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BAYESIAN INTERIM ANALYSIS REGARDING A FUTURE OBSERVATION FROM TWO-PARAMETER PARETO AND EXPONENTIAL DISTRIBUTIONS

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ABSTRACT

Based on an interim type II censored sample of size n, a Bayesian predictive approach is adopted to obtain the predictive probability that if testing is continued for a further type II censored sample of size m, a decision will be reached regarding a future observation from the two parameter Pareto distribution. Since the Pareto distribution is transformation equivalent to the exponential distribution, the results can also be applied to the latter distribution.

1. INTRODUCTION

Geisser (1992) addressed the problem of curtailment or continuation of an experiment or trial at some interim point where n observations are at hand and at least $c > n$ observations had originally been scheduled for a decision . A Bayesian predictive approach was used to determine the probability that if the trial was continued with a further sample of size m where $n + m \geq c$, a particular decision would be reached regarding a parameter. He also considered the same problem when it is required to reach a decision about a future observation. This problem is important when the experimental procedures are costly or time-consuming . In such a case if it is assumed that a fixed-size experiment requires say c trials for concluding the effectiveness of a new treatment, drug or therapy, it will be of great interest for the investigator to know after observing same unplanned interim sample size n whether or not to continue the experiment until its prescribed minimal size c or beyond. This minimum sample size may be required to minimize a 'preposterior' measure of loss [see Martz and Waller (1982)] or alternatively to maximize a certain measure of information.

Geisser (1992) discussed this problem for a number of random sampling distributions including the one-parameter exponential distributions with all values fully observed. Geisser (1993) dealt with the same distribution when the observations are subject to censoring and loss to follow up. Papandonatos and Geisser (1999) considered the problem of Bayesian interim analysis of lifetime data that are independently distributed according to an accelerated-failure time model. A possible solution was suggested based on Laplace approximation to the posterior distribution of the parameters of interest and on Markov-Chain Monte Carlo. Both Geisser (1993) and Papandonatos and Geisser (1999) considered hypotheses regarding parameter (s) of the assumed lifetime distribution .

In this article, Bayesian interim analysis of censored observations from the two-parameter Pareto distribution is considered. Since the Pareto distribution is transformation equivalent to the exponential distribution, the results can also be applied to the latter distribution.

In section 2, some results relating to Bayesian analysis about the Pareto distribution are reviewed. In section 3, the following problem is considered. After observing a type II censored interim sample of size n, it is required to calculate the predictive probability of accepting a hypothesis regarding a future observation Z from the two parameter Pareto distribution if tesing is continued for a further type II censored sample of size m. In section 4 application of the results to the two-parameter exponential distribution is discussed. Finally numerical examples are provided in section 5 to illustrate the theoretical results.

2. PARETO POSTERIOR AND PREDICTIVE DENSITIES

Consider data from a two-parameter Pareto type I distribution with shape parameter α and scale parameter σ and with probability density function,

$$f(x;\alpha,\sigma)=\alpha\,\sigma^{\alpha}\,x^{-(\alpha+1)} \quad x>\sigma, \tag{2.1}$$

denoted by $PI(\alpha,\sigma)$ where $\alpha>0,\ 0<\sigma<L_o$: It is assumed that α and σ are random variables . For other types of Pareto distributions , see for example Johnson , Kotz and Balakrishnan (1994).

The two-parameter Pareto distribution has been subjected to a Bayesian analysis regarding the Pareto parameters by some authors e.g. Lwin (1972), Arnold and Press [(1983),(1989)]. Problems related to Bayesian prediction of future observations from the Pareto distribution have been studied by Geisser ((1984), (1985)), Nigm and Hamdy (1987), Arnold and Press (1989) and Dunsmore and Amin (1997). For recent articles on prediction , refer to AL-Hussaini [(1999),(2001)].

If both α and σ are unknown, Arnold and Press (1983) suggested for the $PI(\alpha,\sigma)$ distribution a power –gamma prior , denoted by $PG(g,a,\mu,L_o)$, by assuming that α has a gamma distribution , while the conditional distribution of σ given α is of the power function form, see Arnold (1983) . The density function of
(α,σ) is then given by:

$$\pi(\alpha,\sigma)\propto \quad \alpha^{g}\,\mu^{-\alpha}\sigma^{a\alpha-1}, \quad \alpha>0,\ 0<\sigma<L_0\ , \tag{2.2}$$

where a,μ,L_0 are positive constants, $g>-1$ and $L_0^a<\mu$.

