THE EGYPTION STATSTICAL JOURNAL
ISSR, CAIRO UNIV., VOL . 44, NO. 2, 2000

Characterization of Some Continuous Distribution Based on Upper Record Values

S. E. Abu-Youssef and A. A. Alzaid
Department of Statistics and Operations Research
King Saud University
P. O. Box 2455, Riyadh 11451
Saudi Arabia

Abstract

This paper deals with characterization problem for some continuous distributions based on the conditional moments of the record values. We give two theorems which will be used to characterize some arbitrary continuous distribution based on the regression of records.

1 Introduction

Record values and associated statistics are great importance in several life problems involving weather, economic, sports data and etc...

The statistical study of record values started with Chandler (1952) and has now spread in different direction. Feller (1966) gave some examples of record values with respect to gambling problem. Properties of record values of independent identically distributed random variables have been extensively studied in the literature. See Ahsanullah (1988), Arnold, et al. (1992), Nagaraja (1988) and Nevzorove (1987).

Some relation for single and product moments of upper record values for specific continuous distributions are established by Balakrishnan and Ahsanullah (1994, 1995), Balakrishnan et al. (1993), Balakrishnan and Sandha (1995) and Balakrishnan and Chan (1993).

Gupta (1984) proved that for given positive integers k and n_0, the sequences $\{E[X_{L_{(n+k)}}] - E[X_{L_{(n)}}]\}_{n=n_0}^{\infty}$ characterized the distribution F up to a location parameter provided $E\{X^p\} < \infty$ for some $p > 1$, where the sequence $\{X_{L_{(n)}}\}_{n=1}^{\infty}$ is the sequence of record values. Lin (1988) discussed characterization of general continuous distribution via the expected spacing of order statistics and via the moments of record values.

In this paper we prove two theorems which will be used to characterize some arbitrary continuous distributions based on the regression of records.

Let $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ be the first n upper record values from a population whose p.d.f. $f(x)$ and c.d.f. $F(x)$. Let $\overline{F}(x) = 1 - F(x)$ be the survival function.

Let $h(x) = \frac{f(x)}{1-F(x)}$ and $H(x) = -\log[\overline{F}(x)]$ be the failure rate and hazard rate of function F. Then the p.d.f. of $X_{U(m)}$, $m = 0, 1, 2, \ldots$ is

$$f_m(x) = \frac{H^{m-1}(x) f(x)}{(m-1)!}, \tag{1.1}$$

and the joint density of two upper records $X_{U(m)}$ and $X_{U(n)}$, $m = 0, 1, 2, \ldots$, $m < n$ is

$$f_{m,n}(x,y) = \frac{1}{(m-1)!(n-m-1)!} H^{m-1}(x)[H(y) - H(x)]^{n-m-1} h(x) f(y),$$
$$-\infty < x < y < \infty \tag{1.2}$$

2 Main Results

Let $X_{U(1)}, X_{U(2)}, \ldots$ be defined as in the previous Section and let ϕ be a real function such that $P[\phi(X) = \text{constant}] = 0$. We now give two representative of survival function in terms of the conditional expectation of two consecutive records.

ISSR, CAIRO UNIV., VOL. 44, NO. 2, 2000, PP.150-158

Theorem (1): Let $g(x)$ be a real function. Then

$$E\{\phi(X_{U(r+1))}|X_{U(r)} = x)\} = g(x) \tag{2.1}$$

if and only if

$$\overline{F}(x) = ce^{-\int \frac{dg(x)}{g(x)-\phi(x)}} \tag{2.2}$$

for some norming constant c.

Proof:

From (1.1) and (1.2), it follows that the conditional probability density of $X_{U(r+1)}$ given $X_{U(r)} = x$ is

$$f_{r+1|r}(y|x) = \frac{f(y)}{\overline{F}(x)}, \qquad y > x. \tag{2.3}$$

This implies that (2.1) is equivalent to

$$\int_x^\infty \phi(y)f(y)dy = g(x)\overline{F}(x).$$

Taking the derivative we get

$$\phi(x)f(x) = f(x)g(x) - g'(x)\overline{F}(x). \tag{2.4}$$

In view of assumption that $P(\phi(X) = \text{constant}) = 0$ and X is a continuous random variables, the above equation implies that $\phi(x) \neq g(x)$ a.s.

Hence

$$h(x) = \frac{g'(x)}{g(x) - \phi(x)} \qquad a.s. \tag{2.5}$$

This complete the proof.

Theorem 1 is an extension of Nagaraja (1977) result who studied the case $r = 1$.

Theorem (2.2): Let ϕ be as defined in Theorem (2.1). Then

$$E\{\phi(X_{U(r)})|X_{U(r+1)} = y)\} = g(y) \tag{2.6}$$

if and only if

$$\overline{F}(y) = e^{-ce^{-\int \frac{dg(y)}{r(g(y)-\phi(y))}}}, \tag{2.7}$$

for some norming constant c.

