THE EGYPTINA STATISTICA JOURNAL
ISSR. CAIRO UNIV., VOL. 45. NO.2. 2001

BAYESIAN ESTIMATION FOR THE PARETO LIFE TIME MODEL WITH DOUBLY CENSORED OBSERVATIONS

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Key Words and Phrases: Pareto distribution; Bayesian estimation; censored sampling; conjugate priors; reliability.

ABSTRACT

Bayesian estimation of the parameters of Pareto life time distribution is considered. The parameters of interest are the shape and scale parameters, reliability and hazard rate. Inferences are based on doubly censored observations when the data is both left and right censored. In addition to point estimators, credible regions for the parameters of interest are considered.

1. INTRODUCTION

The life distribution under consideration in this study is the two parameter Pareto distribution with probability density function

$$f(x;\alpha,\sigma)=\alpha\sigma^{\alpha}x^{-(\alpha+1)} \qquad (x\geq\sigma) \qquad (1.1)$$

where $\alpha>0$ and $\sigma>0$.

The Pareto distribution has found wide spread use as a model for various socio-economic phenomena; such as city population sizes, occurrence of natural resources, stock price fluctuations, size of firms and personal incomes which appear to have remarkably regular distributions with very long right tails; see, for example, Johnson et al. (1994). The Pareto distribution has also been used in reliability and lifetime modeling; see, for example, Davis and Feldstein (1979).

Bayesian inference procedures for samples from the Pareto distribution have been the focus of attention of a number of authors; for example, Malik (1970) and Sinha and Howlader (1980) considered estimation of the shape parameter under complete sampling when the scale parameter is known. Lwin (1972) presented the estimation of the survival probability when either or both parameters are unknown in the case of complete sampling under both squared and absolute error loss functions using the Natural conjugate prior (NCP). Arnold and Press (1983) had some attempts considering the estimation of α and σ as well as the survival probability using both loss functions under complete sampling using a different prior specification. Upadhyay and Shastri (1997) considered Bayesian analysis of the Pareto distribution, under the Non-informative prior (NIP), when the observations are both left and right censored by providing sample-based estimates of posterior distributions using Gibbs sampler algorithm. The estimated marginal posteriors of α and σ were shown using frequency curves. Using the samples of α and σ, Upadhyay and Shastri (1997) obtained the corresponding samples for the survival probability at mission time t, $S(t)$, by substitution and the estimated posterior of $S(t)$ was illustrated using a frequency curve.

This article is concerned with estimation of the parameters of the density function (1.1), reliability as well as the hazard rate. The analysis is carried out under double censoring when lifetimes are both left and right censored, of which complete sampling and Type II censoring are special cases. Under this type of double censoring n items are simultaneously put on test and observed until there have been r failures. However $k-1$ lifetimes are censored on the left. The actual observed lifetimes are the middle $r-k+1$ observations. Denote by $X_{(j)}$ the lifetime of the j-th item to fail. For the derivation of point estimators of the parameters of interest squared error loss and absolute loss will be used as an extension of the work of Arnold and Press (1983). As an extension to the work of Upadhyay and Shastri (1997) closed forms of the estimators will be provided in this article.

However, the existence of a loss function is a big assumption. Unless there is a very clear need for a point estimator and a strong rationale for a specific loss function, the summary of the information available about a single random quantity in a single value to summarize the posterior density may be misleading, especially if the posterior density is markedly skew. Bayes probability intervals for the parameters of interest will be derived. To minimize the size of this interval, only points with the largest posterior density are included arriving at a highest posterior density credible region; Bernardo and Smith (1994).