For a vague prior we have $g=-1,\ \mu=1,\ a=0$ *and* $L_0\to\infty$. Consider the case of type II censoring where only the first r ordered observations $X_{(1)}$, $X_{(2)},\dots,X_{(r)}$ $(r\le n)$ in a sample of n items are measured in an experiment in which all items are tested simultaneously. In this case the results of the informative experiment can be summarized by the jointly sufficient statistics

$$X_{(1)} \text{ and } Q=\sum_{i=1}^{r}\ln X_{(i)}+(n-r)\ln X_{(r)}-n\ln X_{(1)}.$$

Based on the results of Epstein and Sobel (1954), $X_{(1)}$ and Q are independent with conditional joint probability density function given by

$$f(x_{(1)},q \mid \alpha,\sigma)=\frac{n\alpha^{r}}{\Gamma(r-1)}q^{r-2}\exp(-\alpha q)\sigma^{n\alpha}x_{(1)}^{-(n\alpha+1)} \qquad q>0, x_{(1)}>\sigma, \tag{2.3}$$

where q is a realization of Q.

The posterior distribution of α and σ is a Power – Gamma density of the form

$$\pi^{*}(\alpha,\sigma \mid x_{(1)},q)=PG(r+g, n+a, \mu\, x_{(r)}^{n-r}\prod_{i=1}^{r} x_{(i)}, w) \quad , \tag{2.4}$$

where $w = \min(L_0, x_{(1)})$.

Based on the first r ordered observations of the informative experiment, consider predicting a future observation, Z, from the Pareto distribution. From Dunsmore and Amin (1997), the predictive density function of Z is given by

$$f(z \mid x_{(1)},q)=\frac{(a+n)(g+r)}{(a+n+1)zL}\begin{cases}\left[1-\left(\frac{a+n}{L}\right)\ln\left(\frac{z}{w}\right)\right]^{-(g+r+1)} & ,z<w,\\ \left[1+\left(\frac{1}{L}\right)\ln\left(\frac{z}{w}\right)\right]^{-(g+r+1)} & ,z>w,\end{cases} \tag{2.5}$$

where

$$L=\sum_{i=1}^{r}\ln\ x_{(i)}+\ln\ \mu+(n-r)\ln\ x_{(r)}-(n+a)\ln w.$$

3. PREDICTIVE PROBABILITY OF ACCEPTING THE NULL HYPOTHESIS THAT A FUTURE OBSERVATION z IS AT LEAST z_0

The problem of interest is of testing a hypothesis regarding a future observation z, namely

$$H_0: z \geq z_0 \ \ versus\ \ H_1: z < z_0\ . \tag{3.1}$$

At the interim stage, consider a type II censored sample of size n from the two-parameter Pareto distribution where $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(d)}$ are fully observed while the remaining $(n-d)$ observations are censored. Assume also that a type II censored sample of m further observations from the same distribution are put on test until k items $X_{(n+1)} \leq X_{(n+2)} ... \leq X_{(n+k)}$ are fully observed while the remaining $(m-k)$ observations are censored. Suppose that at the end of the experiment we will decide for H_0 if

$$\Pr(Z > z_0 \mid x_{(1)}, s, x_{(n+1)}, u) \geq p\ , \tag{3.2}$$

where

$$s=\sum_{i=1}^{d}\ln\ x_{(i)}+(n-d)\ln\ x_{(d)}-n\ \ln\ x_{(1)},$$

and

$$u=\sum_{i=n+1}^{n+k}\ln\ x_{(i)}+(m-k)\ln\ x_{(n+k)}-m\ \ln\ x_{(n+1)}.$$

Considering the prior distribution given by (2.2), it follows from (2.5) that the predictive density function of Z is given by

$$f(z \mid x_{(1)}, s, x_{(n+1)}, u) = \frac{A\,G}{(A+1)zH} \begin{cases} \left[1-\left(\frac{A}{H}\right)\ln\left(\frac{z}{w_1}\right)\right]^{-(G+1)} & , z < w_1 , \\ \left[1+\left(\frac{1}{H}\right)\ln\left(\frac{z}{w_1}\right)\right]^{-(G+1)} & , z > w_1 , \end{cases} \quad (3.3)$$

where

$A = a + n + m,\quad G = g + d + k,\quad w_1 = \min\{L_0, x_{(1)}, x_{(n+1)}\}$

$H = s + n \ln x_{(1)} + u + m \ln x_{(n+1)} + \ln \mu - A \ln w_1.$

Equation (3.3) gives the predictive density function of a future observation from the Pareto distribution based on the results of the first $d+k$ ordered observations of an informative experiment of size $n+m$ consisting of two type II censored samples .These results can be summarized by the statistics $X_{(1)}, X_{(n+1)}$, S and U , where S and U are defined by equation (3.2) . It thus follows that the predictive density function given by (3.3) has the same form as the predictive density function given by (2.5) .The quantities n, r, w, L appearing in (2.5) are replaced by $n+m$, $d+k$, w_1 and H respectively in equation (3.3).

