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TESTING WHETHER A SURVIVAL FUNCTION IS NEW BETTER THAN USED OF A SPECIFIED AGE

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SUMMARY

A survival variable is a nonnegative random variable X with distribution function F and a val function $\overline{F} = 1 - F$. This variable is said to be new better than used of a specified age $\overline{F}(x+t_0) \le \overline{F}(x)\overline{F}(t_0)$ for all $x \ge 0$ and a fixed t_0. This is a large and practical class of life utions. Its properties, applicability, and testing was discussed by Hollander, Park and (1986), (HPP). In the current investigation , it is demonstrated that a goodness of fit h is possible to carry out this testing problem and that it results in simpler and totically equivalent procedure to the HPP test.

: New better than used at specified age, life distributions, goodness of fit, hypothesis , asymptotic normality.

I. INTRODUCTION

A life is represented by a nonnegative random variable X with distribution functi and survival function $\bar{F} = 1 - F$. This variable is said to be new better than used (NBU) i $\bar{F}(x+t) \le \bar{F}(x)\bar{F}(t)$, for all $x, t_0 \ge 0$. Hollander, Park and Proschan (1986) introduced a l class than that of the new better than used lives. The distribution F is said to be new better used of a specified age t_0 (NBU. t_0) if $\bar{F}(x+t_0) \le \bar{F}(x)\bar{F}(t_0)$ for all $x \ge 0$ and a fixed t_0. that the border class where

$\bar{F}(x+t_0) = \bar{F}(x)\bar{F}(t_0)$ includes only the following members:

i) All exponential distributions.
ii) All life distributions defined on $[0, t_0]$.
iii) All life distributions defined freely on $[0, t_0]$ and defined as

$$\bar{F}^j(t_0)\bar{F}(x - jt_0) \text{ for all } jt_0 \le x < (j+1)t_0, j \ge 1.$$

We shall call this border class $\wp$. Note that the class $\wp$ is a neighborhood of the exponentials.

Basic properties of the above class were discussed by HPP (1986), where they also present a test procedure for testing $H_0 : F \in \wp$ against $H_1 : F$ is NBU. t_0 where t_0 is a known value. This test was extended to the case when t_0 is unknown by Ahmad (1998). It was also extended to a class of tests by Ebrahimi and Habbibullah (1990). In contrast to goodness of fit problems, where the test statistic is based on a measure of departure from H_0 that depends on both H_0 and H_1, most tests in lifetesting settings did not use the null distribution in devising the test statistics. This resulted in test statistics that are often difficult to work with and require programming to calculate.

Ahmad and Mugdadi (2001), applied this methodology for testing the increasing failure rate, new better than used and new better than used in convex ordering, and obtained an equivalent, but simpler, test statistics to those already available in the literature for the latter classes of life distributions.

In the current work, we use similar methodology to obtain a very simple statistic for testing H_0 against H_1 . In section 2 we present a test statistic based on the goodness of fit approach, which is simpler to compute and has equal asymptotic Pitman relative efficiency f several alternatives, Including three alternatives given by Hollander et al. (1986) . In section 3, two example from the literature in the medical science, are presented as an application of the proposed test on a real life data.

2. TESTING AGAINST NBU_t0 ALTERNATIVES

We consider the problem of testing

$$H_0 : F \in \wp \quad (2.1)$$

against

$$H_1 : F \text{ is NBU. } t_0 \quad (2.2)$$

The class $\wp$ is defined in section1. On the basis of a random sample $X_1, X_2, \cdots, X_n$ from a continuous distribution F, HPP (1986) based their test statistic for testing H_0 against H_1 on the following parameter:

$$v(F) = \int_0^\infty \{\overline{F}(x+t_0) - \overline{F}(x)\overline{F}(t_0)\} \, dF(x) \quad (2.3)$$

Now, If we denote by F_0 the null distribution, we can take in place of (2.3),

$$\delta(F) = \int_0^\infty \{\overline{F}(x+t_0) - \overline{F}(x)\overline{F}(t_0)\} \, dF_0(x). \quad (2.4)$$

Since $\delta(F)$ is a scale invariant, we can take $F_0(x) = 1 - e^{-x}, x \ge 0$. The following lemma is essential for the development of our test statistic.

<u>LEMMA 2.1</u>. <u>Let</u> X <u>be a random variable with distribution</u> F. <u>Then</u>

$$\delta(F) = F(t_0)\, Ee^{-X} - E\left(e^{-(X-t_0)}\, I(X > t_0)\right) \quad (2.5)$$

<u>PROOF</u>: Note that $\delta(F)$ can be written as

$$\delta(F) = E\int_0^\infty \overline{F}(t_0)\, I(X > x) e^{-x} \, dx - E\int_0^\infty I(X > x + t_0) e^{-x} \, dx$$

$$= \overline{F}(t_0) E\int_0^X e^{-x} \, dx - E\int_{t_0}^\infty I(X > u) e^{-(u-t_0)} \, du$$

$$= \overline{F}(t_0) E(1 - e^{-x}) - E\int_{t_0}^X I(X > t_0) e^{-(u-t_0)} \, du$$

$$= \overline{F}(t_0) E(1 - e^{-X}) - E\left(I(X > t_0)(1 - e^{-(X-t_0)})\right)$$

The result now follows by simple algebra.

