AVRDv	MAXDv	AVRDv
2	2.27822E-07	2.27822E-07	6.3875E-06	6.3875E-06	3.9056E-05	3.9056E-05
3	4.45174E-05	4.45174E-05	4.98436E-05	4.98436E-05	7.97923E-05	7.97923E-05
4	0.000134005	0.000134005	7.28268E-05	7.28268E-05	0.000461102	0.000461102
5	0.000437856	0.000437856	0.001880824	0.001880824	0.001014754	0.001014754
6	0.003175321	0.003175321	0.004965222	0.004965222	0.010550329	0.010550329
7	0.005299528	0.005299528	0.014529689	0.014529689	0.02348847	0.02348847
8	0.014347817	0.002946011	0.033027082	0.007142035	0.060430368	0.012725836
9	0.025829259	0.005422265	0.052631021	0.013338257	0.101375433	0.023933249
10	0.037801243	0.009730802	0.102076181	0.021309409	0.182091228	0.038777541

Table 4 Accuracy of the computation for the Weibull moments

α	$max_n \lvert \Sigma X_{(i)} - n\mu \rvert$	$max_n \lvert \Sigma X^2_{(i)} - nE(X^2) \rvert$	$max_n \lvert \Sigma\Sigma cov(x_{(i)}, x_{(j)}) - n\sigma^2 \rvert$
1	1.243078E-3	3.871262E-3	8.879795E-3
2	6.901261E-7	1.175743E-8	2.753822E-5
4	2.611964E-10	2.700925E-10	4.009964E-6
6	1.682100E-10	1.657130E-10	1.236547E-07

Table 5 Selected values of $\max_r \lvert E_r^k - \mu_r^k \rvert$, k=1,2, α=2 for the Weibull moments

n	MAXm(1)	MAXm(2)
5	3.97E-08	1.43E-10
10	7.95E-08	5.62E-10
15	2.98E-08	1.18E-09
20	3.97E-08	5.23E-10
25	6.75E-07	1.33E-07
30	0.000159	1.55E-05
35	0.012799	0.002402
40	4.525267	0.882569
50	281115.1	18990.8
60	6.57E+10	1.23E+10
70	5.21E+16	8.54E+15
80	1.59E+21	2.49E+20
100	7.02E+30	9.92E+29

Table 6. Accuracy of the computation for the extreme value moments

n	$max_n \lvert \Sigma X_{(i)} - n\mu \rvert$	$max_n \lvert \Sigma X^2_{(i)} - nE(X^2) \rvert$	$max_n \lvert \Sigma\Sigma cov(x_{(i)}, x_{(j)}) - n\sigma^2 \rvert$
2≤n<20	0.0000978	0.0015703	0.00065502
20≤n<60	0.000298	0.004795605	0.015919896
60≤n<100	0.000489	0.007861647	0.047806417
100≤n<150	0.000733	0.011792471	0.113456224

Table 7. The nonlinear least squares fit for $E(X_{(r)})$ for the gamma distribution