From (3.3), it follows that :

$$\Pr(Z > z_0 \mid x_{(1)}, s, x_{(n+1)}, u) = \begin{cases} 1-\left(\frac{1}{A+1}\right)\left[1+\left(\frac{A}{H}\right)\ln\left(\frac{w_1}{z_0}\right)\right]^{-G} & , z_0 < w_1 , \\ \left(\frac{A}{A+1}\right)\left[1+\frac{1}{H}\ln\left(\frac{z_0}{w_1}\right)\right]^{-G} & , z_0 > w_1 \end{cases} \quad (3.4)$$

Now consider the probability defined by (3.2) .

For the case $z_0 > w_1$, it follows from equation (3.4) that (3.2) is satisfied if :

$$\left[1+\frac{1}{H}\ln\left(\frac{z_0}{w_1}\right)\right]^{-G} \geq \frac{(A+1)p}{A} ,$$

which can be written as

$$\frac{1}{H}\ln\left(\frac{z_0}{w_1}\right) \leq \left[\left[\frac{A}{(A+1)p}\right]^{\frac{1}{G}} - 1\right]$$

Noting that the left hand side of the above inequality is always positive , it follows that :

For the case $z_0 > w_1$,

if $\left[\frac{A}{(A+1)p}\right]^{\frac{1}{\sigma}} - 1 \le 0$ *or* $p \ge \frac{A}{(A+1)}$, (3.2) will not be satisfied

If $p < \frac{A}{A+1}$, (3.2) is given by,

$$H\left[\left(\frac{A}{p(A+1)}\right)^{\frac{1}{\sigma}} - 1\right] \ge \ln\left(\frac{z_0}{w_1}\right). \tag{3.5}$$

Similarly it can be shown that for the case $z_0 < w_1$, if $p \le \frac{A}{A+1}$, (3.2) will always be satisfied whereas if $p > \frac{A}{A+1}$, (3.2) is given by

$$H\left[\left((1-p)(A+1)\right)^{-\frac{1}{\sigma}} - 1\right] \le A\left[\ln\left(\frac{w_1}{z_0}\right)\right]. \tag{3.6}$$

Now suppose that only an interim type II censored sample of size n is available. Based on this sample, it is required to calculate the predictive probability that if testing is continued for further m observations Z exceeds z_0 with a probability greater than or equal to p. This probability is given by

$$P^* = \Pr\{\Pr(Z > z_0 \mid x_{(1)}, s, x_{(n+1)}, u) \ge p\}. \tag{3.7}$$

The inner probability is defined by equation (3.2) whereas the outer probability is obtained using the joint predictive density function of $X_{(n+1)}$ and U.

From Dunsmore (1974), $X_{(n+1)}$ and U are independently distributed, the conditional joint probability density function of $V = \ln X_{(n+1)}$ and U is given by :

$$f(v,u \mid \alpha,\sigma) = \frac{m}{\Gamma(k-1)} \alpha^k u^{k-2} \exp\{-\alpha[u + m(v - \ln\sigma)]\}, \quad u > 0, v > \ln\sigma. \tag{3.8}$$

Integrating (3.8) with respect to the posterior distribution of (α,σ) which is given by (2.4), with r being replaced by d, it follows that the joint predictive density function of (v,u) is given by

$$f(v,u \mid x_{(1)}, s) = \frac{m(a+n)}{\Gamma(k-1)\Gamma(g+d)} \frac{H_1^{g+d} u^{k-2} \Gamma(G) H^{-G}}{A}, u > 0, v > -\infty \tag{3.9}$$

where

$H_1 = \sum_{i=1}^{d} \ln x_{(i)} + \ln \mu + (n-d)\ln x_{(d)} - (a+n)\ln w$, H, G and A are defined in (3.3).

It is clear that the calculation of P^* given by (3.7) differs according to whether $z_0 < w_1$ or $z_0 > w_1$. At the interim stage, however, w_1 is unknown since $x_{(n+1)}$ has not yet been observed. Hence, to calculate P^*, two cases have to be considered : $z_0 > w$ and $z_0 < w$, where w is defined in equation (2.4). These cases will be investigated in the following two subsections.

3.1 The case $z_0 > w$

Since w_1 can be expressed as minimum $(w, x_{(n+1)})$, then $z_0 > w$ implies that $z_0 > w_1$.