Proof:

The conditional probability density of $X_{U(r)}$ given $X_{U(r+1)} = y$ is

$$f_{r|r+1}(x|y) = r\left[\frac{H(x)}{H(y)}\right]^{r-1}\frac{h(x)}{H(y)}. \tag{2.8}$$

Hence (2.6) is equivalent to

$$r\int_0^y \phi(x)H^{r-1}(x)h(x)dx = H^r(y)g(y). \tag{2.9}$$

Hence following same argument used to prove Theorem 1 we get

$$H(y) = ce^{-\int \frac{dg(y)}{r[g(y)-\phi(y)]}}.$$

This complete the proof.

3 Some Applications

Theorem 1 and Theorem 2 give representation of survival function in terms of the functions ϕ and g. In general, it is not easy to compute the integral appearing in (2.2) and (2.6). However the special choice of ϕ as linear function of g leads to characterization of some well known probability distributions. In the following we restate Theorem 1 and Theorem 2 for these special cases.

Corollary 1: Suppose that $g(x) = A\phi(x) + B$ for some real constants $A \neq 0$ and B. Then (2.1) is satisfied if and only if one of the following holds:

(i) $A = 1$ and $\overline{F}(x) = ce^{\frac{-\phi(x)}{B}}$

or

(ii) $A \neq 1$ and $\overline{F}(x) = c[(A-1)\phi(x) + B]^{-\frac{A}{A-1}}$

Corollary 2: Suppose that $g(x) = A\phi(x) + B$ for some real constants $A \neq 0$ and B. Then (2.6) is satisfied if and only if one of the following holds:

(i) $A = 1$ and $\overline{F}(x) = e^{-ce^{-\frac{\phi(x)}{B}}}$

(ii) $A \neq 1$ and $\overline{F}(x) = e^{-c[(A-1)\phi(x)+B]^{-\frac{A}{r(A-1)}}}$

The proof of two corollaries are straight forward and hence omitted. Table 1 summarizes special cases of Corollary 1 with corresponding probability distribution. The corresponding results based on (2.6) are given in Table 2.

Table 1: $g(x) = E\{\phi(X_{U(r+1)}|X_{U(r)} = x)\}$

A	B	C	$\phi(x)$	$g(x)$	$\bar{F}(x)$	Name
1	θ	1	$x^{\alpha}, \alpha > 0$	$x^{\alpha} + \theta, \theta > 0$	$e^{-x^{\alpha}/\theta}$	Weibull
1	$\beta \geq 0$	1	$ax + bx^2$	$ax + bx^2 + \beta$	$\exp\{\frac{-(ax+bx^2)}{\beta}\}$	Linear exponential if $\beta = 2$ exponential if $b = 0, a = 1$ Rayleigh if $a = 0; b = 1, \beta = 2$
1	$1/\alpha$	$a^{-\alpha}, \alpha > 0$	lnx^{α}	$lnx^{\alpha} + 1$	$(\frac{x}{a})^{-\alpha}$	Pareto
1	-1	$a^{-\alpha}$	$ln(a^{\alpha} - x^{\alpha}), \alpha > 0$	$ln(a^{\alpha} - x^{\alpha}) - 1$	$1 - (\frac{x}{a})^{-\alpha}$	Power function
1/2	0	$-2a^{-\alpha}$	$x^{-\alpha}$	$\frac{1}{2}x^{\alpha}$	$(\frac{x}{a})^{-\alpha}$	Pareto
1/2	1/2	$2a^{-\alpha}$	x^{α}	$\frac{1}{2}x^{\alpha} + \frac{1}{2}$	$1 - (\frac{x}{a})^{\alpha}$	Power function
1/2	1/2	2	$\frac{e^{x}}{1+e^{x}}$	$\frac{e^{x}}{2(1+e^{x})} + \frac{1}{2}$	$\frac{e^{-x}}{1+e^{-x}}$	Logistic
1/2	0	2	$\frac{2}{1+e^{-x}}$	$(1 + e^{-x})^{-1}$	$\frac{2e^{-x}}{1+e^{-x}}$	Half-logistic
> 1	1	$c > 0$	$x^{\alpha}, \alpha > 0$	$Ax^{\alpha} + 1$	$[c(A-1)x^{\alpha} + 1]^{-\frac{1}{A-1}}$, $x\epsilon[(\frac{c^{\frac{A-1}{A}} - 1}{A-1})^{1/\alpha}, \infty]$	Burr
< 1	1	$c > 0$	$x^{\alpha}, \alpha > 0$	$Ax^{\alpha} + 1$	$[c(A-1)x^{\alpha} + 1]^{-\frac{A}{A-1}}$, $x\epsilon[(\frac{c^{\frac{A-1}{A}} - 1}{A-1})^{1/\alpha}, \infty]$	-
1/2	1/2	2	$e^{-e^{-x}}$	$\frac{1}{2}e^{-e^{-x}} + \frac{1}{2}$	$1 - e^{-e^{-x}}$	Gumble
1/2	0	2	$e^{-e^{x}}$	$\frac{1}{2}e^{-e^{-x}}$	$e^{-e^{x}}$	extrem value:1