$$E(X_{(r)}|\theta) \approx r/(a + b\,r + c\,r^2)$$

θ	n	b	c	a	R^2	MSE
1	10	-0.41244	-0.0259	10.43126	99.99742	0.000254
1	20	-0.43399	-0.01249	20.43479	99.99813	0.00042
1	30	-0.44409	-0.00808	30.44332	99.99861	0.000505
1	40	-0.45131	-0.0059	40.45856	99.99887	0.000571
1	50	-0.45781	-0.00459	50.4837	99.99903	0.000635
1	60	-0.46412	-0.00371	60.51912	99.99911	0.000714
1	70	-0.47072	-0.00307	70.56708	99.99914	0.000822
1	80	-0.47799	-0.00258	80.63102	99.99912	0.00098
1	90	-0.48572	-0.00219	90.70967	99.99905	0.001205
1	100	-0.49412	-0.00187	100.8061	99.99893	0.001523
2	10	0.596059	-0.05428	1.612099	99.96649	0.00013
2	20	0.747969	-0.03521	2.513486	99.80245	0.001096
2	30	0.844488	-0.02706	3.261114	99.59173	0.002748
2	40	0.915464	-0.02234	3.921702	99.37155	0.00481
2	50	0.971603	-0.01919	4.523321	99.15342	0.007121
2	60	1.01801	-0.01691	5.081193	98.94133	0.009589
2	70	1.057532	-0.01517	5.604752	98.7366	0.012156
2	80	1.091923	-0.0138	6.100359	98.5395	0.014782
2	90	1.122338	-0.01268	6.572563	98.34985	0.017443
2	100	1.149585	-0.01175	7.024758	98.16729	0.020122
4	10	0.348648	-0.0246	0.321638	99.9543	1.25E-05
4	20	0.388428	-0.01458	0.453867	99.74149	8.24E-05
4	30	0.411907	-0.01064	0.555073	99.45484	0.000189
4	40	0.42835	-0.00847	0.640021	99.15083	0.00031
4	50	0.440888	-0.00708	0.714482	98.84836	0.000437
4	60	0.450952	-0.00611	0.781442	98.55422	0.000564
4	70	0.459315	-0.00538	0.842689	98.27081	0.000691
4	80	0.466441	-0.00482	0.899395	97.99872	0.000815
4	90	0.472627	-0.00437	0.95238	97.73776	0.000936
4	100	0.478077	-0.004	1.002242	97.48746	0.001053

Table8(a) Fitted regression $E(X_{(r:n)})=b_0+b_1r+b_2r^2+b_3r^3$ for the Weibull distribution with α =4 The proportion of trimming is [0.05n] from both sides

n	# trim	b_0	b_1	b_2	b_3	R^2	MSE
10	0	0.349407876	0.183668466	-0.021906666	0.001289999	99.975	2.14E-0
	1	0.281453427	0.163220173	-0.015525386	0.001501482	99.995	8.98E-0
15	0	0.349407876	0.130614779	-0.011152867	0.000452132	99.926	5.58E-0
	1	0.231720618	0.122676493	-0.0090559	0.000545105	99.977	3.81E-0
20	0	0.349407876	0.101409811	-0.006739792	0.000208323	99.875	8.96E-0
	1	0.2026682	0.099122421	-0.005938459	0.000259323	99.952	8.09E-0
30	0	0.349407876	0.070109039	-0.003227096	6.76E-05	99.788	0.000145
	2	0.24467904	0.062161903	-0.002391911	7.42E-05	99.964	5.04E-0
40	0	0.349407876	0.053584281	-0.001886225	2.99E-05	99.722	0.000188
	2	0.212685103	0.050099375	-0.001549152	3.46E-05	99.935	9.86E-0
50	0	0.349407876	0.043366466	-0.001235754	1.57E-05	99.67	0.000220
	3	0.23679526	0.0383918	-0.000919303	1.70E-05	99.951	6.83E-0
75	0	0.349407876	0.029369277	-0.00056688	4.85E-06	99.582	0.000276
	4	0.225533206	0.02635655	-0.0004337	5.27E-06	99.936	8.99E-0
100	0	0.349407876	0.022203761	-0.000324019	2.09E-6	99.526	0.000311
	5	0.219707135	0.020072726	-0.000251523	2.28E-06	99.928	0.000103

Table8(b) Fitted regression $E(X_{(r:n)})=b_0+b_1r+b_2r^2+b_3r^3$ for the Weibull distribution with α =2. The proportion of trimming is [0.06n]

n	#trim	b_0	b_1	b_2	b_3	R^2	MSE
10	0	0.063974234	0.237438014	-0.03009503	0.00223141	99.918	0.000232
	1	0.099066037	0.200117163	-0.02027913	0.00151567	99.996	7.22E-06
15	0	0.063974233	0.169173626	-0.01550735	0.00078209	99.814	0.000464
	1	0.096131304	0.143886733	-0.0108769	0.00055221	99.981	3.46E-05
20	0	0.063974294	0.131514658	-0.00943016	0.00036035	99.717	0.00067
	2	0.10315965	0.106824822	-0.00592131	0.00022654	99.987	1.94E-05
30	0	0.063974295	0.091061451	-0.00454444	0.00011697	99.563	0.000993
	2	0.099809307	0.075134516	-0.00300612	7.76E-05	99.967	5.40E-05
40	0	0.06397431	0.069659159	-0.00266495	5.17E-05	99.45	0.001229
	3	0.102337267	0.056420642	-0.00169093	3.28E-05	99.967	4.98E-05
50	0	0.06397431	0.056407825	-0.00174941	2.72E-05	99.364	0.001407
	3	0.100098452	0.046264885	-0.00114965	1.79E-05	99.951	7.78E-05
75	0	0.06397431	0.038231793	-0.00080467	8.39E-06	99.219	0.001708
	5	0.102355072	0.030836159	-0.00050943	5.30E-06	99.952	7.26E-05
100	0	0.063974314	0.028916116	-0.00046056	3.61E-06	99.127	0.001897
	6	0.101510341	0.023423122	-0.00029544	2.31E-06	99.942	8.93E-05