For this case, when $p < \frac{A}{A+1}$,

$\Pr(Z > z_0 | x_{(1)}, s, x_{(n+1)}, u) \geq p$ is defined by equation (3.5). It is noted that :

If $\ln w < v$, then $w_1 = w$ and it follows from the definition of H given in equation (3.3) that (3.5) can be put in the form

$$u + mv \geq \frac{K_1}{K}, \tag{3.10}$$

where

$$K = \left[\frac{A}{p(A+1)}\right]^{\frac{1}{G}} - 1,$$

and

$$K_1 = -K\left[\ln \mu + \sum_{i=1}^{d} \ln x_{(i)} + (n-d)\ln x_{(d)} - (a+n+m)\ln w\right] + \ln\left(\frac{z_0}{w}\right).$$

If $\ln w > v$, then $w_1 = x_{(n+1)}$ and (3.5) is expressed as

$$u - v\left[a + n - \frac{1}{K}\right] \geq \frac{\ln z_0 - K_2}{K}, \tag{3.11}$$

where

$$K_2 = K\left[\ln \mu + \sum_{i=1}^{d} \ln x_{(i)} + (n-d)\ln x_{(d)}\right].$$

For $z_0 > w$, the following cases have to be considered :

(i) If $p \geq \frac{A}{A+1}$, then $P^* = 0$.

This follows from noting that (3.2) is not satisfied in this case as explained above.

(ii) If $p < \frac{A}{A+1}$, $(a + n - \frac{1}{K}) > 0$ and

$$-\left[\ln \mu + \sum_{i=1}^{d} \ln x_{(i)} + (n-d)\ln x_{(d)}\right] + (a+n)\ln w + \frac{1}{K}\ln\left(\frac{z_0}{w}\right) \leq 0,$$

then $P^* = 1$.

This can be shown as follows. If condition (ii) is satisfied, the following inequality is obtained for values of $v < \ln w$.

$$-\ln(w)\left[a + n - \frac{1}{K}\right] \geq \frac{\ln z_0 - K_2}{K}.$$

But since $-\ln(w)\left[a + n - \frac{1}{K}\right]$ is a lower bound for the left hand side of (3.11), it follows that (3.11) will always be satisfied. Applying a similar argument, it can be shown that for values of $v > \ln w$, equation (3.10) will also be always satisfied under condition (ii).

(iii) If $p < \frac{A}{A+1}, (a+n-\frac{1}{K}) > 0$

$$-\left[\ln\mu + \sum_{i=1}^{d} \ln x_{(i)} + (n-d)\ln x_{(d)}\right] + (a+n)\ln w + \frac{1}{K}\ln\left(\frac{z_0}{w}\right) > 0,$$

then

$$P^* = \int\int_{(v,u)\in R_1} f(v,u|x_{(1)},s)\,dv\,du + \int\int_{(v,u)\in R_2} f(v,u|x_{(1)},s)\,dv\,du\,,$$

where R_1 is the region of ($u > 0$, $v > \ln w$) values satisfying (3.10) and R_2 is the region of ($u > 0$, $v < \ln w$) values satisfying (3.11).

It can be shown that

$$P^* = P_1^* + P_2^* + P_3^* \,. \tag{3.12}$$

where

$$P_1^* = 1 - \frac{\beta_{C_1}(k-1, g+d)}{\beta(k-1, g+d)}$$

$$P_2^* = \frac{(a+n)}{(a+n+m)}\left[C_1^{k-1}\left(\frac{1}{K}\ln\left(\frac{z_0}{w}\right)\right)^{-(g+d)} \frac{H_1^{g+d}}{\beta(k,g+d)(g+d+k-1)}\right],$$

$$P_3^* = \frac{m}{(a+n+m)} \frac{H_1^{g+d}}{\beta(k-1,g+d)}[K_4[K(a+n)-1]]^{k-1}$$
$$\times\left[\sum_{i=0}^{k-2} \frac{(-1)^i\binom{k-2}{i}}{(G-i-2)}\left\{\left(1-\frac{K_3}{K_4[K(a+n)-1]}\right)^{-G+i+2}-1\right\}\right],$$

$$C_1 = \left(\frac{1}{K}\ln\left(\frac{z_0}{w}\right) - H_1\right) \Big/ \left(\frac{1}{K}\ln\left(\frac{z_0}{w}\right)\right),$$

$$K_3 = [\ln z_0 - K_2 + (\ln(w))(K(a+n)-1)]/K\,,$$

$$K_4 = \frac{-\left[\ln\mu + \sum_{i=1}^{d}\ln x_{(i)} + (n-d)\ln x_{(d)}\right] + (a+n)\ln z_0}{[K(a+n)-1]},$$

$$\beta_x(a_1,a_2)=\int_0^x y^{a_1-1}(1-y)^{a_2-1}dy,\quad a_1>0\ ,\ a_2>0\ ,\ 0<x<1.$$

