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Multivector Stochastic Rearrangement Inequalities

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Abstract

Chan, D'Abadie and Proschan develop a unified theory for obtaining stochastic rearrangement inequalities based on two vectors. We extend the stochastic rearrangement inequalities for more than two vectors.

1. INTRODUCTION AND SUMMARY

Rearrangement inequalities compare the values of a function of vector arguments with the value of the same function after the components of vectors have been rearranged .The classical example of a rearrangement inequality involving a function of two vector arguments is the well known inequality of Hardy,Littlewood and Polya(1952) for sums of products. For a given vector $\underline{x} = (x_1, \ldots, x_n) \in R^n$, we let $\underline{x}_\uparrow = (x_{[n]}, x_{[n-1]}, \cdots, x_{[1]})$ and $\underline{x}^\downarrow = (x_{[1]}, x_{[2]}, \cdots, x_{[n]})$ be respectively the vectors with components of $\underline{x}$ arranged in increasing and decreasing order and for any permutation $\underline{\pi} = (\pi_1, \ldots, \pi_n)$ of $(1, 2, \ldots, n)$ let $\underline{x}_{\underline{\pi}} = (x_{\pi(1)}, \cdots, x_{\pi(n)})$. This inequality states that

$$\sum_{i=1}^{n} x_{[i]} y_{[i]} \geq \sum_{i=1}^{n} x_{[i]} y_{\pi(i)} \geq \sum_{i=1}^{n} x_{[i]} y_{[n-i+1]}.$$

Since the work of Hardy, Littlewood and Polya (1952) many papers on rearrangement inequalities on R^n have appeared. Jurkat and Ryser (1966) show that if $\underline{x}_\uparrow$ and $\underline{y}_\uparrow$ are non-negative increasing components of n-tuples then

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$$\prod_{i=1}^{n} \min(x_{[i]}, y_{[i]}) \geq \prod_{i=1}^{n} \min(x_{[i]}, y_{\pi(i)}) \geq \prod_{i=1}^{n} \min(x_{[i]}, y_{[n-i+1]})$$

$$\sum_{i=1}^{n} \min(x_{[i]}, y_{[i]}) \geq \sum_{i=1}^{n} \min(x_{[i]}, y_{\pi(i)}) \geq \sum_{i=1}^{n} \min(x_{[i]}, y_{[n-i+1]})$$

and Minc (1971) shows that

$$\prod_{i=1}^{n} \max(x_{[i]}, y_{[i]}) \leq \prod_{i=1}^{n} \max(x_{[i]}, y_{\pi(i)}) \leq \prod_{i=1}^{n} \max(x_{[i]}, y_{[n-i+1]})$$

$$\sum_{i=1}^{n} \max(x_{[i]}, y_{[i]}) \leq \sum_{i=1}^{n} \max(x_{[i]}, y_{\pi(i)}) \leq \sum_{i=1}^{n} \max(x_{[i]}, y_{[n-i+1]})$$

for all permutations $\underline{\pi}$. The Marshall and Olkin (1978) book presents comprehensivly the study of deterministic rearrangement inequalities developed by London (1970), Lorenz (1952) and others. Chan, D'Abadie and Proschan (1987) develop a general theory for obtaining stochastic versions of inequalities of this type involving inequalities of two vectors. However, when $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m$ are m vectors, then some of the deterministic inequalities have generalisations. Ruderman (1952) has shown that, if $\underline{x}_j j = 1, \ldots, m$ are m vectors with positive components and if $(x_{[1]j}, x_{[2]j}, \ldots, x_{[n]j})$ denotes decreasing rearrangement of $\underline{x}_j$ then

$$\sum_{i=1}^{n} \prod_{j=1}^{m} x_{ij} \leq \sum_{i=1}^{n} \prod_{j=1}^{m} x_{[i]j}$$

$$\prod_{i=1}^{n} \sum_{j=1}^{m} x_{[i]j} \leq \prod_{i=1}^{n} \sum_{j=1}^{m} x_{ij}$$