Table8(c) Fitted regression $E(X_{(r:n)})=b_0+b_1r+b_2r^2+b_3r^3$ for the Weibull distribution with α =1. The proportion of trimming is [0.15n]

n	#trim	b_0	b_1	b_2	b_3	R^2	MSE
5	0	-0.23796196	0.58544746	-0.18431103	0.033467	99.97	0.0007887
10	0	-0.23796	0.360635	-0.07093	0.006552	99.57	0.0049860
	2	-0.03135753	0.13716842	-0.01082136	0.0020543	99.99	3.62E-05
15	0	-0.23795964	0.26183772	-0.03707219	0.002297	99.11	0.0093157
	3	-0.025136969	0.090315385	-0.004684562	0.00063236	99.98	4.32E-05
20	0	-0.23795979	0.20572476	-0.02270793	0.001058	98.71	0.0129731
	3	-0.03135753	0.13716842	-0.01082136	0.0020543	99.99	3.62E-05
30	0	-0.23795991	0.14409441	-0.01102372	0.000344	98.11	0.0184811
	5	-0.02659282	0.04820917	-0.00154029	9.42E-05	99.96	9.62E-05
40	0	-0.2379601	0.11090379	-0.00648854	0.000152	97.69	0.0223485
	6	-0.02936537	0.03774699	-0.00100642	4.33E-05	99.95	0.000142
50	0	-0.23796065	0.09014740	-0.00426896	8.00E-05	97.39	0.0251978
	8	-0.02533219	0.02917618	-0.00057762	2.11E-05	99.95	0.000111
75	0	-0.23796004	0.06141789	-0.00196946	2.46E-05	96.90	0.0298485
	12	-0.023813065	0.019339783	-0.000252962	6.28E-06	99.96	0.00010924
100	0	-0.23795998	0.04657642	-0.00112893	1.06E-05	96.60	0.0326565
	15	-0.0256339	0.01489494	-0.0001562	2.79E-06	99.94	0.000137

Table 9(a): The best approximate fit for the extreme value first moment.