(iv) If $p<\frac{A}{A+1}, (a+n-\frac{1}{K})<0$ *and*

$$-\left[\ln\mu+\sum_{i=1}^{d}\ln x_{(i)}+(n-d)\ln x_{(d)}\right]+(a+n)\ln w+\frac{1}{K}\ln\left(\frac{z_0}{w}\right)<0,$$

then

$$P^*=1-\frac{m}{a+n+m}\left(\frac{H_1}{H_2}\right)^{g+d}\left[1-(a+n)K\right]^{g+d+k-1}, \tag{3.13}$$

where

$$H_2=\ln\mu+\sum_{i=1}^{d}\ln x_{(i)}+(n-d)\ln x_{(d)}-(a+n)\ln\ z_0 .$$

(v) If $p<\frac{A}{A+1}, (a+n-\frac{1}{K})<0$ *and*

$$(a+n)\ln\ z_0<\ln\mu+\sum_{i=1}^{d}\ln x_{(i)}+(n-d)\ln x_{(d)}<\frac{1}{K}\ln\left(\frac{z_0}{w}\right)+(a+n)\ln w ,$$

then

$$P^*=P_1^*+P_2^*+P_4^* , \tag{3.14}$$

where P_1^* and P_2^* are defined in (3.12)

$$P_4^*=-\left\{\left(\frac{H_1}{H_2}\right)^{g+d}\frac{m}{(a+n+m)}\left[1-(a+n)K\right]^{g+d+k-1}\right.$$
$$\left.\left[1-\frac{\beta_{C2}(k-1,g+d)}{\beta(k-1,g+d)}\right]\right\} ,$$

$$C_2=\left[\frac{1}{K}\ln\left(\frac{z_0}{w}\right)-H_1\right]\Big/\left[\ln\left(\frac{z_0}{w}\right)\left(\frac{1}{K}-(a+n)\right)\right]$$

Theorem: A theoretical limit for P^* given by (3.14) when both m and k tend to infinity with their ratio kept constant is given by:

$$\lim_{\substack{m\to\infty\\k\to\infty}} P^{*} = 1 - D\left[\frac{g+d}{H_1}\,\frac{\ln\left(\frac{z_0}{w}\right)}{(-\ln p)}\right] - \left(\frac{H_1}{H_2}\right)^{g+d} p^{a+n}$$

$$\cdot\left\{1 - D\left[\frac{(g+d)\ln\left(\frac{z_0}{w}\right)}{\left[H_1-(a+n)\ln\left(\frac{z_0}{w}\right)\right](-\ln p)}\right]\right\}, \tag{3.15}$$

where D(.) is the distribution function of an F variate with $2(k\text{-}1)$ and $2(g+d)$ degrees of freedom.

Proof

By using the relation between the incomplete beta function and the distribution function of an F variate , it can be noted that

$$\frac{\beta_{c_1}(k-1,g+d)}{\beta(k-1,g+d)} = D\left[\frac{(g+d)}{H_1}\,\frac{\left(\frac{1}{K}\ln\left(\frac{z_0}{w}\right)-H_1\right)}{(k-1)}\right].$$

Using the formula $a^x = \exp(x \ln a)$ in expressing K defined in equation (3.1) and applying the Maclaurin's formula for expanding the function $\exp(x)$, it can be shown that

$$\lim_{\substack{m\to\infty\\k\to\infty}} \frac{g+d}{H_1}\left[\frac{\frac{1}{K}\ln\left(\frac{z_0}{w}\right)-H_1}{(k-1)}\right] = \frac{(g+d)\ln\left(\frac{z_0}{w}\right)}{H_1(-\ln p)}.$$

Hence

$$\lim_{m\to\infty} P_1^{*} = 1 - D\left[\frac{g+d}{H_1}\;\frac{\ln(z_o/w)}{(-\ln p)}\right]. \tag{3.16}$$

Similarly it can be shown that if $\left(\frac{k}{m}\right)$ is constant, then

$$\lim_{m\to\infty}\left[1-\frac{\beta_{C2}(k-1,g+d)}{\beta(k-1,g+d)}\right] = 1 - D\left[\frac{(g+d)\ln\left(\frac{z_0}{w}\right)}{\left(H_1-(a+n)\ln\left(\frac{z_0}{w}\right)\right)(-\ln p)}\right].$$