2. BAYESIAN MODELS FOR THE PARETO DISTRIBUTION

When both α and σ are unknown, a NCP for (α,σ) was first suggested by Lwin (1972) and later generalized by Arnold and Press (1983) to include broader classes of prior distributions. The kernel for generalized Lwin priors or the Power-Gamma prior, denoted by $PG(\nu,\lambda,\mu,\theta)$, is given by

$$g(\alpha,\sigma) \propto \sigma^{\lambda\alpha-1}\alpha^{\nu}\mu^{-\alpha} \quad (\alpha>0,\ 0<\sigma<\theta) \quad (2.1)$$

where μ, θ, ν and λ are positive constants, and $\theta^{\lambda}<\mu$. Such a prior specifies $g(\alpha)$ as $Ga(\nu,\ln\mu-\lambda\ln\theta)$ and $g(\sigma|\alpha)$ as a power function distribution $PF(\lambda\alpha,\theta)$ of form $\lambda\alpha\sigma^{\lambda\alpha-1}\theta^{-\lambda\alpha}$ $(0<\sigma<\theta)$.

Under double censoring, when the lifetimes are both left and right censored, $k-1$ lifetimes are left censored and $n-r$ lifetimes are right censored and the $r-k+1$ middle observations are the actual observed lifetimes. The likelihood for this data configuration assumes the form

$$L(\alpha,\sigma)=\frac{n!}{(k-1)!(n-r)!}\left\{1-\left(\frac{\sigma}{x_{(k)}}\right)^{\alpha}\right\}^{k-1}\left(\frac{\sigma}{x_{(r)}}\right)^{(n-r)\alpha}\alpha^{r-k+1}\sigma^{(r-k+1)\alpha}\prod_{l=k}^{r}x_{(l)}^{-(\alpha+1)}$$

$$\left(\sigma\le x_{(k)}<x_{(k+1)}<\dots<x_{(r)}\right).$$

Applying the power gamma prior given in (2.1), the corresponding posterior density under this sampling plan is given by

$$g\left(\alpha,\sigma\middle|\mathbf{x}^{(r-k+1)}\right)=\frac{\alpha^{r+v-k+1}\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{i}\sigma^{(n+\lambda+i-k+1)\alpha-1}\left(\mu x_{(r)}^{(n-r)}x_{(k)}^{i}\prod_{l=k}^{r}x_{(l)}\right)^{-\alpha}}{\Gamma(r+v-k+1)\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^{j}}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+v-k+1)}}.$$

$$(\alpha>0,\sigma<w)\quad(2.2)$$

where $\mathbf{X}^{(r-k+1)}=\left(X_{(k)},X_{(k+1)},\dots,X_{(r)}\right)$, $w=\min\left(\theta,x_{(k)}\right)$ and

$A(j)=\sum_{l=k}^{r}\ln x_{(l)}+\ln\mu+(n-r)\ln x_{(r)}+j\ln x_{(k)}-(n+\lambda+j-k+1)\ln w$ for

$j=0,1,\dots,k-1$.

The posterior density (2.2) specifies

$$g\left(\alpha\middle|\mathbf{x}^{(r-k+1)}\right)=\frac{\alpha^{r+v-k}\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{(-1)^{i}}{(n+\lambda+i-k+1)}\left[\exp\{A(i)\}\right]^{-\alpha}}{\Gamma(r+v-k+1)\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^{j}}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+v-k+1)}}\quad(\alpha>0),$$

$$(2.3)$$

$$g\left(\sigma\middle|\mathbf{x}^{(r-k+1)}\right)=\frac{(r+v-k+1)\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{i}\sigma^{-1}\left\{A(i)-(n+\lambda+i-k+1)\ln\left(\frac{\sigma}{w}\right)\right\}^{-(r+v-k+2)}}{\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^{j}}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+v-k+1)}}$$

$$(0<\sigma<w),\quad(2.4)$$

and

$$g\left(\sigma \mid \alpha, \mathbf{x}^{(r-k+1)}\right)=\frac{\alpha\sigma^{(n+\lambda-k+1)\alpha-1}\left\{1-\left(\frac{\sigma}{x_{(k)}}\right)^{\alpha}\right\}^{k-1}}{\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^{j}}{(n+\lambda+j-k+1)}x_{(k)}^{-j\alpha}w^{(n+\lambda+j-k+1)\alpha}} \qquad (0<\sigma<w).$$

When we are in a situation where little is known a priori about the values of α and σ; that is, information concerning α and σ comes primarily from the sample. The following NIP will then be adopted

$$g(\alpha,\sigma)\propto\frac{1}{\alpha\sigma} \qquad (\alpha,\sigma>0).$$

Results under the NIP are equivalent to those obtained under the Power Gamma prior (2.1) on setting $\nu=-1$, $\lambda=0$, $\mu=1$ and $\theta\to\infty$.

