Bayesian and Non-Bayesian Estimation for Exponentiated-Weibull Distribution Based on Record Values

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Abstract This article deals with the problem of estimating parameters for the two-parameter exponentiated Weibull distribution based on a set of lower record values. Maximum likelihood and Bayesian estimators, either point or interval, for the two shape parameters of the exponentiated-Weibull distribution are derived based on lower record values. Bayes estimators have been developed under symmetric (squared error) and asymmetric (LINEX) loss functions. Numerical computations are given to illustrate these results.

Keywords Exponentiated Weibull distribution; Record values; Maximum likelihood estimation; Bayesian estimation; Numerical example.

1. Introduction

The exponentiated Weibull (EW) family was introduced by Mudholkar and Srivastava (1993) as a simple generalization of the well-known Weibull distribution by introducing one additional shape parameter. The main feature of this family from other lifetime distribution is that it accommodates nearly all types of failure rates both monotone and non-monotone (unimodal and bathtub) and includes a number of distributions as particular cases.

The two shape parameters EW distribution has a cumulative distribution function (cdf) of the form

$$F(x) = [1 - \exp(-x^{\alpha})]^{\theta}, \quad x > 0, \quad \alpha, \theta > 0 \qquad (1.1)$$

and hence the probability density function (pdf) is given by

$$f(x) = \alpha\theta[1 - \exp(-x^{\alpha})]^{\theta-1}\exp(-x^{\alpha})x^{\alpha-1}, x > 0, \quad \alpha, \theta > 0 \qquad (1.2)$$

where α and θ are the two shape parameters. The reliability and hazard (failure rate) functions of the EW distribution are given, respectively, by

$$R(t) = 1 - [1 - \exp(-t^{\alpha})]^{\theta} \tag{1.3}$$

$$H(t) = \frac{\alpha\theta[1 - \exp(-t^{\alpha})]^{\theta-1}\exp(-t^{\alpha})t^{\alpha-1}}{\left[1 - \left(1 - \exp(-t^{\alpha})\right)^{\theta}\right]} \tag{1.4}$$

The principal applications of the exponentiated Weibull (EW) distribution are in the areas of reliability and survival analysis, extreme value analysis. isotones and distribution approximations. Its applications in reliability and survival studies and the extreme-value analysis are illustrated in Mudholkar et al. (1995) and Mudholkar and Hutson (1996).

Let $X_1, X_2, \ldots X_n, \ldots$ be a sequence of independent and identically distributed random variables having cumulative distribution function $F(x)$ and probability density function $f(x)$. Let $Y_n = max(min)\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. We say X_j is an upper (lower) record value of this sequence if $Y_j > (<) Y_{j-1}, j > 1$. Thus X_j will be called an upper (lower) record value if its value exceeds (is lower than) that of all previous observations. By definition X_1 is an upper as well as a lower record value. One can transform from upper records to lower records by replacing the original sequence of random variables by $\{-X_j, j \geq 1\}$ or by $\{1/X_j, j \geq 1\}$ (if $P(X_i > 1) = 1$ for all i) the lower record values of this sequence will correspond to the upper record values of the original sequence. For more details, see for example, Ahsanullah (1995) and Arnold et al. (1998).

Record data arise in several real-life problems including industrial stress testing, meteorological analysis, hydrology, seismology, athletic events, oil and mining surveys. The formal study of record value theory probably started with the pioneering paper by Chandler (1952). After that many authors have discussed estimation problems for record values based on certain distribution. Mousa, et al. (2002) obtained the Bayesian estimators for the two parameters of the Gumbel distribution based on lower record values. Jaheen (2004) derived Bayes and empirical Bayes estimators for the one-parameter of the generalized exponential distribution based on lower record values. Malinowska and Szynal (2004) derived a family of Bayesian estimators and predictors for the Gumbel model based on lower records. Ashour and Amine (2005) obtained Bayesian and non-Bayesian estimators of the unknown parameters of Weibull distribution based on upper record values. Doostparast (2009) derived Bayesian and non-Bayesian estimates for the two parameters of the Exponential distribution based on lower record values, with respect to the squared error and Linear-Exponential loss functions.