$$E(X_r) \approx b_0 + b_1 r^3 + b_2 ln(r)$$

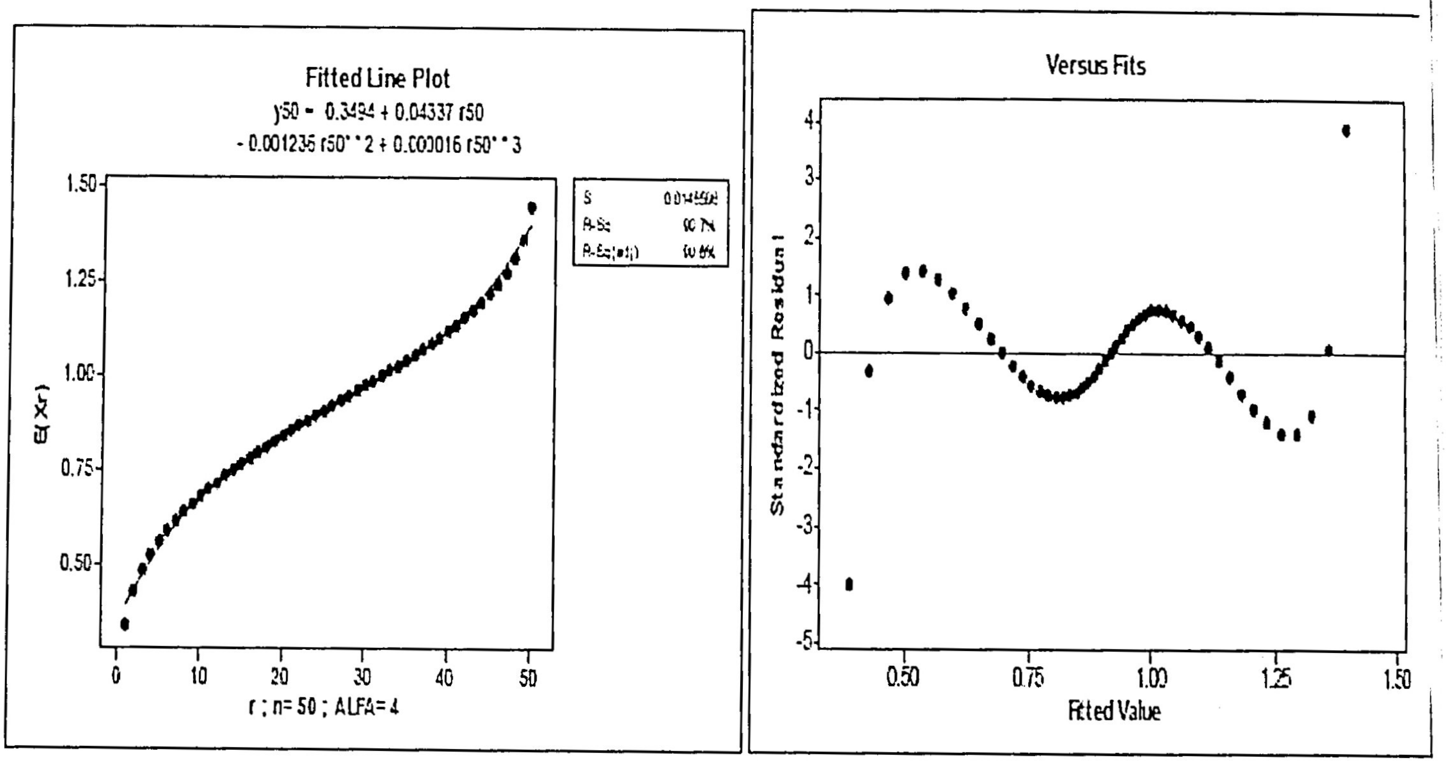
n	b0	b1	b2	R2	mse
5	-2.177602305	0.003329024	1.514974368	99.95822751	0.001029
10	-2.827792478	0.000606989	1.36845269	99.87570393	0.002205
15	-3.200186065	0.000208882	1.302425924	99.83421427	0.002792
20	-3.462203118	9.58E-05	1.263138909	99.81164086	0.003117
25	-3.664755047	5.18E-05	1.23646825	99.79830441	0.003312
30	-3.83007548	3.11E-05	1.216898268	99.78996539	0.003435
35	-3.969868165	2.02E-05	1.201779973	99.78455014	0.003516
40	-4.091054461	1.38E-05	1.189664737	99.78094833	0.003570
45	-4.198068788	9.91E-06	1.179685883	99.77852236	0.003607
50	-4.293923862	7.34E-06	1.171289446	99.77688548	0.003632
55	-4.380760488	5.58E-06	1.164102886	99.7757926	0.003650
60	-4.460154879	4.35E-06	1.157865356	99.77508284	0.0036613
65	-4.533301445	3.46E-06	1.152388133	99.77464742	0.0036685
70	-4.601127107	2.79E-06	1.1475309	99.77441099	0.0036726
75	-4.664365765	2.29E-06	1.143186913	99.77432026	0.0036745
80	-4.601127107	2.79E-06	1.1475309	99.77441099	0.0036726
85	-4.664365765	2.29E-06	1.143186913	99.77432026	0.0036745
90	-4.831955871	1.35E-06	1.132489148	99.7745875	0.003671
95	-4.881795566	1.15E-06	1.129523683	99.77478528	0.0036691
100	-4.929141013	9.93E-07	1.126793621	99.77501453	0.003666