3. ESTIMATION OF THE SHAPE PARAMETER

The Bayes estimator for α with respect to the quadratic form loss is the mean of (2.3) given by

$$\hat{\alpha}=\frac{(r+\nu-k+1)\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{(-1)^{i}}{(n+\lambda+i-k+1)}\{A(i)\}^{-(r+\nu-k+2)}}{\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^{j}}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+\nu-k+1)}}.$$

With respect to the absolute loss function, the corresponding estimator for α, $\tilde{\alpha}$, is the median of the posterior density; that is, $P\left(\alpha\le\tilde{\alpha} \mid \mathbf{x}^{(r-k+1)}\right)=\frac{1}{2}$. From (2.3), $\tilde{\alpha}$ satisfies the equation

$$\frac{\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{(-1)^{i}}{(n+\lambda+i-k+1)}\{A(i)\}^{-(r+\nu-k+1)}\Gamma\{r+\nu-k+1,\tilde{\alpha}A(i)\}}{\Gamma(r+\nu-k+1)\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^{j}}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+\nu-k+1)}}=\frac{1}{2},$$

where $\Gamma(q,y)=\int_{y}^{\infty}u^{q-1}e^{-u}du \quad (q>0, y>0)$ is the incomplete gamma function.

A $100(1-\gamma)\%$ Bayes probability interval for α is obtained by solving the following equation

$$\int_{\alpha_*}^{\alpha^*} g\left(\alpha \middle| \mathbf{x}^{(r-k+1)}\right) d\alpha = 1-\gamma$$

for the lower and upper limits α_* and α^* respectively. From (2.3) α_* and α^* satisfy the following equation

$$\frac{\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{(-1)^i}{(n+\lambda+i-k+1)}\{A(i)\}^{-(r+v-k+1)}\left[\Gamma\{r+v-k+1,\alpha_* A(i)\}-\Gamma\{r+v-k+1,\alpha^* A(i)\}\right]}{\Gamma(r+v-k+1)\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+v-k+1)}}=1-\gamma \qquad (3.1).$$

In practice one would pick those α_* and α^* that satisfy (3.1) for which $\alpha^*-\alpha_*$ is shortest.

The hazard rate at time t, $h(t)=\alpha/t$, specifies the instantaneous rate of death or failure at time t given that the component survived up till t. Point estimators as well as Bayesian probability intervals for $h(t)$ could be derived directly from the corresponding estimators and intervals for α.

4. ESTIMATION OF THE SCALE PARAMETER

In their study under complete sampling, Arnold and Press (1983) presented no estimators for σ stating that the marginal posterior distribution for σ (a special case of 2.4 when $k=1$ and $r=n$) is not as simple as one might hope. To overcome this difficulty estimators for σ will be derived here using (2.2). Under squared loss the Bayes estimator of σ is of the form

$$\hat{\sigma} = E\left(\sigma \middle| \mathbf{x}^{(r-k+1)}\right) = \iint \sigma g\left(\alpha, \sigma \middle| \mathbf{x}^{(r-k+1)}\right) d\sigma d\alpha .$$

From (2.2) this is given by

$$\frac{w\sum_{i=0}^{k-1}\sum_{m=0}^{r+v-k+1}\binom{k-1}{i}\binom{r+v-k+1}{m}\frac{(-1)^{m+i}}{(n+\lambda+i-k+1)^{m+1}}\left[\exp\left\{\frac{A(i)}{(n+\lambda+i-k+1)}\right\}\right]\{A(i)\}^{-(r+v-k-m+1)}\Gamma\left\{r+v-k-m+1,\frac{A(i)}{(n+\lambda+i-k+1)}\right\}}{\Gamma(r+v-k+1)\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^{j}}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+v-k+1)}}$$