The aim of this paper is to develop estimators for the two parameter exponentiated Weibull distribution based on record values using different methods. In Section 2, estimation of parameters using maximum likelihood estimation are obtained. In Section 3, Bayes estimators for the unknown two shape parameters are obtained based on squared error and LINEX loss functions. Finally, a numerical example is provided in Section 4.

2. Maximum Likelihood Estimation

Suppose we observe m lower record values $X_{L(1)} = x_1, X_{L(2)} = x_2, \dots,$ $X_{L(m)} = x_m$ from the EW distribution with pdf (1.2). The likelihood function of the m lower records is given by (see Ahsanullah (1995))

$$L(\underline{x} \mid \theta) = f(x_m) \prod_{i=1}^{m-1} r(x_i) \tag{2.1}$$

where $\underline{x} = (x_1, x_2, \dots, x_m)$ and $r(x_i) = f(x_i)/F(x_i)$

Substituting equations (1.1) and (1.2) in equation (2.1), the likelihood function based on the m lower record values from EW distribution is

$$L(\underline{x} \mid \alpha, \theta) = \alpha^m \theta^m \exp\left\{(\alpha - 1)\sum_{i=1}^{m} \ln x_i - \sum_{i=1}^{m} x_i^{\alpha} + \sum_{i=1}^{m} q_i - \theta q_m\right\} \tag{2.2}$$

where

$$q_i = -\ln\left(1 - e^{-x_i^{\alpha}}\right) \qquad i = 1, \dots, m$$

$$q_m = -\ln\left(1 - e^{-x_m^{\alpha}}\right)$$

Taking the logarithm of the likelihood function (2.2), we have

$$\ln L(\underline{x} \mid \alpha, \theta) = m\ln(\alpha) + m\ln\theta - U \tag{2.3}$$

where

$$U = \left\{-(\alpha-1)\sum_{i=1}^{m}\ln x_i + \sum_{i=1}^{m} x_i^{\alpha} - \sum_{i=1}^{m} q_i + \theta q_m\right\}$$

To obtain the MLE's of α and θ, we can solve the following two non-linear equations:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{m}{\alpha} - u_1 - u_3 + \theta u_2 = 0 \tag{2.4}$$

and

$$\frac{\partial \ln L}{\partial \theta} = \frac{m}{\theta} - q_m = 0 \tag{2.5}$$

where

$$u_1 = \sum_{i=1}^{m}(x_i^{\alpha} - 1)\ln x_i, \qquad u_2 = \omega_m[(1 - e^{-x_m^{\alpha}})]^{-1},$$

$$u_3 = \sum_{i=1}^{m}\omega_i[(1 - e^{-x_i^{\alpha}})]^{-1} \text{ and } \qquad \omega_i = e^{-x_i^{\alpha}} x_i^{\alpha}\ln(x_i)$$

From (2.5), we obtain the MLE of θ as a function of $\hat{\alpha}$, as follow

$$\hat{\theta}_{ML} = \frac{m}{-\ln(1 - e^{-x_m^{\hat{\alpha}}})} \tag{2.6}$$

where $\hat{\alpha}_{ML}$is the MLE of the parameter α, which can be obtained as a solution of the following non-linear equation

$$\hat{\alpha}_{ML} = m/[u_1 + u_3 - m(u_2/q_m)] \tag{2.7}$$

It may be noted here that a closed form for α, from likelihood equation given in (2.7) is not possible. Therefore, the solution can be obtained by using Newton–Raphson method.

The asymptotic variance–covariance of the MLE for parameters α and θ are given by the elements of the inverse of the Fisher information matrix

$$I_{ij} = -E\left(\frac{\partial \ln L(\alpha,\theta \mid x)}{\partial\alpha\partial\theta}\right), \qquad i,j = 1,2.$$

