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A Testing for New Renewal Better than Used Class of Survival Functions

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Abstract

A U-statistic is derived for testing exponentiality against new renewal better (worse) than used. For this class of life distributions, a nonparametric procedure (U-statistic) is presented in this investigation. Selected critical values are tabulated for sample size $n = 5(1)39$. The Pitman asymptotic efficiency relative to tests of other classes such as "decreasing mean residual life and new better than used in expectation" are discussed. Our procedure is simple to implement and is asymptotically normal both under the null hypothesis and alternative hypothesis with good efficiency against other procedures for some commonly used alternatives.

Key Words: Efficiency, exponentiality, asymptotic normality, U-test, class of life distributions, Pitman asymptotic efficiency.

Solution of undamped vibration system:

The undamped forced oscillator is a representative of a structure where damping plays a small role in the response. It is governed by the equation:

$$x^{(2)}(t)+\omega^2 x(t)=F(t) \qquad \text{.....(1)}$$

where x(0) and $x^{(1)}(0)$ are random initial displacement and initial velocity, respectively, and F(t) is the random excitation of this system. There are two components to any forced vibration response: transient or free vibration, and steady state or forced vibration. In mathematical terms, the transient response is the homogeneous solution to the differential equation $X_H(t)$[6] such that ;

$$x_H(t) = x(0)Cos(\omega t) + \frac{1}{\omega}x^{(1)}(0)Sin(\omega t) \qquad \text{.....(2)}$$

And the steady state is the particular solution $X_P(t)$ [6] such that ;

$$x_p(t) = \int_0^t h(t-s)F(s)ds \qquad \text{.....(3)}$$

$$h(t) = \frac{1}{\omega}Sin(\omega t) \qquad \text{.....(4)}$$

Where h(t) is the impulse response function. By linear superposition, each problem is solved separately, then add both solutions to obtain the complete response $x(t)$:

$$x(t) = x(0)Cos(\omega t) + x^{(1)}(0)\frac{1}{\omega}Sin(\omega t) + \frac{Sin(\omega t)}{\omega}I_1 - \frac{Cos(\omega t)}{\omega}I_2 \qquad \text{.....(5)}$$

Where;

$$\left.\begin{aligned} I_1 &= \int_0^t F(s)Cos(\omega s)ds \\ I_2 &= \int_0^t F(s)Sin(\omega s)ds \end{aligned}\right] \qquad \text{.....(6)}$$

by differentiating equation (5) the velocity function will be in the form

$$x^{(1)}(t) = -\omega x(0)Sin(\omega t) + x^{(1)}(0)Cos(\omega t) + Cos(\omega t).I_1 + Sin(\omega t).I_2$$

$$\text{.....(7)}$$

Let x(0) and $x^{(1)}(0)$ are random initial conditions with normal distribution

Where ;

$$\left.\begin{array}{ll} E[x(0)] = \mu_1 & , Var[x(0)] = \sigma_1^2 \\ E[x^{(1)}(0)] = \mu_2 & , Var[x^{(1)}(0)] = \sigma_2^2 \\ E[(x(0)-\mu_1)(x^{(1)}(0)-\mu_2)] = \upsilon & \end{array}\right] \qquad \text{........................(8)}$$

υ is the covariance between x(0) and $x^{(1)}(0)$

And consider F(t) is a Gaussian process and is independent with x(0) and $x^{(1)}(0)$ where ;

$$\left.\begin{array}{l} E[F(t)] = \mu_F(t) \\ E[(F(t)-\mu_F(t))(F(s)-\mu_F(s))] = K_{FF}(t,s) \end{array}\right] \qquad \text{........................(9)}$$

$K_{FF}(t, s)$ is the auto covariance function of F(t)

Using equations [5-9], the statistical moments of x(t) and $x^{(1)}(t)$ are obtained as the following :

$$\mu_x(t) = \mu_1 Cos(\omega t) + \frac{\mu_2}{\omega} Sin(\omega t) + \frac{Sin(\omega t)}{\omega}\int_0^t \mu_F(s) Cos(\omega s) ds - \frac{Cos(\omega t)}{\omega}\int_0^t \mu_F(s) Sin(\omega s) ds \qquad \text{.....................(10)}$$

$$\mu_{x^{(1)}}(t) = -\mu_1 \omega Sin(\omega t) + \mu_2 Cos(\omega t) + Cos(\omega t)\int_0^t \mu_F(s) Cos(\omega s) ds + Sin(\omega t)\int_0^t \mu_F(s) Sin(\omega s) ds \qquad \text{.......................(11)}$$