Minc (1971) has also shown that

$$\prod_{i=1}^{n} \min_j x_{ij} \leq \prod_{i=1}^{n} \min_j x_{[i]j}$$

$$\sum_{i=1}^{n} \min_j x_{ij} \leq \sum_{i=1}^{n} \min_j x_{[i]j}$$

$$\prod_{i=1}^{n} \max_j x_{ij} \geq \prod_{i=1}^{n} \max_j x_{[i]j}$$

$$\sum_{i=1}^{n} \max_j x_{ij} \geq \sum_{i=1}^{n} \max_j x_{[i]j}$$

In this paper, we obtain stochastic version of rearrangement inequalities for the deterministic inequalities of Ruderman (1952) and Minc(1971) which involve more than two vectors. In section 2 we present some definitions and properties of multivector rearrangement functions. In section 3, we present multivector stochastic rearrangement inequalities.

2. DEFINITIONS AND BASIC PROPERTIES OF MULTIVECTOR ARRANGEMNT INCREASING FUNCTIONS

Let S_n be the group of all permutations of $(1,2,\ldots,n)$. A member of S_n is denoted by $\underline{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$. Let $\underline{\pi}$ and $\underline{\pi}'$ be two members of S_n such that $\underline{\pi}'$ contains exactly one inversion of a pair of co-ordinates which occur in the natural order in $\underline{\pi}$. For example, $\underline{\pi} = (\pi_1, \pi_2, \ldots, \pi_i, \ldots, \pi_j, \ldots, \pi_n)$ and $\underline{\pi}' = (\pi_1, \pi_2, \ldots, \pi_j, \ldots, \pi_i, \ldots, \pi_n)$ where $i < j$ and $\pi_i < \pi_j$. We say that $\underline{\pi}'$ is a simple transposition of $\underline{\pi}$, written $\underline{\pi} >^t \underline{\pi}'$. Let $\underline{\pi}$ and $\underline{\pi}^*$ be two arbitrary elements in S_n. We say that $\underline{\pi}^*$ is a transposition of $\underline{\pi}$ if there exists a finite number of elements $\underline{\pi}_0, \underline{\pi}_1, \ldots, \underline{\pi}_k$ in S_n satisfying $\underline{\pi} = \underline{\pi}_0 >^t \underline{\pi}_1 >^t \ldots >^t \underline{\pi}_k = \underline{\pi}^*$. That is $\underline{\pi}^*$ is obtained from $\underline{\pi}$ by finite number of simple transpositions. In other words $\underline{\pi}$ is better arranged than $\underline{\pi}^*$. Hollander, Proschan and Sethuraman (1977) use the terminology decreasing in transposition. However, Marshall and Olkin (1979) use the terminology $\underline{\pi}$ is *better arranged* than $\underline{\pi}^*$ and written $\underline{\pi} \geq^a \underline{\pi}^*$.

DEFINITION 2.1

A function $f : S_n \rightarrow R$ is said be Arrangement Increasing (AI) if $\underline{\pi} \geq^a \underline{\pi}^*$ implies $f(\underline{\pi}) \geq f(\underline{\pi}^*)$ for $\underline{\pi}, \underline{\pi}^* \epsilon S_n$ and f is said to be arrangement decreasing (AD) if $-f$ is arrangement increasing.

The concept of arrangement increasing can be extended to functions defined on $R^n, S_n \times R^n$ and $R^n \times R^n$. For a detailed exposition, see Hollander, Proschan and Sethuraman (1977). Note that they use the terminology Decreasing in Transposition (DT), instead of AI.

Boland and Proschan (1988) extend the concept of arrangement increasing function to vectors involving more than two vectors and call that a Multivariate arrangement increasing function.