But, it is difficult to get the exact expressions of the above expectation. Alternatively, we can use the approximate asymptotic variance–covariance matrix for MLE as follows

$$\hat{\Sigma} = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial\alpha^2} & -\frac{\partial^2 \ln L}{\partial\alpha\partial\theta} \\ -\frac{\partial^2 \ln L}{\partial\alpha\partial\theta} & -\frac{\partial^2 \ln L}{\partial\theta^2} \end{bmatrix}^{-1}_{\theta=\hat{\theta}_{ML},\ \alpha=\hat{\alpha}_{ML}} = \begin{bmatrix} \hat{\sigma}^2_{\alpha} & \hat{\sigma}_{\alpha,\theta} \\ \hat{\sigma}_{\alpha,\theta} & \hat{\sigma}^2_{\theta} \end{bmatrix} \tag{2.8}$$

The elements of the sample Fisher information matrix are obtained by obtaining the second derivatives of the log-likelihood function (2.2) and evaluating them at the MLEs. This elements can be written as follow

$$\frac{\partial^2 \ln L}{\partial\alpha^2} = -\frac{m}{\alpha^2} - \sum_{i=1}^{m}\left[x_i^{\alpha} + \frac{e^{-x_i^{\alpha}}x_i^{\alpha}\left[1 - x_i^{\alpha} - e^{-x_i^{\alpha}}\right]}{\left(1 - e^{-x_i^{\alpha}}\right)^2}\right](\ln(x_i))^2$$
$$+\frac{\theta\, e^{-x_m^{\alpha}}x_m^{\alpha}(\ln(x_m))^2\left[1 - x_m^{\alpha} - e^{-x_m^{\alpha}}\right]}{(1 - e^{-x_m^{\alpha}})^2},$$

$$\frac{\partial^2 \ln L}{\partial\theta^2} = -\frac{m}{\theta^2}$$

and

$$\frac{\partial^2 \ln L}{\partial\alpha\partial\theta} = \frac{e^{-x_m^{\alpha}}x_m^{\alpha}\ln(x_m)}{(1 - e^{-x_m^{\alpha}})}. \tag{2.9}$$

The asymptotic normality of the MLE can be used to compute the approximate confidence intervals for the parameters α and θ, as follow

$$\hat{\alpha}_{ML} \pm Z_{\tau/2}\sqrt{\hat{\sigma}^2_{\alpha}} \tag{2.10}$$

$$\hat{\theta}_{ML} \pm Z_{\tau/2}\sqrt{\hat{\sigma}^2_{\theta}} \tag{2.11}$$

where $Z_{\tau/2}$ is an upper $(\tau/2)100\%$ of the standard normal distribution.

When α is known, the maximum likelihood estimator $\hat{\theta}_{ML}$ of the shape parameter θ based on the first m lower record values is given by

$$\hat{\theta}_{ML} = \frac{m}{-\ln\left(1 - e^{-x_m^{\alpha}}\right)} \tag{2.12}$$

The mean and the variance of the MLE $\hat{\theta}_{ML}$ are

$$E\left[\hat{\theta}_{ML}\right] = E(Y) = \theta\frac{m}{m-1}, \quad m > 1. \tag{2.13}$$

$$Var\left[\hat{\theta}_{ML}\right] = \frac{(m\theta)^2}{(m-1)^2(m-2)}, \quad m > 2. \tag{2.14}$$

This means that the maximum likelihood estimator $\hat{\theta}_{ML}$ is a biased estimator of θ and the unbiased estimator is

$$\hat{\theta}_{MU} = \frac{m-1}{-\ln\left(1 - e^{-x_m^{\alpha}}\right)} \tag{2.15}$$

By putting $\alpha = 1$ in equation (2.6), we get the MLE for the parameter θ for generalized exponential distribution based on lower record values, which coincide with the result obtained by Jaheen (2004).

3. Bayesian Estimation for the Parameters α and θ

Under the assumption that the parameters α and θ are unknown, we assume a gamma (conjugate prior) density for θ with parameters *a, b* and the following pdf

$$\pi(\theta) \propto \theta^{a-1}e^{-b\theta}, \qquad \theta > 0, \qquad a, b > 0 \tag{3.1}$$

and the prior density function of the parameter α, is the uniform distribution with density function

$$\pi(\alpha) = \frac{1}{d-c}, \qquad c < \alpha < d. \tag{3.2}$$

Then, the joint prior density functions of α and θ will be obtained from multiplying equations (3.1) and (3.2) as follows

$$\pi(\alpha, \theta) \propto \frac{\theta^{a-1}e^{-b\theta}}{d-c} \qquad c < \alpha < d, \ \ \theta > 0, \ \ a, b > 0 \tag{3.3}$$