Noting that

$$\lim_{m\to\infty} \frac{m}{a+n+m}[1-(a+n)K]^{g+d+k-1} \doteq p^{a+n},$$

it follows that

$$\lim_{m\to\infty} P_4^* = -\left(\frac{H_1}{H_2}\right)^{g+d} p^{a+n}\left\{1 - D\left[\frac{(g+d)\ln\left(\frac{z_0}{w}\right)}{\left(H_1-(a+n)\ln\left(\frac{z_0}{w}\right)\right)(-\ln p)}\right]\right\}. \tag{3.17}$$

It can be shown that

$$\lim_{m\to\infty} C_1^{k-1} = \exp[-H_1 \ln(1/p)/\ln(z_0/w)],$$

and that

$$\lim_{m\to\infty} \frac{(a+n)H_1^{g+d}}{(a+n+m)\ \beta(k,g+d)(g+d+k-1)\left(\frac{1}{K}\ln\left(\frac{z_0}{w}\right)\right)^{g+d}}$$

$$= \lim_{m\to\infty} \frac{(a+n)H_1^{g+d}K^{g+d}\Gamma(g+d+k-1)}{(a+n+m)\Gamma(g+d)\ \Gamma(k)\ln\left(\frac{z_0}{w}\right)^{g+d}} = 0.$$

Hence

$$\lim_{m\to\infty} P_2^* = 0. \tag{3.18}$$

Using (3.16) , (3.17) and (3.18), the theoretical limit given in (3.15) is established.

(vi) If $p<\frac{A}{A+1}$; $(a+n-\frac{1}{K})<0$ *and*

$$\left[\ln\mu+\sum_{i=1}^{d} \ln x_{(i)} +(n-d)\ln x_{(k)}\right] < [(a+n)\ln z_0],$$

then

$$P^*=P_1^*+P_2^*+P_5^*, \tag{3.19}$$

where $P_5^* = -\frac{m(H_1)^{g+d}[1-(a+h)k]^{g+d+k-1}}{\beta(k-1,g+d)(a+n+m)}\sum_{i=0}^{k-2} \frac{\binom{k-2}{i}[-H_2]^{k-2-i}}{(G-i-2)\left(\ln\left(\frac{z_0}{w}\right)\right)^{G-i-2}\left[\frac{1}{K}-(a+n)\right]^{G-i-2}}.$

Note that the condition $(a+n-\frac{1}{K})<0$ implies that

$$(a+n)\ln z_0 < \left[\frac{\ln\left(\frac{z_0}{w}\right)}{K} + (a+n)\ln w\right]$$

The derivations of equations (3.12) , (3.13) , (3.14) and (3.19) are given in the Appendix.

3.2 The case $z_0 < w$

It is noted that for the case $z_0 < w$:

$z_0 > w_1$ for values of u and v satisfying $-\infty < v < \ln z_0$, $u > 0$.

$z_0 < w_1$ for values of u and v satisfying $\ln z_0 < v < \ln w$, $u > 0$ *and* $v > \ln w, u > 0$.

It can be shown that :

(i) If $p \le \frac{A}{A-1}$ *and* $\left(a+n-\frac{1}{k}\right) > 0$, *then*

$$P^* = 1. \tag{3.20}$$

(ii) If $p \le \frac{A}{A-1}$ *and* $\left(a+n-\frac{1}{k}\right) < 0$

then

$$P^* = 1 - \frac{m}{(a+n+m)}\left(\frac{H_1}{H_2}\right)^{g+d}\left[1-K(a+n)\right]^{g+d+k-1}. \tag{3.21}$$

(iii) If $p > \frac{A}{A-1}$ *and*

$$-\left[\ln\mu + \sum_{i=1}^{d}\ln x_{(i)} + (n-d)\ln x_{(d)}\right] + (a+n)\ln w + \frac{(a+n+m)\ln\left(\frac{w}{z_0}\right)}{K_5} \le 0,$$

then

$$P^* = 0 \tag{3.22}$$

where

$$K_5 = \left[(1-p)(1+A)\right]^{-\frac{1}{g+k+d}} - 1.$$

(iv) If $p > \frac{A}{A+1}$ *and*

$$-\left[\ln\mu + \sum_{i=1}^{d}\ln x_{(i)} + (n-d)\ln x_{(d)}\right] + (a+n)\ln w + \frac{(a+n+m)\ln\left(\frac{w}{z_0}\right)}{K_5} > 0,$$