Based on a random sample $X_1, \ldots, X_n$ from a distribution F. We wish to test H_0 against H_1. We note that, under $H_0, \delta(F) = 0$, while it is negative under H_1. Thus, we may be testing on its estimate. A direct empirical estimate of $\delta(F)$ is:

$$\hat{\delta}(F) = \frac{2}{n(n-1)} \sum_{i<j}^{n} \sum^{n} \left\{ e^{-X_i} I(X_j > t_0) - e^{-(X_i - t_0)} I(X_i > t_0) \right\} \quad (2.6)$$

$$= \frac{2}{n(n-1)} \sum_{i<j}^{n} \sum^{n} \varphi_{t_0}(X_i, X_j), \text{say.}$$

We now state and prove the following theorem.

THEOREM 2.1. As $n \to \infty$, $\sqrt{n}\left(\hat{\delta}(F) - \delta(F)\right)$ is as m toticall normal with me variance σ^2 where σ^2 is given in (2.9). Under H_0, σ_0^2 is given in (2.10).

PROOF. Using standard U-statistics theory, cf. Lee (1989), we need only to evaluate the asymptotic variance, which is equal to

$$\sigma^2 = V\{E[\varphi(X_1, X_2) \mid X_1] + E[\varphi(X_2, X_1) \mid X_1]\}.$$

Now,

$$E[\varphi(X_1, X_2) \mid X_1] = e^{-X_1}\overline{F}(t_0) - e^{-(X_1 - t_0)} I(X_1 > t_0)$$

and,

$$E[\varphi(X_2, X_1) \mid X_1] = I(X_2 > t_0)\ Ee^{-X_1} - E\left[e^{-(X_1 - t_0)} I(X_1 > t_0)\right]$$

Hence,

$$\sigma^2 = V\left[e^{-X_1}\overline{F}(t_0) - e^{-(X_1 - t_0)} I(X_1 > t_0) + I(X_1 > t_0)\ Ee^{-X_1} - E\left(e^{-(X_1 - t_0)} I(X_1 > t_0)\right)\right] \quad (2.$$

Under H_0,

$$\sigma_0^2 = \left(\frac{1}{12}\right)\overline{F}(t_0) + \left(\frac{1}{12}\right)\overline{F}^2(t_0) - \left(\frac{1}{6}\right)\overline{F}^3(t_0) \quad (2$$

Note that this is exactly the same result obtained by HPP (1986). To perform the above test, calculate $\sqrt{\frac{n}{\sigma_0^2}}\hat{\delta}$ and reject H_0, if this value exceeds $-Z_\alpha$ the lower α – percentile of the standard normal variate.

Next, we show that $\hat{\delta}_F$ has the same asymptotic Pitman's efficacy as that of the HPP (1986). However the calculations here are a lot easier. The asymptotic Pitman efficacy of a statistic T_n is defined by: $eff(T) = \left|\frac{\partial}{\partial\theta} ET_n\right|_{\theta \to \theta_0} \Big/ \sigma_0$. But $E\hat{\delta}(F) = \delta(F)$. Hence, we g

$$AE(\hat{\delta}(F)) = \left|\frac{\partial}{\partial\theta}\delta(F_\theta)\right|_{\theta \to \theta_0} \Big/ \sigma_0 \quad (2.11)$$

The following three families of alternatives are used by HPP (1986) for efficiency comparis

(i) The Linear Failure Rate Distribution:

$$\overline{F}_\theta(x)=\exp\{-x-\theta x^2/2\},\quad x\geq 0,\theta\geq 0.$$

(ii) The Makeham Distribution:

$$\overline{F}_\theta(x)=\exp\{-x-\theta[x+e^{-x}-1]\},\quad x\geq 0,\theta\geq 0.$$

(iii) The c^* Distribution:

$$\overline{F}_\theta(x)=\begin{cases}\exp\{-x+\theta(2t_0)^{-1}x^2\}, & 0\leq x<t_0, 0\leq\theta\leq 1\\ \exp\{-x+\theta(2)^{-1}t_0\}, & x\geq t_0, 0\leq\theta\leq 1\end{cases}.$$

The null exponential is attained at $\theta=0$.

Carrying out the efficacy calculations for the above three alternatives, we get, $-4^{-1}t_0\exp(-t_0)$, $6^{-1}\exp(-t_0)[-1+\exp(-t_0)]$ and $[2t_0\exp(-3t_0)-\exp(-t_0)+\exp(-3t_0)]/(8t_0)$ respectively, which are exactly the values of HPP (1986) test but are much simpler to calculate.