Table 9(b) The best approximate fit for the extreme value first moment. [0.01n] from the left and [0.07n] from the right

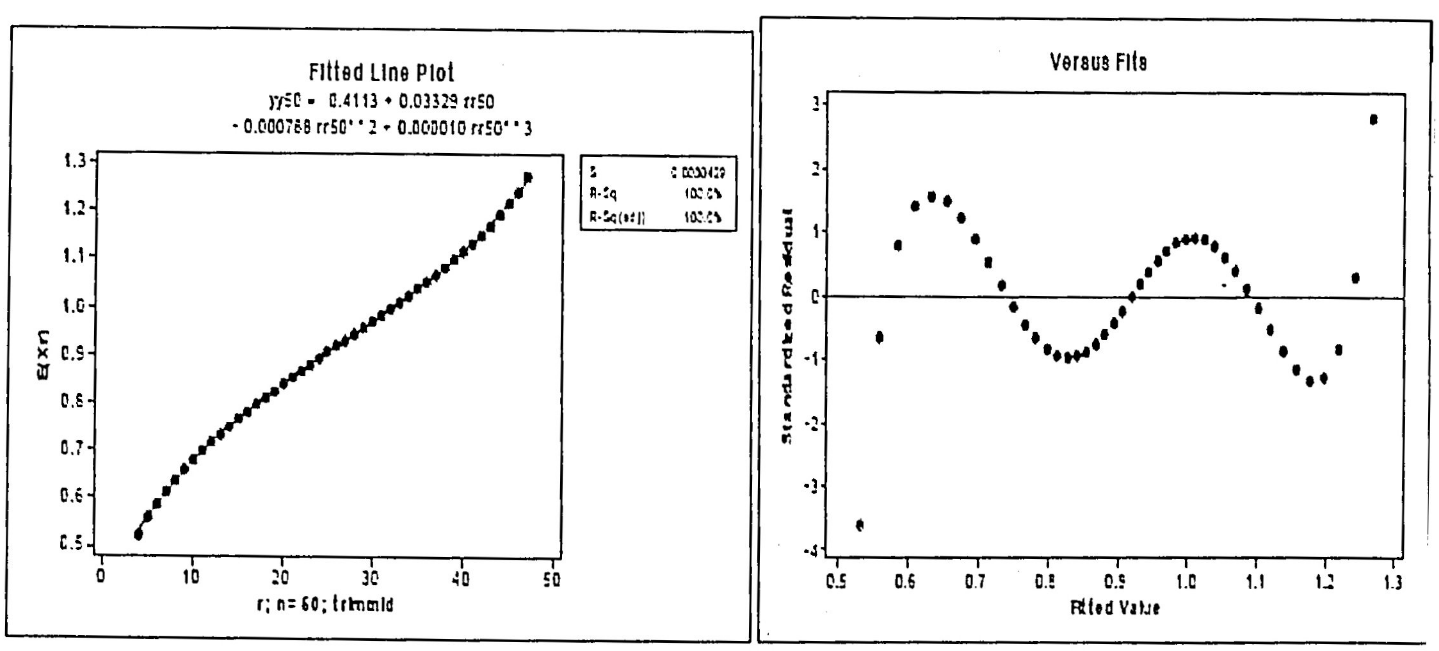
$$E(X_{(r)}) \approx b_0 + b_1 r^3 + b_2 ln(r)$$