DEFINITION 2.2 (Boland and Proschan (1988))

Let f be a function from $(R^n)^m \rightarrow R$. We say that f is Common Permutation Invariant (CPI) if for any $(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m) \in (R^n)^m$ and for any permutation $\underline{\pi} \in S_n$,

$$f(\underline{x}_1.\underline{\pi}, \underline{x}_2.\underline{\pi}, \ldots, \underline{x}_m.\underline{\pi}) = f(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m).$$

Extending the concept of better arranged to m vectors we have, $(\underline{x}_1, \ldots, \underline{x}_m) \overset{a}{\leq} (\underline{z}_1, \ldots, \underline{z}_m)$ implies and is implied by that there exists a finite number of elements $(\underline{y}_2^1, \underline{y}_3^1, \ldots \underline{y}_m^1), \ldots, (\underline{y}_2^p, \ldots, \underline{y}_m^p)$ in $(R^n)^{m-1}$ such that

i) $(\underline{x}_1, \ldots, \underline{x}_m) \overset{a}{=} (\underline{x}_{1\uparrow}, \underline{y}_2^1, \ldots, \underline{y}_m^1)$ and

$(\underline{z}_1, \ldots, \underline{z}_m) \overset{a}{=} (\underline{x}_{1\uparrow}, \underline{y}_2^p, \ldots, \underline{y}_m^p)$

ii) for each $j = 2, \ldots, p$ there exists a pair of co-ordinate indices l and k $(l < k)$ such that $(\underline{x}_{1\uparrow}, \underline{y}_2^j, \ldots, \underline{y}_m^j)$ may be obtained from $(\underline{x}_{1\uparrow}, \underline{y}_2^{j-1}, \ldots, \underline{y}_m^{j-1})$ by interchanging the l and k co-ordinates of any vector $\underline{y}_t^{j-1}$ such that $y_{tl}^{j-1} > y_{tk}^{j-1}$.

It is to be noted that $\overset{a}{\leq}$ is a pre ordering on $(R^n)^m$ and that if $(\underline{x}_1, \ldots, \underline{x}_m) \overset{a}{\leq} (\underline{z}_1, \ldots, \underline{z}_m)$ then the components of the vectors $(\underline{x}_1, \ldots, \underline{x}_m)$ are relatively less similarly arranged than the components of the vectors $(\underline{z}_1, \ldots, \underline{z}_m)$ .If $(\underline{x}_1, \ldots, \underline{x}_m) \overset{a}{=} (\underline{z}_1, \ldots, \underline{z}_m)$ then the relative arrangement of the components in the vectors $\underline{x}_1, \ldots, \underline{x}_m$ is equivalent to that of the components in $\underline{z}_1, \ldots, \underline{z}_m$. For any $(\underline{x}_1, \ldots, \underline{x}_m) \in (R^n)^m$ it follows that

$$(\underline{x}_1, \ldots, \underline{x}_m) \overset{a}{\leq} (\underline{x}_{1\uparrow}, \ldots, \underline{x}_{m\uparrow}) \overset{a}{=} (\underline{x}_{1\downarrow}, \ldots, \underline{x}_{m\downarrow})$$

DEFINITION 2.3 A function f from $(R^n)^m \to R$ is said to be Multivariate Arrangement Increasing (MAI) if

i) f is CPI

ii) $f(\underline{x}_1, \ldots, \underline{x}_m) \geq f(\underline{z}_1, \ldots, \underline{z}_m)$ whenever $(\underline{x}_1, \ldots, \underline{x}_m) \geq^a (\underline{z}_1, \ldots \underline{z}_m)$

DEFINITION 2.4 (Proschan (1989))

A function $f(\underline{\lambda}, \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m) : (R^n)^{m+1} \to R$ is said to be arrangement increasing with respect to $\underline{\lambda}$, if f is CPI and

$$f(\underline{\lambda}, \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_i, \ldots, \underline{x}_m) \geq f(\underline{\lambda}, \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_i', \ldots, \underline{x}_m)$$

where $\underline{x}_i'$ is obtained from $\underline{x}_i$ by interchanging the j and k elements in $\underline{x}_i$ whenever $x_{ij} < x_{ik}$ for $j < k$ and $\lambda_j < \lambda_k$.

We may interpret $\underline{\lambda}$ as a parameter vector and $\underline{x}_i, i = 1, 2, \ldots, m$ are observation vectors similarly arranged to $\underline{\lambda}$. One may have different parameter vectors for each observation vector and hence the following definition.