(4.1)

For the evaluation of $\hat{\sigma}$ from (4.1), we need to evaluate $\Gamma(q, y)$ for $q = 0,1,2,\ldots r+v-k+1$ and $y = \frac{A(i)}{(n+\lambda+i-k+1)}$ for $i = 0,1,\ldots,k-1$. The following recurrence relation could be used for $q = 1,2,\ldots r+v-k$

$$\Gamma(q+1, y) = q\Gamma(q, y) + y^{q}e^{-y} \quad \text{(Erdelyi et al. (1953))},$$

where $\Gamma(1, y) = e^{-y}$. The special incomplete gamma function $\Gamma(0, y)$ is given by the exponential integral, $E_1(y) = \int_y^{\infty} u^{-1}e^{-u}du$, tables of which can be found in Abramowitz and Stegun (1965).

Under absolute loss the Bayes estimator $\tilde{\sigma}$, the median of the posterior density (2.4), satisfies the equation

$$\frac{\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{(-1)^{i}}{(n+\lambda+i-k+1)}\left\{A(i)-(n+\lambda+i-k+1)\ln\left(\frac{\tilde{\sigma}}{w}\right)\right\}^{-(r+v-k+1)}}{\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^{j}}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+v-k+1)}} = \frac{1}{2}.$$

From the posterior density (2.4), a $100(1-\gamma)\%$ Bayes probability interval for σ is obtained by solving the following equation

$$\int_{\sigma_*}^{\sigma^*} g\left(\sigma \middle| \mathbf{x}^{(r-k+1)}\right) d\sigma = 1-\gamma \ ,$$

for the lower and upper limits σ_* and σ^* respectively. Hence σ_* and σ^* satisfy the following equation

$$\frac{\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{(-1)^i}{(n+\lambda+i-k+1)}\left[\left\{A(i)-(n+\lambda+i-k+1)\ln\left(\frac{\sigma^*}{w}\right)\right\}^{-(r+v-k+1)} - \left\{A(i)-(n+\lambda+i-k+1)\ln\left(\frac{\sigma_*}{w}\right)\right\}^{-(r+v-k+1)}\right]}{\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+v-k+1)}} = 1-\gamma \qquad (4.2)$$

In practice one would pick those σ_* and σ^* that satisfy (4.2) for which $\sigma^* - \sigma_*$ is shortest.

5. ESTIMATION OF RELIABILITY AT TIME t

The probability that a component survives mission time, t, for a given t is $S(t)$ which for (1.1) is given by

$$S(t) = P(X_i > t) = \begin{cases} \left(\frac{\sigma}{t}\right)^{\alpha} & \sigma < t < w \text{ or } t \geq w, \\ 1 & t < \sigma < w. \end{cases}$$

Under squared error loss

$$\hat{S}(t) = \begin{cases} \int_0^{\infty}\int_0^{t}\left(\frac{\sigma}{t}\right)^{\alpha} g\left(\alpha,\sigma\middle|\mathbf{x}^{(r-k+1)}\right)d\sigma d\alpha + \int_0^{\infty}\int_t^{w} g\left(\alpha,\sigma\middle|\mathbf{x}^{(r-k+1)}\right)d\sigma d\alpha & (t < w) \\ \int_0^{\infty}\int_0^{w}\left(\frac{\sigma}{t}\right)^{\alpha} g\left(\alpha,\sigma\middle|\mathbf{x}^{(r-k+1)}\right)d\sigma d\alpha & (t \geq w). \end{cases}$$