N	b0	b1	b2	R2	MSE
5	-2.096645126	3.17E-03	1.443208348	100	1.70E-14
10	-2.732539902	5.79E-04	1.312733468	99.98945831	8.07E-05
15	-3.060498593	2.11E-04	1.230602968	99.97932443	0.000135727
20	-3.328502211	9.62E-05	1.202263469	99.96822866	0.000242574
25	-3.536894808	5.19E-05	1.183030317	99.95862147	0.000349271
30	-3.684264538	3.17E-05	1.159141737	99.94289036	0.00044001
35	-3.831120181	2.05E-05	1.149647947	99.93505439	0.00053689
40	-3.958409997	1.40E-05	1.141960102	99.92824396	0.000627143
45	-4.055878567	1.01E-05	1.130196872	99.91267653	0.000715132
50	-4.157912267	7.47E-06	1.125458989	99.90729194	0.000794037
55	-4.250177918	5.68E-06	1.121338136	99.90251027	0.000867378
60	-4.323673858	4.44E-06	1.114194393	99.88828582	0.000945058
65	-4.402152153	3.52E-06	1.111373302	99.88452089	0.001009304
70	-4.474746251	2.84E-06	1.108820979	99.88111944	0.001069368
75	-4.533965371	2.34E-06	1.103950541	99.86829905	0.001136947
80	-4.597858848	1.93E-06	1.102093503	99.86562684	0.001190039
85	-4.657806387	1.62E-06	1.100370261	99.86318119	0.001239996
90	-4.707486029	1.38E-06	1.096795928	99.85160633	0.001298997
95	-4.761435817	1.17E-06	1.095491869	99.84969686	0.001343583
100	-4.806291265	1.01E-06	1.092498917	99.83888785	0.001397691