DEFINITION 2.5 (Proschan (1989))

A function $f(\underline{\lambda}_1, \underline{\lambda}_2, \cdots, \underline{\lambda}_m; \underline{x}_1, \underline{x}_2, \cdots, \underline{x}_m) : (R^n)^m \times (R^n)^m \to R$ is said to be Paired Arrangement Increasing (PAI) if f is AI in each pair $(\underline{\lambda}_i, \underline{x}_i)$ for fixed value of other vectors.

It may be noted that if $\phi : R^m \to R$ is increasing in each component and if $f_i(\cdot), i = 1, 2, \ldots, m$, is AI then

(i) $g(\underline{\lambda}, \underline{x}_1, \underline{x}_2, \cdots, \underline{x}_m) = \phi(f_1(\underline{\lambda}, \underline{x}_1), \cdots, f_m(\underline{\lambda}, \underline{x}_m))$ is AI with respect to $\underline{\lambda}$ and

(ii) $h(\underline{\lambda}_1, \cdots, \underline{\lambda}_m; \underline{x}_1, \cdots, \underline{x}_m) = \phi(f_1(\underline{\lambda}_1, \underline{x}_1), \cdots, f_m(\underline{\lambda}_m, \underline{x}_m))$
is PAI. From the above definition it follows that if $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_m$ are m *i.i.d.* random samples from a distribution with parameter vector $\underline{\lambda}$, then the joint density $\prod_{i=1}^{m} f(\underline{\lambda}, \underline{x}_i)$ is MAI with respect to $\underline{\lambda}$. Analogously, if $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_m$ are m independent random samples with respective parameter vector $\underline{\lambda}_1, \underline{\lambda}_2, \ldots, \underline{\lambda}_m$ then the joint density of $\prod_{i=1}^{m} f(\underline{\lambda}_i, \underline{x}_i)$ is PAI.

2.6 *Note.*

It is to be understood that if

$g(\underline{\pi}_1, \underline{\pi}_2, \cdots, \underline{\pi}_m; \underline{x}_1, \underline{x}_2, \cdots, \underline{x}_m)$ are PAI on $(R^n)^m \times (R^n)^m$ then the function defined by

$g(\underline{\pi}_1, \underline{\pi}_2, \cdots, \underline{\pi}_m; \underline{x}_1, \underline{x}_2, \cdots, \underline{x}_m) = f(\underline{x}_1 \cdot \underline{\pi}_1^{-1}, \cdots, \underline{x}_m \cdot \underline{\pi}_m^{-1})$ is PAI on $(R^n)^m$.

EXAMPLE 2.7

Let f be a function from $(R^4)^3 \times (R^4)^3 \to$ R. Then,

$$f[(2,4,5,3),(1,4,3,5),(2,5,6,4);(2,6,9,5),(3,2,4,5),(1,2,3,7)]$$

$$= f[(5,4,2,3),(1,3,5,4),(2,5,4,6);(9,6,2,5),(3,4,5,2),(1,2,7,3)]$$

This property is common permutation invariance property .

The PAI function satisfies the following inequalities

$$f[(2,4,5,3),(1,4,3,5),(2,5,6,4);(2,6,9,5),(3,2,4,5),(1,2,3,7)]$$

$$\geq f[(2,4,5,3),(1,4,3,5),(2,5,6,4);(6,2,9,5),(3,2,4,5),(1,2,3,7)]$$

$$\geq f[(2,4,5,3),(1,4,3,5),(2,5,6,4);(6,2,9,5),(3,2,4,5),(7,2,3,1)]$$

$$\geq f[(2,4,5,3),(1,4,3,5),(2,5,6,4);(6,2,9,5),(5,2,4,3),(7,2,3,1)]$$

Also, the PAI satisfies this equation.

$$f[(2,4,5,3),(1,4,3,5),(2,5,6,4);(6,2,9,5),(3,2,4,5),(1,2,3,7)]$$

$$-f[(2,4,5,3),(1,4,3,5),(2,5,6,4);(5,6,9,2),(3,2,4,5),(1,2,3,7)]$$

$$-f[(2,4,5,3),(1,4,3,5),(2,5,6,4);(2,6,9,5),(3,2,4,5),(1,2,3,7)]$$

$$+f[(2,4,5,3),(1,4,3,5),(2,5,6,4);(5,6,9,2),(3,2,5,4),(1,2,3,7)] \leq 0$$

Here we make the transposition only on the first vector and next on the second vector on the resulting function.