From (2.2) this is given by

$$\hat{S}(t)=\begin{cases} 1-\dfrac{\sum_{i=0}^{k-1}\binom{k-1}{i}\dfrac{(-1)^{i}}{(n+\lambda+i-k+1)(n+\lambda+i-k+2)}\{B(i)\}^{-(r+v-k+1)}}{\sum_{j=0}^{k-1}\binom{k-1}{j}\dfrac{(-1)^{j}}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+v-k+1)}} & (t<w) \\[2ex] \dfrac{\sum_{i=0}^{k-1}\binom{k-1}{i}\dfrac{(-1)^{i}}{(n+\lambda+i-k+2)}\left\{A(i)-\ln\left(\dfrac{w}{t}\right)\right\}^{-(r+v-k+1)}}{\sum_{j=0}^{k-1}\binom{k-1}{j}\dfrac{(-1)^{j}}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+v-k+1)}} & (t\geq w) \end{cases}$$

where $B(i) \doteq A(i)-(n+\lambda+i-k+1)\ln\left(\dfrac{t}{w}\right)$.

Under the absolute loss function, the Bayes estimator $\tilde{S}$, the median of the posterior density of S, satisfies the equation $P\left(S \geq \tilde{S} \mid \mathbf{x}^{(r-k+1)}\right)=\dfrac{1}{2}$. The posterior density of S will be derived and hence the posterior probability of the event $S \geq \tilde{S}$. For a given t, Figures (1) and (2) give $S(t)$ as a function of α and σ for the two cases $t < w$ and $t \geq w$ respectively.

FIG. (1) $S(t)$ as a function of α and σ for $t < w$

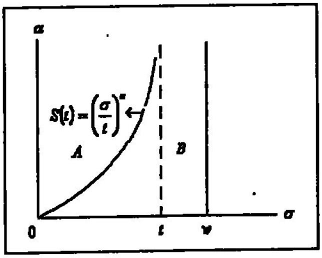
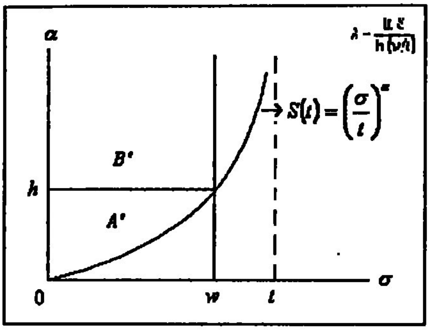


FIG. (2) $S(t)$ as a function of α and σ for $t \geq w$



For $t < w$, from Figure (1), we have that

$$P\left(S = 1 \middle| \mathbf{x}^{(r-k+1)}\right) = P(B) = \int_0^{\infty}\int_t^{w} g\left(\alpha, \sigma \middle| \mathbf{x}^{(r-k+1)}\right) d\sigma d\alpha$$

$$= 1 - \frac{\sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^i}{(n+\lambda+i-k+1)} \{B(i)\}^{-(r+v-k+1)}}{\sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{(n+\lambda+j-k+1)} \{A(j)\}^{-(r+v-k+1)}}.$$

From Figure (1), for $0 < s < 1$, we have that

$$P\left(S \leq s \middle| \mathbf{x}^{(r-k+1)}\right) = P(A) = \int_0^{\infty}\int_0^{ts^{\frac{1}{\alpha}}} g\left(\alpha, \sigma \middle| \mathbf{x}^{(r-k+1)}\right) d\sigma d\alpha$$

$$= \frac{\sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^i}{(n+\lambda+i-k+1)} s^{(n+\lambda+i-k+1)} \{B(i)\}^{-(r+v-k+1)}}{\sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{(n+\lambda+j-k+1)} \{A(j)\}^{-(r+v-k+1)}}.$$

Hence for a given $t < w$, $p\left(s \middle| \mathbf{x}^{(r-k+1)}\right)$ is given as follows

$$p\left(s\middle|\mathbf{x}^{(r-k+1)}\right)=\begin{cases}1-\dfrac{\sum_{i=0}^{k-1}\binom{k-1}{i}\dfrac{(-1)^{i}}{(n+\lambda+i-k+1)}\{B(i)\}^{-(r+\nu-k+1)}}{\sum_{j=0}^{k-1}\binom{k-1}{j}\dfrac{(-1)^{j}}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+\nu-k+1)}} & (s=1)\\[2ex] \dfrac{\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{i}s^{(n+\lambda+i-k)}\{B(i)\}^{-(r+\nu-k+1)}}{\sum_{j=0}^{k-1}\binom{k-1}{j}\dfrac{(-1)^{j}}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+\nu-k+1)}} & (0<s<1)\end{cases} \qquad (5.1)$$