$$\begin{aligned} \sigma_x^2(t) = {} & \sigma_1^2 Cos^2(\omega t) + \frac{Sin(2\omega t)}{\omega}\upsilon + \sigma_2^2 \frac{Sin^2(\omega t)}{\omega^2} \\ & + \frac{Sin^2(\omega t)}{\omega^2}\int_0^t\int_0^t K_{FF}(u,v) Cos(\omega u) Cos(\omega v) du dv \\ & - \frac{Sin(2\omega t)}{\omega^2}\int_0^t\int_0^t K_{FF}(u,v) Sin(\omega u) Cos(\omega v) du dv \\ & + \frac{Cos^2(\omega t)}{\omega^2}\int_0^t\int_0^t K_{FF}(u,v) Sin(\omega u) Sin(\omega v) du dv \end{aligned} \qquad \text{.......................(12)}$$

$$\sigma^2_{x^{(1)}}(t) = \sigma_1^2\omega^2 Sin^2(\omega t) - \omega\upsilon\, Sin(2\omega t) + \sigma_2^2 Cos^2(\omega t)$$

$$+ Cos^2(\omega t)\int_0^t\int_0^t K_{FF}(u,v)Cos(\omega u)Cos(\omega v)dudv$$

$$+ Sin(2\omega t)\int_0^t\int_0^t K_{FF}(u,v)Sin(\omega u)Cos(wv)dudv$$

$$+ Sin^2(\omega t)\int_0^t\int_0^t K_{FF}(u.v)Sin(\omega u)Sin(\omega v)dudv$$

..............................(13)

$$\beta = -\frac{\omega}{2}\sigma_1^2 Sin(2\omega t) + \upsilon Cos(2\omega t) + \frac{\sigma_2^2}{2\omega}Sin(2\omega t)$$

$$-\frac{Cos(2\omega t)}{\omega}\int_0^t\int_0^t K_{FF}(u,v)Sin(\omega u)Cos(\omega v)dudv$$

$$+\frac{Sin(2\omega t)}{2\omega}\int_0^t\int_0^t K_{FF}(u,v)Cos\,\omega(u+v)dudv$$

........................(14)

Where β is the auto covariance between $x(t)$ & $x^{(1)}(t)$

Solution of deterministic differential equation:

The deterministic differential equation $x^{(2)}(t) + \omega^2 x(t) = F(t)$ with initial conditions $x(0) = x_0$ and $x^{(1)}(0) = v_0$ has a solution in the form :

$$x(t) = x_0 Cos(\omega t) + \frac{v_0}{\omega}Sin(\omega t) + \frac{1}{\omega}\int_0^t F(s)Sin\,\omega(t-s)ds \quad \text{.............(15)}$$

$$x^{(1)}(t) = -\omega x_0 Sin(\omega t) + v_0 Cos(\omega t) + Cos(\omega t)\int_0^t F(s)Cos(\omega s)ds$$

$$+ Sin(\omega t)\int_0^t F(s)Sin(\omega s)ds$$

...............................(16)

Deterministic and random excitation

Case(I)

(a) Let $F(t) = C$ where C is constant and $x_0 = v_0 = 0$
Using equations (15,16)

$$\left.\begin{aligned} x(t) &= \frac{c}{\omega^2}(1 - Cos(\omega t)) \\ x^{(1)}(t) &= \frac{c}{\omega} Sin(\omega t) \end{aligned}\right] \qquad \text{.....(17)}$$

(b) If $C \in N[\mu, \sigma_F]$ so $\mu_F(t) = \mu$, $K_{FF}(t, s) = \sigma_F^2$.
Using equations (10-13)

$$\left.\begin{aligned} \mu_x(t) &= \frac{\mu}{\omega^2}(1 - Cos(\omega t)) \\ \mu_{x^{(1)}}(t) &= \frac{\mu}{\omega} Sin(\omega t) \\ \sigma_x^2(t) &= \frac{\sigma_F^2}{\omega^4}(1 - Cos(\omega t))^2 \\ \sigma_{x^{(1)}}^2(t) &= \frac{\sigma_F^2}{\omega^2} Sin^2(\omega t) \end{aligned}\right] \qquad \text{.....(18)}$$

Case (II)