Table 2: $g(x) = E\{\phi(X_{U(r)}|X_{U(r+1)} = x)\}$

A	B	C	$\phi(x)$	$g(x)$	$F(x)$	Name
1	$-1/r\theta$	1	lnx	$l_n - \frac{1}{r\theta}, \theta \geq 0$	$e^{-x^{\theta}}$	Weibull
1	$-1/r$	1	$ln(\alpha x - \frac{\nu x^2}{2})$	$ln(\alpha x + \frac{\nu x^2}{2}) + 1$ $\alpha, \nu > 0$	$e^{-\alpha x - \frac{\nu x^2}{2}}$	Linear exponential
1	$-1/r$	α	$ln(ln(\frac{x}{a}))$	$ln(ln(\frac{x}{a})) - 1$	$(\frac{x}{a})^{-\alpha}$, $a \leq x \leq \infty$	Pareto
1	$-1/r$	-1	$ln(ln(1-(\frac{x}{a})^{\alpha}))$	$ln(ln(1-(\frac{x}{a})^{\alpha})) - 1$	$1-(\frac{x}{a})^{\alpha}$, $0 \leq x \leq a$	Power function
1	$1/r$	1	x	$x+1$	$e^{-e^{-x}}$	Gumble
1	$-1/r$	≥ 0	$ln(ln(1+\theta x^{P}))$	$ln(ln(1+\theta x^{P}))$	$(1+\theta xP)^{-c}$	Burr
1/2	0	$-2/\theta$	$x^{\alpha r}$	$\frac{1}{2}x^{\alpha r}$	$e^{-\frac{x^{\alpha}}{\theta}}$, $\alpha, \theta > 0$	Weibull
1/2	0	-2	$(\lambda x + \frac{\nu x^2}{2})^r$	$\frac{1}{2}(\lambda x + \frac{\nu x^2}{2})^r$	$e^{-\alpha x - \frac{\nu x^2}{2}}$, $\lambda r > 0$	Linear exponential
1/2	0	-2	$ln(\frac{x}{a})^{\alpha r}$	$\frac{1}{2}ln(\frac{x}{a})^{\alpha r}$	$(\frac{x}{a})^{-\alpha}$	Pareto
1/2	0	2	$(ln(1-(\frac{x}{a})^{\alpha})^{-r}$	$\frac{1}{2}(ln(1-(\frac{x}{a})^{\alpha}))^{-r}$	$1-(\frac{x}{a})^{\alpha}$	Power function
1/2	0	1/2	e^{-rx}	$\frac{1}{2}e^{-rx}$	$e^{-e^{-x}}$	Gumble
1/2	0	$-1/2$	$(ln(\frac{e^{-x}}{1+e^{-x}}))^{-r}$	$\frac{1}{2}(ln(\frac{e^{-x}}{1+e^{-x}}))^{-r}$	$\frac{e^{-x}}{1+e^{-x}}$	Logistic

REFERENCES

Ahsanullah, M. (1988). Introduction to record values, Ginn Press, Needham Heights, M. A

Arnold, B. C. and Balakrishnan, N. and Nagaraja, H. N. (1992). A First Course in order statistics, John Wiley & Sons, New York.

Balakrishnan, N. and Ahsanullah, M. (1995). Relations for single and product moments of record values from exponential distribution. J. Appl. Statist. Sci., 1, 13, 73-80.

Balakrishnan, N. and Ahsanullah, M. (1994). Recurrence relations for single and product moments of record values from exponential generalized pareto distribution. Meth., 23(10), 2841-2852.

Balakrishnan, N. (1993). Record values from Rayleigh and Weibull distribution and associated inference. National Institute of Standards and technology, J. of Research, Special publications 866, 41-51.

Balakrishnan, N. Chan, P. S. and Ahsanullah, M. (1993). Recurrence relations for moments of record values from generalized extreme value distribution. Commun. Statist. Theor. Meth., 22(5), 1471-1482.

Balakrishnan, N. and Sandhu, K. S. (1995). Recurrence relation for single and product moments of order statistics from a generalized half-logistic with application to inference. J. Statist. Commun. Simul., Vol., 52, 385-398.

Gupta, R. C. (1984). Relationships between order statistics and record values and some characterization results. J. Appl. Prob., 21, 425-430.

Lin, G. D. (1987). On characterization of distribution via moments of record values. Prob. Theor. Rel. Fields, 74, 479-483.

Lin, G. P. (1987). Characterization of distribution via relationships between two moments of order statistics. J. Statist. Plann. and Infere., 19, 73-80.

Nagaraja, H. N. (1977). On characterization based on record values. Austral. J. Statist., 19, 1, 70-73.

Nagaraja, H. N. (1988). Record values and related statistics a review. Commun. Statist. Theory Meth., 17, 2223-2228.

Nagaraja, H. N. (1982). Record values and extreme value distribution. J. Appl. Prob., 19, 233-239.

Nevzorove, V. B. (1981). Records. Theor, Probab. Appl., 32, 201-228.