For $t \geq w$, from Figure (2), we have that

$$P\left(S\leq s\middle|\mathbf{x}^{(r-k+1)}\right)=\iint_{A'+B'} g\left(\alpha,\sigma\middle|\mathbf{x}^{(r-k+1)}\right)d\sigma d\alpha$$

$$P\left(S\leq s\middle|\mathbf{x}^{(r-k+1)}\right)=\int_{0}^{\frac{\ln s}{\ln\left(\frac{w}{t}\right)}}\int_{0}^{\alpha^{\frac{1}{\alpha}}} g\left(\alpha,\sigma\middle|\mathbf{x}^{(r-k+1)}\right)d\sigma d\alpha+\int_{\frac{\ln s}{\ln\left(\frac{w}{t}\right)}}^{\infty}\int_{0}^{w} g\left(\alpha,\sigma\middle|\mathbf{x}^{(r-k+1)}\right)d\sigma d\alpha$$

From the posterior density (2.2), we find that

$$P\left(S\leq s\middle|\mathbf{x}^{(r-k+1)}\right)=\frac{\sum_{i=0}^{k-1}\binom{k-1}{i}\dfrac{(-1)^{i}}{(n+\lambda+i-k+1)}s^{(n+\lambda+i-k+1)}\{B(i)\}^{-(r+\nu-k+1)}\gamma\left[r+\nu-k+1,\{B(i)\}\left\{\dfrac{\ln s}{\ln\left(\frac{w}{t}\right)}\right\}\right]}{\Gamma(r+\nu-k+1)\sum_{j=0}^{k-1}\binom{k-1}{j}\dfrac{(-1)^{j}}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+\nu-k+1)}}$$

$$+\frac{\sum_{i=0}^{k-1}\binom{k-1}{i}\dfrac{(-1)^{i}}{(n+\lambda+i-k+1)}\{A(i)\}^{-(r+\nu-k+1)}\Gamma\left[r+\nu-k+1,\{A(i)\}\left\{\dfrac{\ln s}{\ln\left(\frac{w}{t}\right)}\right\}\right]}{\Gamma(r+\nu-k+1)\sum_{j=0}^{k-1}\binom{k-1}{j}\dfrac{(-1)^{j}}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+\nu-k+1)}}$$

(5.2)

where $\gamma(a,y)=\int_{0}^{y}t^{a-1}e^{-t}dt=\Gamma(a)-\Gamma(a,y)$.

Hence, $p\left(s\middle|\mathbf{x}^{(r-k+1)}\right)=\frac{\partial}{\partial s}\left\{P\left(S\leq s\middle|\mathbf{x}^{(r-k+1)}\right)\right\}$.

For $t \geq w$, it follows that

$$p\left(s \middle| \mathbf{x}^{(r-k+1)}\right) = \frac{\sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i s^{(n+\lambda+i-k)} \{B(i)\}^{-(r+v-k+1)} \gamma\left[r+v-k+1, \{B(i)\}\left\{\frac{\ln s}{\ln\left(\frac{w}{t}\right)}\right\}\right]}{\Gamma(r+v-k+1) \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{(n+\lambda+j-k+1)} \{A(j)\}^{-(r+v-k+1)}} \qquad (5.3)$$

Now the posterior median $\tilde{S}$ for a given t could be derived for the two different cases as follows