The following definitions are useful in the sequel.

Let $\underline{x}^{\downarrow} = (\underline{x}_{[1]}, \underline{x}_{[2]}, \ldots, \underline{x}_{[n]})$ be the decreasing rearrangements of $\underline{x} = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)$.

DEFINITION 2.8

We say that the vector $\underline{x}$ majorizes $\underline{x}'$, written $\underline{x} > \underline{x}'$ if

$$\sum_{i=1}^{j} x_{[i]} \geq \sum_{i=1}^{j} x'_{[i]} \text{ for } j = 1, \ldots, n-1$$

and
$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} x'_{[i]}.$$

DEFINITION 2.9

The function $g : R^n \rightarrow R$ is said to be Schur-convex (Schur-concave) if $\underline{x} >^m \underline{x}'$ implies $g(\underline{x}) \geq (\leq) g(\underline{x}')$. The Schur functions have plenty of applications in reliablity, statistics and inequalities. Some of the work in these areas among others are : Pledger and Proschan (1971) obtain comparisons of the reliablity function $h_k(p)$ for a *k out of n* system. Proschan and Sethuraman (1976) extend work of Pledger and Proschan (1971). Barlow and Proschan (1965) show that the mean life of serial system with IFR components exceeds the mean life of a similar system with exponential components. Gopal and Rajalakshmi (1999) study the Schur property of L- superadditive functions.

DEFINITION 2.10

Let ϕ be a function on $R^n \to$ R. ϕ is said to be L-superadditive if ϕ is a positive set function on each pair of the arguments $x_1, \ldots, x_n$. That is, for every pair (x_i, x_j),

$$\phi(x_i+\delta_1, x_j+\delta_2)+\phi(x_i-\delta_1, x_j-\delta_2) \geq \phi(x_i+\delta_1, x_j-\delta_2)+\phi(x_i-\delta_1, x_j+\delta_2),$$

for $\delta_1, \delta_2 > 0$.

PROPOSITION 2.11

(i) Let $f(\underline{\lambda}_1, \underline{\lambda}_2, \ldots, \underline{\lambda}_m; \underline{x}_1, \ldots, \underline{x}_m) = g(\sum_{i=1}^{m}(\underline{\lambda}_i + \underline{x}_i))$.

Then f is PAI on $(R^n)^m \times (R^n)^m$ iff g is Schur convex on R^n.

(ii) Let $f(\underline{\lambda}_1, \underline{\lambda}_2, \ldots, \underline{\lambda}_m; \underline{x}_1, \ldots, \underline{x}_m) = g(\sum_{i=1}^{m}(\underline{\lambda}_i - \underline{x}_i))$. Then f is PAI on $(R^n)^m \times (R^n)^m$ iff g is Schur concave on R^n.

(iii) Let $f(\underline{\lambda}_1, \underline{\lambda}_2, \ldots, \underline{\lambda}_m; \underline{x}_1, \ldots, \underline{x}_m)$

$$= \sum_{i=1}^{n} \phi(\lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{mi}; x_{1i}, x_{2i}, \ldots, x_{mi})$$

Then f is PAI on $(R^n)^m \times (R^n)^m$ iff ϕ is L-superadditive.

Proof. (i) Let f be PAI on $(R^n)^m \times (R^n)^m$. Let $\underline{x}'_i i = 1, 2, \ldots, m$ be the vectors obtained by transposing the components of $\underline{x}_i$. Therefore, $f(\underline{\lambda}_1, \underline{\lambda}_2, \ldots, \underline{\lambda}_m; \underline{x}_1, \ldots, \underline{x}_m) \geq f(\underline{\lambda}_1, \underline{\lambda}_2, \ldots, \underline{\lambda}_m; \underline{x}'_1, \ldots, \underline{x}'_m)$ also $\sum_{i=1}^{m}(\underline{\lambda}_i + \underline{x}_i) >^m \sum_{i=1}^{n}(\underline{\lambda}_i + \underline{x}'_1)$. By definition of g, it is Schur convex on R^n. The converse is immediate. Parts (ii) and (iii) can be proved on similar lines.