then

$$P^* = P_6^* + P_7^* - P_8^* . \tag{3.23}$$

where

$$P_6^* = \frac{\beta_{C_3}(k-1, g+d)}{\beta(k-1, g+d)},$$

$$P_7^* = -\left\{\frac{m}{(a+n+m)^G}\left(\frac{H_1}{H_2}\right)^{g+d}\frac{\beta_{C_4}(k-1, g+d)}{\beta(k-1, g+d)}[(a+n)K_5+(a+n+m)]^{G-1}\right\},$$

$$P_8^* = \left\{\frac{(a+n)H_1^{g+d}}{(a+n+m)(g+d+k-1)B(k, g+d)}\right.$$

$$\left.\left[\frac{K_5}{(a+n+m)\ln\left(\frac{w}{z_0}\right)}\right]^{g+d+k-1}\left[\frac{(a+n+m)\ln\left(\frac{w}{z_0}\right)}{K_5}-H_1\right]\right\}$$

$$C_3 = \left(\frac{(a+n+m)\ln\left(\frac{w}{z_0}\right)}{K_5}-H_1\right)\Bigg/\frac{(a+n+m)\ln\left(\frac{w}{z_0}\right)}{K_5},$$

$$C_4 = \left(\frac{(a+n+m)\ln\left(\frac{w}{z_0}\right)}{K_5}-H_1\right)\Bigg/K_6\ \ln\left(\frac{w}{z_0}\right),$$

$$K_6 = (a+n)+\frac{(a+n+m)}{K_5}.$$

The derivations of equations (3.20) ,(3.21) , (3.22) and (3.23) are given in the appendix .

4. APPLICATION OF THE RESULTS TO THE TWO-PARAMETER EXPONENTIAL DISTRIBUTION

Making the transformation $T = \ln X$ and using the reparametrization $\delta = \ln\ \sigma$ in (2.1), we obtain the two-parameter exponential distribution with probability density function given by

$$f(t\mid\alpha,\delta) = \alpha\ exp[-\alpha(t-\delta)],\quad t \geq \delta, \alpha > 0. \tag{4.1}$$

The natural conjugate prior for (α,δ) is given by Dunsmore (1974) as

$$\pi(\alpha,\delta) = \frac{a\ h^g}{\Gamma(g)}\alpha^g\ exp\{-\alpha[h+a(b-\delta)]\},\quad \delta < b, \alpha > 0, \tag{4.2}$$

where $a > 0$, $h > 0$ and $g > -1$.

Note that (4.2) may be obtained from (2.2) by making the transformation $\delta = \ln\ \sigma$ and setting $h + ab = \ln\ \mu$ where $b = \ln L_0$.

Given a type II censored sample of size n where $t_{(1)} \le t_{(2)} \le \dots \le t_{(r)}$ denote the first r ordered observations, the posterior distribution of (α,δ) is given by

$$\pi^{*}(\alpha,\delta \mid t_{(1)},q) = \frac{A_1 L^{G_1}}{\Gamma(G_1)} \alpha^{G_1} \exp\{-\alpha[L + A_1(B-\delta)]\},\ \delta < B. \tag{4.3}$$

where

$$Q = \sum_{i=1}^{r} t_{(i)} + (n-r)t_{(r)} - nt_{(1)},$$ q is a realization of Q .

$$A_1 = a + n,$$

$$H + A_1 B = h + ab + \sum_{i=1}^{r} t_{(i)} + (n-r)t_{(r)},$$

and

$$B = \min\ (b, t_{(1)}) .$$

Consider a future observation from the two-parameter exponential distribution . It can be easily shown that if we apply the transformation $Y = \ln Z$ where $Z \sim PI(\alpha,\sigma)$, then $Y \sim$exponential $(\alpha,\ln\sigma)$ so that $Pr[Y \le t_0] = Pr(Z \le z_0)$ where $t_0 = \ln z_0$.

It follows that all of the results presented for the two parameter Pareto distribution may be applied to the two-parameter exponential distribution by setting

$$T_i = \ln X_i \qquad i = 1,2,\dots,n+m,$$

$$\delta = \ln\sigma,$$

$$b = \ln L_0,$$

$$B = \min\left[\ln L_0, \ln x_{(1)}\right] = \ln w,$$

$$Y = \ln Z,$$

$$t_0 = \ln z_0,$$

$$B_1 = \min\left[\ln L_0, \ln x_{(1)}, \ln x_{(n+1)}\right] = \ln w_1.$$

REMARK

It is to be noted that for the prior given by (4.2) there is no restriction for δ to be non negative. In life testing experiments, taking $\delta > 0$ is a natural constraint. The results presented in this article however , may still be used as good approximations for the corresponding results for the case where $\delta > 0$ if the probability that δ takes negative values is negligibly small. From (4.3), the posterior probability that δ takes negative values is given by $\left(\frac{L}{L + A_1 B}\right)^{G_1}$ which is likely is be small in life testing experiments (refer to Evans and Nigm (1980) for a discussion of this problem).