For $t < w$, if $\tilde{S}$ is the posterior median of S and if $P\left(S = 1 \middle| \mathbf{x}^{(r-k+1)}\right) < \frac{1}{2}$ that is

$$\frac{\sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^i}{(n+\lambda+i-k+1)} \{B(i)\}^{-(r+v-k+1)}}{\sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{(n+\lambda+j-k+1)} \{A(j)\}^{-(r+v-k+1)}} > \frac{1}{2},$$

thus from (5.1), $\tilde{S}$ satisfies the equation

$$\frac{\sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^i}{(n+\lambda+i-k+1)} \tilde{S}^{(n+\lambda+i-k+1)} \{B(i)\}^{-(r+v-k+1)}}{\sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{(n+\lambda+j-k+1)} \{A(j)\}^{-(r+v-k+1)}} = \frac{1}{2} \qquad (5.4)$$

A search program can be used to arrive at the value of S, $\tilde{S}$, satisfying (5.4).

For $t \geq w$, if $\tilde{S}$ is the posterior median of S then $P\left(S \leq \tilde{S} \middle| \mathbf{x}^{(r-k+1)}\right) = \frac{1}{2}$. A search program can be used to arrive at the value of S, $\tilde{S}$, from equating (5.2) to 0.5.

A $100(1-\gamma)\%$ Bayesian probability interval for S could be derived from the posterior distribution for the two separate cases as follows:

For $t < w$, let $P\left(s_{*} < S < s^{*} \middle| \mathbf{x}^{(r-k+1)}\right) = 1-\gamma$, by setting $P\left(0 < S < s_{*} \middle| \mathbf{x}^{(r-k+1)}\right) = \frac{\gamma}{2}$ and $P\left(s^{*} < S < 1 \middle| \mathbf{x}^{(r-k+1)}\right) + P\left(S = 1 \middle| \mathbf{x}^{(r-k+1)}\right) = \frac{\gamma}{2}$.

This for a given t will provide the limits for S given by the following equations

$$\frac{\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{(-1)^i}{(n+\lambda+i-k+1)}s_*^{(n+\lambda+i-k+1)}\{B(i)\}^{-(r+v-k+1)}}{\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+v-k+1)}}=\frac{\gamma}{2} \tag{5.5}$$

$$\frac{\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{(-1)^i}{(n+\lambda+i-k+1)}s^{*(n+\lambda+i-k+1)}\{B(i)\}^{-(r+v-k+1)}}{\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(n+\lambda+j-k+1)}\{A(j)\}^{-(r+v-k+1)}}=1-\frac{\gamma}{2} \tag{5.6}$$

For $t \geq w$, let $P\left(s_* < S < s^* \middle| \mathbf{x}^{(r-k+1)}\right) = 1-\gamma$, by setting $P\left(0 < S < s_* \middle| \mathbf{x}^{(r-k+1)}\right) = \frac{\gamma}{2}$ and $P\left(s^* < S < 1 \middle| \mathbf{x}^{(r-k+1)}\right) = \frac{\gamma}{2}$.

For a given t, bounds on S in this case are then obtained from fractiles of (5.2). The lower and upper limits s_* and s^* for S are the unique solutions for S from equating (5.2) to $\frac{\gamma}{2}$ and $1-\frac{\gamma}{2}$ respectively, for some preassigned γ.

For the special case $k=1$ and $r=n$ results of sections 2 to 5 reduce to the case of complete sampling, while for the special case of $k=1$ results are obtained under Type II censoring.

6. NUMERICAL EXAMPLE

Consider a life test where 20 units whose lifetimes follow the same Pareto distribution with both parameters unknown are put on test simultaneously. The times of failure of the third to the eleventh items measured in an informative experiment are shown in Table 1.