The joint posterior density function of α and θ given $\underline{x}$ is obtained by combining equations (2.2) and (3.3) as follow

$$\pi(\alpha,\theta \mid \underline{x}) = \frac{\theta^{m+a-1}\alpha^{m}e^{-\theta(q_m+b)}}{k_1\Gamma(m+a)} \prod_{i=1}^{m} \frac{e^{-x_i^{\alpha}}x_i^{\alpha-1}}{(1-e^{-x_i^{\alpha}})} \tag{3.4}$$

where

$$k_1 = \int_c^d \frac{\alpha^m}{(q_m+b)^{m+a}} \prod_{i=1}^{m} \frac{e^{-x_i^{\alpha}}x_i^{\alpha-1}}{(1-e^{-x_i^{\alpha}})} \, d\alpha \tag{3.5}$$

The marginal posterior density of the parameter α is obtained by integrating the joint posterior density (3.4) with respect to the parameter θ as follows

$$\pi(\alpha \mid \underline{x}) = \frac{\alpha^m}{k_1(q_m+b)^{m+a}} \prod_{i=1}^{m} \frac{e^{-x_i^{\alpha}}x_i^{\alpha-1}}{(1-e^{-x_i^{\alpha}})} \tag{3.6}$$

and the marginal posterior density of the parameter θ is obtained by integrating the joint posterior density (3.4) with respect to the parameter α as follows

$$\pi(\theta \mid \underline{x}) = \frac{\theta^{m+a-1}e^{-b\theta}k_2}{k_1\Gamma(m+a)} \tag{3.7}$$

where

$$k_2 = \int_c^d \alpha^m(1-e^{-x_m^{\alpha}})^{\theta} \prod_{i=1}^{m} \frac{e^{-x_i^{\alpha}}x_i^{\alpha-1}}{(1-e^{-x_i^{\alpha}})} \, d\alpha \tag{3.8}$$

3.1 Bayes Estimator for α and θ Under Squared Error Loss Function (BESF)

Under a squared error loss function, the Bayes estimate is the mean of the posterior distribution. Therefore, the Bayes estimators $\hat{\alpha}_{BS}$ and $\hat{\theta}_{BS}$ for parameters α and θ know as

$$\hat{\alpha}_{BS} = \int_c^d \alpha\pi(\alpha \mid \underline{x})d\alpha$$

and

$$\hat{\theta}_{BS} = \int_0^{\infty} \theta\pi(\theta \mid \underline{x})d\theta$$

where $\pi(\alpha \mid \underline{x})$ and $\pi(\theta \mid \underline{x})$ are the marginal posterior densities for α and θ respectively. Thus, the Bayes estimator under a squared error loss function $\hat{\alpha}_{BS}$ of α can be expressed as

$$\hat{\alpha}_{BS} = \frac{k_3}{k_1} \tag{3.9}$$

where

$$k_3 = \int_c^d \frac{\alpha^{m+1}}{(q_{m+b})^{m+a}} \prod_{i=1}^{m} \frac{e^{-x_i^\alpha} x_i^{\alpha-1}}{(1-e^{-x_i^\alpha})} \, d\alpha \tag{3.10}$$

In a similar way, the Bayesian point estimator of θ is given by

$$\hat{\theta}_{BS} = \frac{(m+a)\, k_4}{k_1} \tag{3.11}$$

where

$$k_4 = \int_c^d \frac{\alpha^{m}}{(q_{m+b})^{m+a+1}} \prod_{i=1}^{m} \frac{e^{-x_i^\alpha} x_i^{\alpha-1}}{(1-e^{-x_i^\alpha})} \, d\alpha \tag{3.12}$$

The estimators of the parameters α and θ do not result in closed forms; hence, we propose numerical integration procedures for their evaluation numerically.