It is also easy to see that our proposed test is consistent and unbiased. For samples 5(1)25(5)50 and using 10000 replications, the upper %90, %95 and %99 percentiles of the statistic $\hat{\delta}$ for t_0=0.25 are given in table 2.1, and its power for samples 5,10, 15 and 20 for the linear failure rate and makeham alternatives, for %95 percentile and for t_0=0.25 and t_0=0.5 are given in tables 2.2 and 2.3 respectively.

TABLE 2.1 : Critical values of $\hat{\delta}$: t_0 =0.25

n	%90	%95	%99
5	-0.08354	-0.07557	-0.05055
6	-0.08595	-0.07735	-0.06219
7	-0.08779	-0.08167	-0.06570
8	-0.08944	-0.08327	-0.06952
9	-0.09078	-0.08514	-0.07282
10	-0.09166	-0.08634	-0.07589
11	-0.09247	-0.08735	-0.07673
12	-0.09335	-0.08852	-0.07980
13	-0.09402	-0.08955	-0.08053
14	-0.09478	-0.09021	-0.08196
15	-0.09512	-0.09093	-0.08327
16	-0.09572	-0.09155	-0.08410
17	-0.09618	-0.09233	-0.08499

TABLE 2.1 : continued

n	%90	%95	%99
18	-0.09659	-0.09267	-0.08566
19	-0.09701	-0.09329	-0.08623
20	-0.09725	-0.09377	-0.08745
21	-0.09761	-0.09420	-0.08804
22	-0.09798	-0.09450	-0.08807
23	-0.09816	-0.09492	-0.08902
24	-0.09842	-0.09506	-0.08891
25	-0.09880	-0.09555	-0.08965
30	-0.09961	-0.09714	-0.09148
35	-0.10064	-0.09791	-0.09284
40	-0.10128	-0.09871	-0.09399
45	-0.10152	-0.09918	-0.09492
50	-0.10368	-0.01015	-0.09310

TABLE 2.2: Power estimates of $\hat{\delta}$: t_0 =0.25

Distribution	Parameter	Sample size			
	θ	5	10	15	20
Linear failure Rate	2	0.9489	0.9607	0.9741	0.9815
	3	0.9620	0.9677	0.9818	0.9880
	4	0.9704	0.9727	0.9858	0.9913
Makeham	2	0.9520	0.9606	0.9754	0.9818
	3	0.9662	0.9668	0.9803	0.9876
	4	0.9608	0.9732	0.9831	0.9893

TABLE 2.3: Power estimates of $\hat{\delta}$: t_0 =0. 5

Distribution	Parameter	Sample size			
	θ	5	10	15	20
Linear failure Rate	2	0.4687	0.5315	0.6035	0.9999
	3	0.6100	0.6501	0.6908	1.0000
	4	0.7438	0.7634	0.7754	1.0000
Makeham	2	0.5973	0.6566	0.7973	0.9999
	3	0.6646	0.6970	0.7313	1.0000
	4	0.7720	0.7879	0.8002	1.0000

Thus we have shown that, based on Monte Carlo methods, the test statistic $\hat{\delta}$ has not only simplicity advantages over earlier ones but also has very good power.

3. APPLICATION

In this section two data sets from the medical field are analyzed. We show that the proposed test gives consistent conclusions about the distribution of the data with previous obtained results.

3.1 EXAMPLE 1

The first data set is from Bryson and Siddiqui (1969) , which represent the survival times of 43 patients suffering from chronic granulocytic leukemia. To apply the test we choose the same value for $t_0 = 1{,}825$ ($\approx$ 5 years) as chosen by HPP. We obtain $\hat{\delta} = -0.007$, $\hat{\sigma}_0^2 = 0.0151$, and

$$\sqrt{\frac{43}{\hat{\sigma}_0^2}}\hat{\delta} = -3.74,$$ with a corresponding one sided P value of 0.0001. Thus the NBU-1825 test strongly suggests that a newly diagnosed patient has stochastically greater residual life than does a patient after five years, in agreement with the conclusion of Bryson and Siddiqui (1969) and others.

3.2 EXAMPLE 2

The second data set consist of the survival times of 40 patients suffering from blood cancer (Leukemia) from one of the hospitals in Saudi Arabia, Hendi and Abouammoh (2001). The ordered life times (in days) are:
115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1165, 1191, 1222, 1122, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578 ,1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852.

To apply the test we choose $t_0 = 1{,}137$ ($\approx$ 3 years). We obtain $\hat{\delta} = -0.033$, $\hat{\sigma}_0^2 = 0.0439$, and

$$\sqrt{\frac{40}{\hat{\sigma}_0^2}}\hat{\delta} = -0.99,$$ with a corresponding one sided P value of 0.16. Hence H_0 is not rejected , agreeing with the conclusion of Hendi and Abouammoh (2001).

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