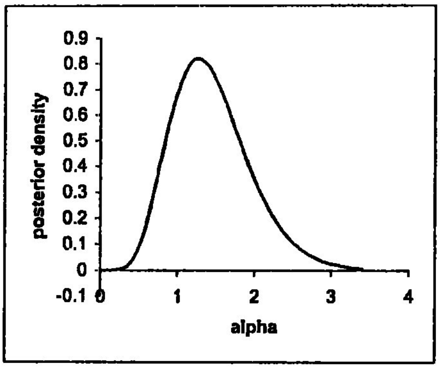
TABLE I Times of failure of the third to the eleventh items

10.425	10.757	10.946	11.433	11.663
11.945	14.712	15.279	16.121	

We use the results of sections 2 to 5 with $n = 20$, $k = 3$ and $r = 11$, and assume little is known a priori about (α, σ); that is, results of these sections are used under the settings $\nu = -1$, $\lambda = 0$, $\mu = 1$ and $\theta \to \infty$.

Figure (3) illustrates the posterior density of α given $\mathbf{x}^{(9)}$ under these settings, $\alpha | \mathbf{x}^{(9)} = Ga(8, 5.506624)$. The posterior density of α is slightly positively skewed.

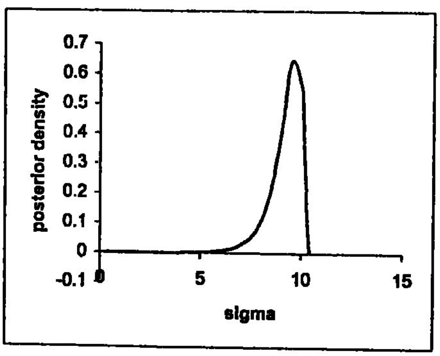
FIG. (3) The posterior density of α given $\mathbf{x}^{(9)}$



The posterior mean of α is given by $\hat{\alpha} = 1.4528$ while $\tilde{\alpha} = 1.39269$. A 95% Bayesian interval for α is of the form $\alpha_* = 0.62724$ and $\alpha^* = 2.61912$.

Figure (4) illustrates the posterior density of σ given $\mathbf{x}^{(9)}$. The posterior density of σ is negatively skewed.

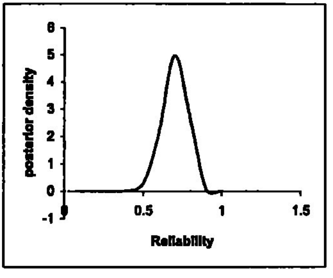
FIG. (4) The posterior density of σ given $\mathbf{x}^{(9)}$



The posterior mean of σ is given by $\hat{\sigma} = 9.24284$ while $\tilde{\sigma} = 9.42128$. A 95% Bayesian interval for σ is of the form $\sigma_* = 7.24481$ and $\sigma^* = 10.21093$.

The probability that a component survives an arbitrarily chosen mission time $t = 12$ $\left(t > w = x_{(3)} = 10.425\right)$ is given by $S(12)$. Figure (5) illustrates the posterior density of S given $\mathbf{x}^{(9)}$ for $t = 12$. The posterior density of S is almost symmetrical (very slightly negatively skewed).

FIG. (5) The posterior density of S given $\mathbf{x}^{(9)}$ for $t = 12$



The posterior mean of S is given by $\hat{S} = 0.70048$ while $\tilde{S} = 0.70543$. A 95% Bayesian interval is of the form $s_* = 0.53369$ and $s^* = 0.83941$.

The posterior mean, median for the failure rate at mission time 12, $h(12)$, are given as $\hat{h} = 0.12107$ and $\tilde{h} = 0.11606$ respectively while a 95% Bayesian interval is of the form $h_* = 0.05227$ and $h^* = 0.21826$.

7. CONCLUSION

Bayesian estimation of the parameters of Pareto life time distribution was considered. Point estimators of the shape and scale parameters as well as reliability at time t were derived under squared and absolute error loss. Bayes probability intervals for the parameters of interest were obtained. Inferences were based on doubly censored observations when the data is both left and right censored. Gaps in earlier work of Arnold and Press (1983) in case of complete sampling was covered as a special case of work presented in this article. Closed forms of the estimators of the parameters of interest were presented in case of Natural conjugate prior as well as non-informative prior as an extension of the work of Upadhyay and Shastri (1997) who provided sample based estimates of posterior distributions of parameters under the non-informative prior.

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