EXAMPLE 2.12

The function $g : (R^+)^n \to R^+$ defined by $g(x) = \prod_{i=1}^{n} x_i$ is Schur concave and hence $-g$ is Schur convex. By Proposition 2.11,

(i) $f(\underline{x}_1, \ldots, \underline{x}_m) = \prod_{i=1}^{n} \sum_{j=1}^{m} x_{ij}$ is arrangement decreasing.

(ii) The function $\min_{1\le i\le n}(\underline{x}_i)$ and $-\max_{1\le i\le n}(\underline{x}_i)$ are L-superadditive Hence, $\sum_{i=1}^{n}\min_j x_{ij}$ is PAI and $\sum_{i=1}^{n}\max_j x_{ij}$ is PAD.

iii) $\log\min_{1\le i\le n}(\underline{x}_i)$ and $-\log\max_{1\le i\le n}(\underline{x}_i)$ are L-superadditive on $((0,\infty)^n)^m$. Hence, $\prod_{i=1}^{n}\max_j x_{ij}$ and $\prod_{i=1}^{n}\min x_{ij}$ are respectively PAI and PAD.

Next we prove an important theorem.

THEOREM 2.13

Let $h(\underline{x}_1,\ldots,\underline{x}_m)$ be MAI and $f(\underline{\lambda}_1,\underline{\lambda}_2,\ldots,\underline{\lambda}_m;\underline{x}_1,\ldots,\underline{x}_m)$ be PAI, then the function g defined by

$$g(\underline{\lambda}_1,\underline{\lambda}_2,\ldots,\underline{\lambda}_m)=\int\int\cdots\int h(\underline{x}_1,\ldots,\underline{x}_m)f(\underline{\lambda}_1,\underline{\lambda}_2,\ldots,\underline{\lambda}_m;\underline{x}_1,\ldots,\underline{x}_m)\, d\mu(\underline{x}_1),\ldots,d\mu(\underline{x}_m)$$ is MAI on $(R^n)^m$,

where μ is a permutation invariant measure on R^n.

Proof.

Let $\underline{\lambda}'_i, i=1,2,\ldots,m$ be the vectors obtained by transposing the components of $\underline{\lambda}_i$. We have to show that

$$g(\underline{\lambda}_1,\underline{\lambda}_2,\ldots,\underline{\lambda}_m)-g(\underline{\lambda}'_1,\underline{\lambda}'_2,\ldots,\underline{\lambda}'_m)=\int\int\cdots\int h(\underline{x}_1,\ldots,\underline{x}_m)[f(\underline{\lambda}_1,\underline{\lambda}_2,\ldots,\underline{\lambda}_m;\underline{x}_1,\ldots,\underline{x}_m)-f(\underline{\lambda}'_1,\underline{\lambda}'_2,\ldots,\underline{\lambda}'_m;\underline{x}_1,\ldots,\underline{x}_m)]d\mu(\underline{x}_1)\ldots d\mu(\underline{x}_m)$$

is non-negative. Breaking up the region of integration into 2^m partitions and making a change of variables in the last 2^{m-1} partitions, we have,