When α is known, we consider the natural conjugate family of prior densities for θ is a gamma prior density function with pdf

$$\pi(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}, \qquad \theta > 0, \qquad a, b > 0 \tag{3.13}$$

Combining the likelihood function (2.2) with the prior density function (3.13) and applying the Bayes theorem, we get the joint posterior density function of θ as follows

$$\pi(\theta \mid \alpha, \underline{x}) = \frac{(q_m + b)^{m+a}}{\Gamma(m+a)} \theta^{m+a-1} e^{-\theta(q_m+b)}, \qquad \theta > 0 \tag{3.14}$$

which is distributed as gamma distribution with parameters $(m + a,\ q_m + b)$. Thus, the Bayes estimator under a squared error loss function $\hat{\theta}_{BS}$ of θ is the mean of the posterior density (3.14), as follows

$$\hat{\theta}_{BS} = \frac{m + a}{b + q_m} \tag{3.15}$$

where q_m as given in (2.2).

By putting $\alpha = 1$ in equation (3.15), we get the Bayes estimator of the parameter θ for generalized exponential distribution based on lower record values, which coincide with the result obtained by Jaheen (2004).

3.2 Bayes Estimator Under LINEX Loss Function (BELF)

The Bayes estimators for the shape parameters α and θ of EW distribution based on lower record values under LINEX loss function can be obtained as (see Zellner (1986))

$$\hat{\alpha}_{BL} = -\frac{1}{\nu} \ln\left(E(e^{-\nu\,\alpha})\right)$$

and

$$\hat{\theta}_{BL} = -\frac{1}{\nu} \ln\left(E(e^{-\nu\,\theta})\right)$$

where *E(.)* the posterior expectation. Therefore, these estimators can be expressed as

$$\hat{\alpha}_{BL} = -\frac{1}{\nu} \ln\left(\frac{k_5}{k_1}\right) \tag{3.16}$$

and

$$\hat{\theta}_{BL} = -\frac{1}{\nu} \ln\left(\frac{k_6}{k_1}\right) \tag{3.17}$$

where

$$k_5 = \int_c^d \frac{\alpha^m e^{-\nu\alpha}}{(q_m + b)^{m+a}} \prod_{i=1}^{m} \frac{e^{-x_i^\alpha} x_i^{\alpha-1}}{(1 - e^{-x_i^\alpha})}\, d\alpha \tag{3.18}$$

and

$$k_6 = \int_c^d \frac{\alpha^m}{(\nu + b + q_m)^{m+a}} \prod_{i=1}^{m} \frac{e^{-x_i^{\alpha}} x_i^{\alpha-1}}{(1 - e^{-x_i^{\alpha}})} \, d\alpha \tag{3.19}$$

The estimators of the parameters α and θ do not result in closed forms; hence, we propose numerical integration procedures for their evaluation.

When α is known, the Bayes estimate $\hat{\theta}_{BL}$ of θ under the LINEX loss function is obtained by using (3.14), as follows

$$\hat{\theta}_{BL} = \frac{m+a}{\nu} \ln\left(1 + \frac{\nu}{b + q_m}\right), \qquad \nu \neq 0 \tag{3.20}$$

3.3 Credible Interval for α and θ

A symmetric $100(\tau)\%$ two sided Bayes probability interval for α can be obtained from the posterior density of the parameter α given in equation (3.6). The lower and upper limits for α are L_α and U_α respectively, can be determined by solving the two equations

$$\int_{L_\alpha}^d \frac{\alpha^m}{(q_m + b)^{(m+a)}} \prod_{i=1}^{m} \frac{e^{-x_i^{\alpha}} x_i^{\alpha-1}}{(1 - e^{-x_i^{\alpha}})} \, d\alpha = k_1 \left(\frac{1+\tau}{2}\right) \tag{3.21}$$

$$\int_{U_\alpha}^d \frac{\alpha^m}{(q_m + b)^{(m+a)}} \prod_{i=1}^{m} \frac{e^{-x_i^{\alpha}} x_i^{\alpha-1}}{(1 - e^{-x_i^{\alpha}})} \, d\alpha = k_1 \left(\frac{1-\tau}{2}\right). \tag{3.22}$$

The two integrals do not result in closed forms; hence, we propose numerical integration procedures to determine the interval (L_α and U_α) for α.