(a) Let $F(t) = C e^{Kt}$ where C and K are constants and $x_0 = v_0 = 0$.
Using equations (15,16)

$$x(t) = \frac{C}{\omega} \cdot \frac{K^2}{\omega^2 + K^2}\left[\frac{\omega}{K^2} e^{Kt} - \frac{1}{K} Sin(\omega t) - \frac{\omega}{K^2} Cos(\omega t)\right]$$

$$x^{(1)}(t) = C \cdot \frac{K^2}{\omega^2 + K^2}\left[\frac{1}{K} e^{Kt} - \frac{1}{K} \cos(\omega t) + \frac{\omega}{K^2} Sin(\omega t)\right] \qquad \text{.....(19)}$$

(b) If $F(t) = C(\theta) e^{Kt}$ where $C(\theta) \in N[\mu, \sigma_F]$, so $\mu_F(t) = \mu e^{Kt}$,

$K_{FF}(t, s) = \sigma_F^2 e^{K(t+s)}$. Using (10,13)

$$\mu_x(t) = \frac{\mu}{\omega} \cdot \frac{K^2}{\omega^2 + K^2}\left[\frac{\omega}{K^2} e^{Kt} - \frac{1}{K} Sin(\omega t) - \frac{\omega}{K^2} Cos(\omega t)\right]$$

$$\mu_{x^{(1)}}(t) = \mu \cdot \frac{K^2}{\omega^2 + K^2}\left[\frac{1}{K} e^{Kt} - \frac{1}{K} Cos(\omega t) + \frac{\omega}{K^2} Sin(\omega t)\right]$$

$$\sigma_x^2(t) = \frac{\sigma_F^2}{\omega^2}\left(\frac{K^2}{\omega^2 + K^2}\right)^2\left(\frac{\omega}{K^2} e^{Kt} - \frac{1}{K} Sin(\omega t) - \frac{\omega}{K^2} Cos(\omega t)\right)^2$$

$$\sigma^2_{x^{(1)}}(t) = \sigma^2_F\left(\frac{K^2}{\omega^2 + K^2}\right)^2\left(\frac{1}{K}e^{Kt} - \frac{1}{K}Cos\,(\omega t) + \frac{\omega}{K^2}Sin\,(\omega t)\right)^2 \qquad \text{.........................(20)}$$

From the previous cases, the excitation force expressed in a separate manner that creates it a product of two functions, one is deterministic and the other is a pure function of random variable. So we can obtain the statistical moments of the solution directly from the deterministic solution in equations(17),(19) instead of the calculations in equations (10-13). As a result we can deal with harmonic excitation.

Harmonic excitation :

Base excitation problems have many mainfestations in applications. These include the variation of structures on foundations, such as in earthquake engineering, where the random loading is through the base, the response of an autombile to road irregularities and bumps, and the interaction between a machine and its support. In such cases we make the assumption that the excitation is harmonic.

Let $F(t) = Sin\,(\omega_F t + \theta)$ is harmonic excitation force where, $x(0) = x^{(1)}(0) = 0$ and ω_F is the frequency of the driving force and $\omega \neq \omega_F$ for no resonance. Using equation (15)and the properties of sine function. The final solution of equation (1) will be ;

$$x(t) = \frac{1}{\omega}\left(\frac{\omega_F Sin\,(\omega t) - \omega Sin\,(\omega_F t)}{\omega_F^2 - \omega^2}\right)Cos\,\theta + \left(\frac{Cos\,(\omega t) - Cos\,(\omega_F t)}{\omega_F^2 - \omega^2}\right)Sin\,\theta \qquad \text{.......................(21)}$$

If θ is a random variable uniformly distributed over the interval $[0,2\pi]$. Using equation (21) and principles of expectation and variance.

$$\mu_x(t) = 0 \qquad \text{.........................(22)}$$

$$\sigma^2_x(t) = \frac{1}{2}\left(\frac{1}{\omega_F^2 - \omega^2}\right)^2 * \left[\left(\frac{\omega_F Sin\,(\omega t) - \omega Sin\,(\omega_F t)}{\omega}\right)^2 + \left(Cos\,(\omega t) - Cos\,(\omega_F t)\right)^2\right] \qquad \text{..................(23)}$$

We obtain the same previous results by using equations (10,12).Knowing that $U_F(t)=0$ and $K_{FF}(t,s)=\frac{1}{2}Cos(\omega_F(t-s))$

Conclusion :

The statistical moments of the solution of stochastic linear vibration with random initial condition and random excitation characterized by second order differential equation can be obtained.

The statistical moments of the solution can be obtained directly from the deterministic solution of the same differential equation when the excitation force expressed in a separate manner that creates it a product of two functions, one is deterministic and the other is a pure function of random variable. As a result we can deal with harmonic excitation.

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