$$\begin{aligned}
g(\underline{\lambda}_1,\underline{\lambda}_2,\ldots,\underline{\lambda}_m)-g(\underline{\lambda}'_1,\underline{\lambda}'_2,\ldots,\underline{\lambda}'_m)=&\int_{x_{1i}<x_{1j}}\int_{x_{2i}<x_{2j}}\cdots\int_{x_{mi}<x_{mj}} h(x_1,\ldots,\underline{x}_m)[f(\underline{\lambda}_1,\underline{\lambda}_2,\ldots,\underline{\lambda}_m;\underline{x}_1,\ldots,\underline{x}_m)\\
&-f(\underline{\lambda}'_1,\underline{\lambda}'_2,\ldots,\underline{\lambda}'_m;\underline{x}_1,\ldots,\underline{x}_m)]\\
&+h(\underline{x}'_1,\ldots,\underline{x}_m)[f(\underline{\lambda}_1,\underline{\lambda}_2,\ldots,\underline{\lambda}_m;\underline{x}'_1,\ldots,\underline{x}_m)\\
&-f(\underline{\lambda}'_1,\underline{\lambda}'_2,\ldots,\underline{\lambda}'_m;\underline{x}'_1,\ldots,\underline{x}_m)]\\
&+h(\underline{x}_1,\underline{x}'_2\ldots,\underline{x}_m)f(\underline{\lambda}_1,\underline{\lambda}_2,\ldots,\underline{\lambda}_m;\underline{x}_1,\underline{x}'_2,\ldots,\underline{x}_m)\\
&-f(\underline{\lambda}'_1,\underline{\lambda}'_2,\ldots,\underline{\lambda}'_m;\underline{x}_1,\underline{x}'_2,\ldots,\underline{x}_m)+\ldots\\
&+h(\underline{x}'_1,\ldots,\underline{x}'_m)f(\underline{\lambda}_1,\underline{\lambda}_2,\ldots,\underline{\lambda}_m;\underline{x}'_1,\ldots,\underline{x}'_m)\\
&-f(\underline{\lambda}'_1,\underline{\lambda}'_2,\ldots,\underline{\lambda}'_m;\underline{x}'_1,\ldots,\underline{x}'_m)\\
&d\mu(\underline{x}_1),\ldots,d\mu(\underline{x}_m).
\end{aligned}$$

$$= \int_{x_{1i}<x_{1j}} \int_{x_{2i}<x_{2j}} \cdots \int_{x_{mi}<x_{mj}} \left\{ T_{\underline{x}_m}, T_{\underline{x}_{m-1}}, \ldots, T_{\underline{x}_1}(h(\underline{x}_1, \ldots, \underline{x}_m) \right\}$$

$$\left\{ T_{\underline{x}_m}, T_{\underline{x}_{m-1}}, \ldots, T_{\underline{x}_1} \left[f(\underline{\lambda}_1, \underline{\lambda}_2, \ldots, \underline{\lambda}_m; \underline{x}_1, \ldots, \underline{x}_m) \right.\right.$$

$$\left.\left. - f(\underline{\lambda}'_1, \underline{\lambda}'_2, \ldots, \underline{\lambda}'_m; \underline{x}_1, \ldots, \underline{x}_m) \right] \right\} d\mu(\underline{x}_1), \ldots, d\mu(\underline{x}_m).$$

Where $T_{\underline{x}_i}(\phi(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m))$

$$= \phi(\underline{x}_1, \ldots, \underline{x}_i, \ldots, \underline{x}_m) - \phi(\underline{x}'_1, \ldots, \underline{x}'_i, \ldots, \underline{x}_m).$$

Since h and f are PAI on $(R^n)^m$ and $(R^n)^m \times (R^n)^m$, the two factors of the integrand are non-negative. Hence the theorem.

3. STOCHASTIC REARRANGEMENT INEQUALITIES

Next we define the concept of stochastically similarly arranged vectors.Chan, D Abadie and Proschan (1987) introduce the concept of stochastically similarly arranged pairs of random vectors.Extending this concept to more than two vectors we have the following definition.

DEFINITION 3.1

We say that $(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m)$ are stochastically similarly arranged if there exists a joint density $K(.)$ of $(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m)$ with respect to a permutation invariant measure such that $K(\underline{x}.\underline{\pi}_1^{-1}, \underline{x}.\underline{\pi}_2^{-1}, \ldots, \underline{x}.\underline{\pi}_m^{-1})$ is PAI on $(R^n)^m$.

If $(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m)$ are stochastically similarly arranged and is degenerate at $(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m)$ then $(x_{1i} - x_{1j}), (x_{2i} - x_{2j}), \ldots, (x_{mi} - x_{mj})$ for all pairs (i, j) .However if for at least one pair (not all) $x_{ki} > x_{kj}, i < j$,then the vector is not stochastically similarly arranged. Moreover if $(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m)$ is degenerate at $(\underline{x}_{1\uparrow} \underline{x}_{2\uparrow} \ldots, \underline{x}_{m\uparrow})$ then $(\underline{x}_1, \ldots, \underline{x}_m)$ is stochastically similarly arranged.