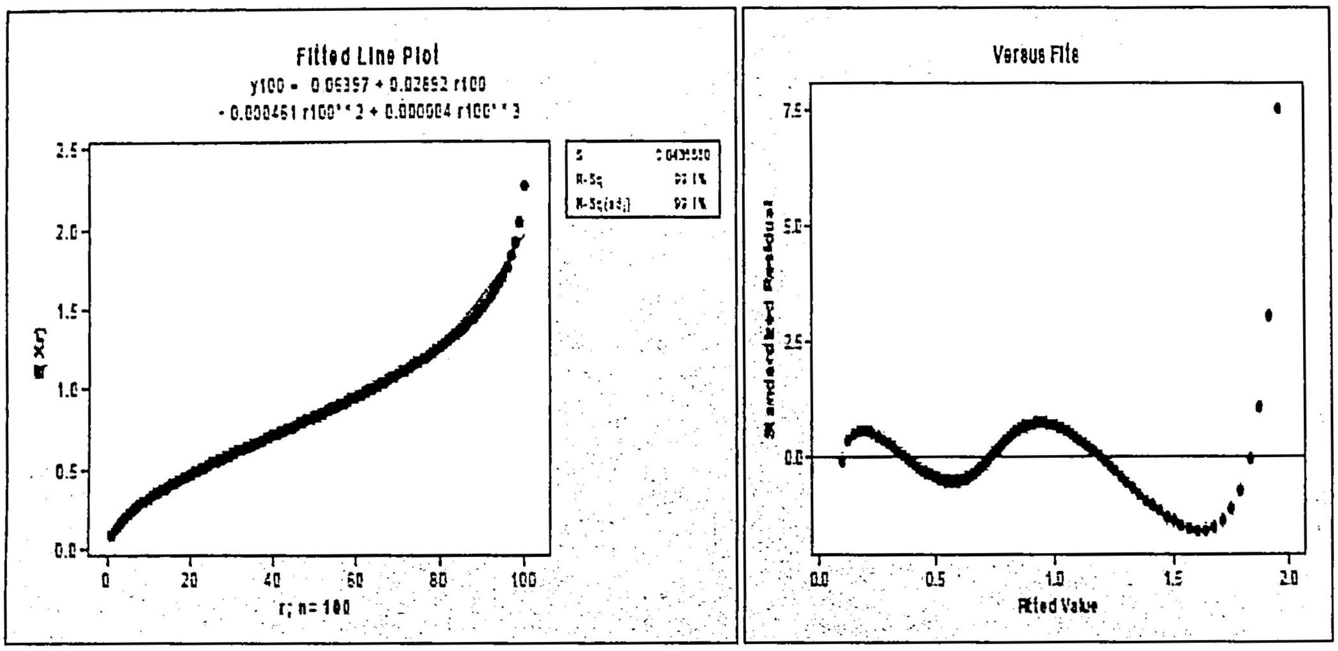
Appendix B



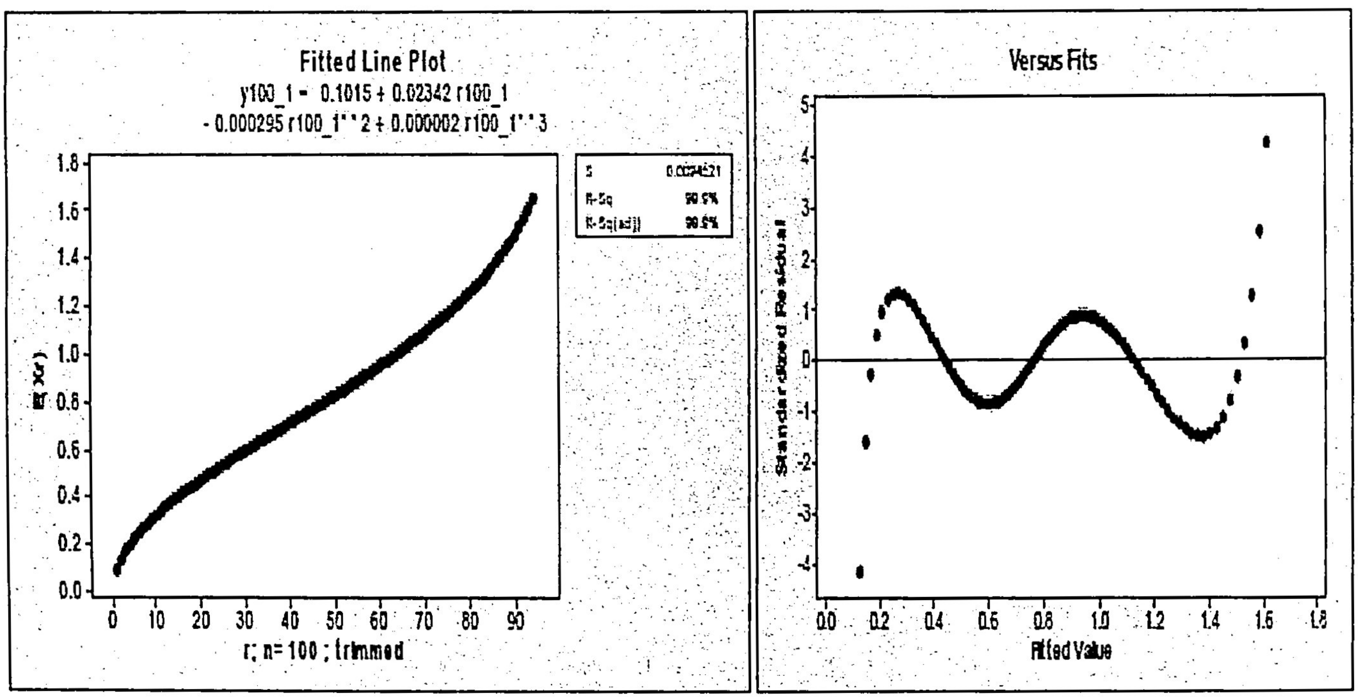
Graph 1: Weibull distribution fit alfa=4; n=50; no trimming



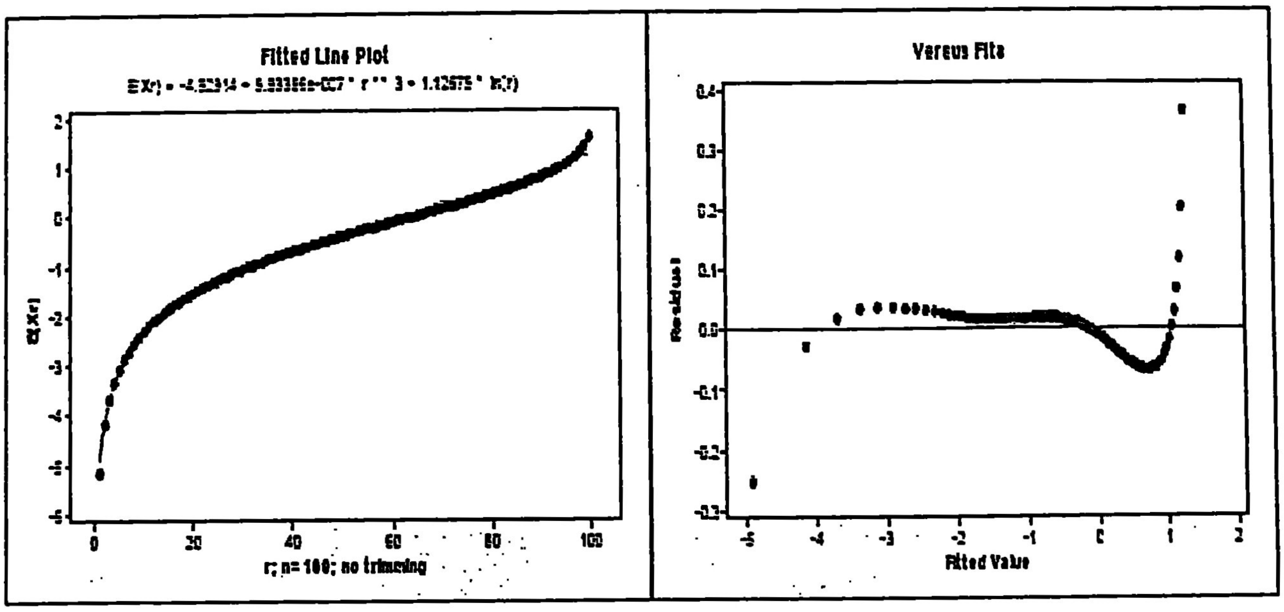
Graph 2: Weibull distribution fit alfa=4; n=50; trim [0.05n] both sides