In a similar way, from equation (3.7) the credible interval of the parameter θ, denoted by $L_{(\theta)}$ and $U_{(\theta)}$, can be obtained by solving the following two equations

$$\int_c^d \int_{L_\theta}^\infty \theta^{m+a-1} e^{-\theta(q_m+b)} \alpha^m \prod_{i=1}^{m} \frac{e^{-x_i^{\alpha}} x_i^{\alpha-1}}{(1 - e^{-x_i^{\alpha}})} \, d\theta d\alpha = k_1 \Gamma(m+a) \left(\frac{1+\tau}{2}\right) \tag{3.23}$$

$$\int_c^d \int_{U_\theta}^\infty \theta^{m+a-1} e^{-\theta(q_m+b)} \alpha^m \prod_{i=1}^{m} \frac{e^{-x_i^{\alpha}} x_i^{\alpha-1}}{(1 - e^{-x_i^{\alpha}})} \, d\theta d\alpha = k_1 \Gamma(m+a) \left(\frac{1-\tau}{2}\right) \tag{3.24}$$

The integrals (3.23) and (3.24) can be solved numerically to get the Bayesian interval estimation for θ.

4. Numerical Illustration

In the following, a numerical example is given to illustrate the developed procedures in this paper. Estimation of the parameters α and θ from the EW distribution based on a lower record values will be computed by using the Mathcad (2001) program according to the following steps.

(1) A lower record sample of size $m = 8$ is generated from the EW ($\alpha = 3$, $\theta = 2$) distribution with pdf given by (1.2). This sample is:

1.138 1.088 1.085 0.826 0.766 0.68 0.625 0.592

(2) Using the previous data, the MLE, approximate variance-covariance matrix and 95% confidence intervals for α and θ are computed from (2.4), (2.5), (2.8), (2.15) and (2.11). The computational results are displayed in Table (1).

Table 1: MLE, approximate variance-covariance matrix and 95% confidence interval for α and θ

Parameter	MLE	Variance-covariance matrix		95% confidence interval
α	4.464	0.155	-0.096	(0.692, 5.236)
θ	3.35	-0.096	1.307	(1.109, 5.591)

(3) For given values of the prior parameters *a, b, c* and *d*, The Bayes estimates of α and θ under the SE loss function are computed from (3.9) and (3.11). While, the Bayes estimates of α and θ under the LINEX loss function when the same prior parameters are used with given values of ν are computed from (3.16) and (3.17). Also, the 95% credible intervals of α and θ are computed from (3.21) to (3.24). The computational results are displayed in Table (2).

Table 2: Bayesian point and interval estimation of the shape parameters α and θ for given values of the prior parameters *a, b ,v, (c=2.5* and *d=3.5).*

a	b	$\hat{\alpha}_{BS}$	$\hat{\alpha}_{BL}$			$\hat{\theta}_{BS}$	$\hat{\theta}_{BL}$		
			v=.5	v=2	v=3		v=.5	v=2	v=3
2	2	3.114 (2.616, 3.981)*	3.095	3.036	2.997	2.685 (1.283, 4.604)	2.517	2.145	1.965
3	3	3.126 (2.564, 3.487)	3.108	3.05	3.01	2.325 (1.158, 3.896)	2.209	1.937	1.798

* The 95% credible intervals for α and θ (in parentheses)

5. Concluding Remarks

In this paper, the maximum likelihood and Bayes methods of estimation are used to estimate the two shape parameters of the exponentiated Weibull distribution based on record values.

Remark 1. The 95% confidence interval in Table 1, and the 95% credible interval for α and θ in Table 2, contain the true values of the parameters $\alpha = 3, \ \theta = 2$. But the 95% confidence interval is wider than the corresponding 95% credible interval.

Remark 2. As can be seen from these results the Bayes estimators of the parameters α and θ are quite better than the maximum likelihood estimators (being closer to the population parameters).

Remark 3. The Bayes estimates of the parameters α and θ that are obtained based on the LINEX loss function are closer to these estimates based on squared error loss when v tends to zero.

Remark 4. Different values of the prior parameters (other than those listed in Table 2) have been considered, but did not change the previous conclusion.

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