THEOREM 3.2

Let $(\underline{x}_1, \underline{x}_2, .., \underline{x}_m)$ be stochastically similarly arranged, with respect to a permutation invariant measure μ .If f is PAI,then

$E[f(\underline{x}_{\underline{\pi}1}, \underline{x}_{\underline{\pi}2}, \ldots, \underline{x}_{\underline{\pi}m})]$ is PAI with respect to $(\underline{\pi}_1, \ldots, \underline{\pi}_m)$.

Proof of this theorem is a direct consequence of Theorem 2.13.

Using the above theorem we prove the generalised stochastic rearrangement inequalities.

THEOREM 3.3

Let $(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m)$ be stochastically similarly arranged. Then the following inequalities hold for all permutations $\underline{\pi}_1, \underline{\pi}_2, \ldots, \underline{\pi}_m$

$$\text{(i)} \quad \sum_{i=1}^{n} x_{1i} \ldots x_{mi} \geq \sum_{i=1}^{n} x_{1\pi_1(i)} x_{2\pi_2(i)} \cdots x_{m\pi_m(i)}$$

$$\text{(ii)} \quad \prod_{i=1}^{n} \min(x_{1i}, \ldots, x_{mi}) \geq \prod_{i=1}^{n} \min(x_{1\pi_1(i)}, x_{2\pi_2(i)}, \ldots, x_{m\pi_m(i)})$$

$$\text{(iii)} \quad \sum_{i=1}^{n} \min(x_{1i}, \ldots, x_{mi}) \geq \sum_{i=1}^{n} \min(x_{1\pi_1(i)}, x_{2\pi_2(i)}, \ldots, x_{m\pi_m(i)})$$

$$\text{(iv)} \quad \sum_{i=1}^{n} \max(x_{1i}, \ldots, x_{mi}) \leq \sum_{i=1}^{n} \max(x_{1\pi_1(i)}, x_{2\pi_2(i)}, \ldots, x_{m\pi_m(i)})$$

$$\text{(v)} \quad \prod_{i=1}^{n} \max(x_{1i}, \ldots, x_{mi}) \leq \prod_{i=1}^{n} \max(x_{1\pi_1(i)}, x_{2\pi_2(i)}, \ldots, x_{m\pi_m(i)})$$

$$\text{(vi)} \quad \prod_{i=1}^{n} (x_{1i} + \ldots + x_{mi}) \leq \prod_{i=1}^{n} (x_{1\pi_1(i)} + x_{2\pi_2(i)} + \ldots + x_{m\pi_m(i)})$$

Proof.

It follows that from Example 2.12 the following functions (i), (ii) and (iii) are PAI and (iv), (v) and (vi) are PAD.

$$\text{(i)} \quad f_1(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m) = \sum_{i=1}^{n} x_{1i} x_{2i} \ldots x_{mi}.$$

$$\text{(ii)} \quad f_2(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m) = \prod_{i=1}^{n} \min_{1 \leq j \leq m} x_{ij}$$

$$\text{(iii)} \quad f_3(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m) = \sum_{i=1}^{n} \min_{1 \leq j \leq m} x_{ij}$$

$$\text{(iv)} \quad f_4(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m) = \sum_{i=1}^{n} \max_{1 \leq j \leq m} x_{ij}$$

$$\text{(v)} \quad f_5(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m) = \prod_{i=1}^{n} \max_{1 \leq j \leq m} x_{ij}$$

$$\text{(vi)} \quad f_6(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m) = \prod_{i=1}^{n} \sum_{j=1}^{m} x_{ij}$$

These inequalities are the generalizations of the inequalities obtained by Chan, D' Abadie and Proschan (1987). In the case involving only two vectors, one can obtain both the maximum and minimum for all the inequalities and in the case of more than two vectors only the maximum for the first three inequalities and minimum for the last three inequalities can be obtained.

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