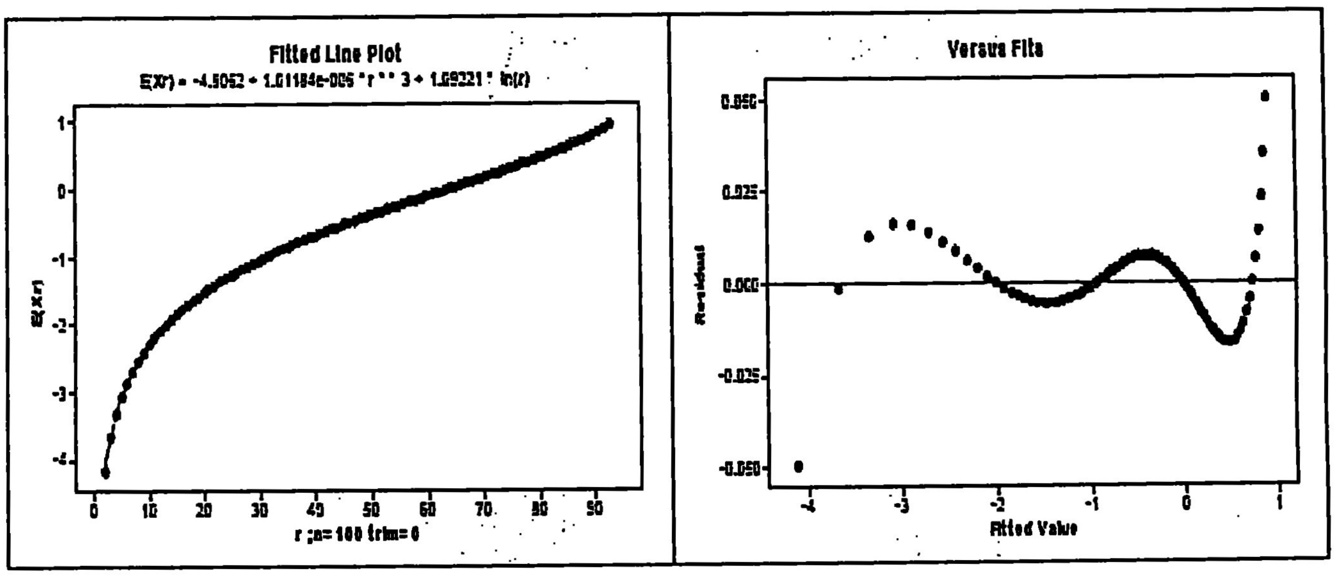
Graph 3: Weibull distribution fit alfa=2; n=100; no trimming



Graph 4: Weibull distribution fit alfa=2; n=100; trim=[0.06] right.



Graph 5 Extreme value distribution fit , n=100; no trimming



Graph 6 Extreme value distribution fit , n=100; trim [0.01n] left and [0.07n] right