5. NUMERICAL EXAMPLES

Example (5.1).

Nigm and Hamdy (1987) used the Pareto distribution to model the survival times (in years) of new small businesses, they gave the following data which represent the operational times of the first 10 of a random sample of 15 businesses .

1.01 , 1.05 , 1.08 , 1.14 , 1.28 , 1.30 , 1.33 , 1.43 , 1.59 , 1.62.

A non-informative prior is assumed for the parameters .

It is noted that at the interim stage, the predictive probability that Z exceeds z_0 given by $\Pr(Z > z_0 | x_{(1)}, s)$ may be calculated for different values of z_0 as follows .

$\Pr(Z > 1.007 | x_{(1)}, s) = 0.943.$

$\Pr(Z > 1.02 | x_{(1)}, s) = 0.920.$

$\Pr(Z > 1.09 | x_{(1)}, s) = 0.809.$

$\Pr(Z > 1.18 | x_{(1)}, s) = 0.696.$

$\Pr(Z > 1.32 | x_{(1)}, s) = 0.565.$

Table (5.1) displays P^* for different values of z_0, p and k when m= 15.

Table (5.1) : P^* for different values of z_0, p and k when $m = 15$

p	z_0 \ k	5	7	10	15
0.5		0.999	0.999	0.999	0.999
0.6		0.999	0.999	0.999	0.999
0.7	1.007	0.998	0.998	0.998	0.998
0.8		0.977	0.976	0.976	0.975
0.9		0.840	0.841	0.842	0.843
0.5		0.999	0.999	0.999	0.999
0.6		0.999	0.999	0.999	0.999
0.7	1.02	0.998	0.997	0.997	0.996
0.8		0.966	0.996	0.965	0.964
0.9		0.767	0.768	0.769	0.770
0.5		0.999	0.999	0.999	0.999
0.6		0.999	0.999	0.999	0.998
0.7	1.09	0.977	0.974	0.970	0.966
0.8		0.635	0.629	0.627	0.626
0.9		0.002	0.003	0.005	0.008
0.5		0.999	0.999	0.999	0.999
0.6		0.981	0.969	0.953	0.935
0.7	1.18	0.397	0.426	0.449	0.469
0.8		0.014	0.022	0.03	0.041
0.9		0	0	0	0
0.5		0.857	0.823	0.800	0.783
0.6		0.210	0.247	0.247	0.309
0.7	1.32	0.015	0.023	0.032	0.044
0.8		0	0	0.001	0.001
0.9		0	0	0	0

Comments

From the numerical results it is observed that :

(1) The values of P^* are sensitive to variations in p and z_0 especially for larger values of either p or z_0.

(2) For values of p less than the predictive probability at the interim stage increasing k may decrease P^*. On the other hand for values of p greater than the predictive probability at the interim stage increasing k may increase P^*. This conclusion is similar to the one reached by Papandonatos and Geisser (1999) who explained this situation as follows. Any value of p less than the predictive probability at the interim stage would have resulted in acceptance of H_0 at the interim stage. Hence by introducing uncertainty in the form of extra observations, we risk altering our conclusion in favor of H_1. Similarly any p greater than the predictive probability at the interim stage would have made acceptance of H_0 impossible at this stage. In this case by increasing k a favorable outcome might then become possible at termination.

Based on the data given in example 5.1, figure (5.1) (page 17), shows P^* for different values of p and m for z_0 fixed at 1.09 when both k and m are increased such that their ratio is always kept constant at 0.5. The value of P^* when $m \to \infty$ is calculated using the theoretical limit given by equation (3.15).

(3) It is observed from figure (5.1) that for values of p given by 0.65, 0.7, 0.75 and 0.8 which are less than the value 0.809, predictive probability at the interim stage, the curve corresponding to $m = 20$ is the highest curve. Whereas for values of p given by 0.85 and 0.9 which are greater than 0.809, the curve corresponding to $m \to \infty$ is the highest curve. This comment agrees with comment (2).

Example (5.2)

In a study of new anticancer drugs in L1210 animal leukemia system, Johnson, Zelen and Kemp (1965) successfully used the two-parameter exponential distribution as the model for survival time. The system consists of injecting a tumor innoculum into imbred mice. These tumor cells then profilerate and eventually kill the animal, but survival time may be prolonged by an active drug (cyclophosphamide)

The following data represent the survival time of 19 mice treated with a dose of 320 mg per kg of cyclophosphamide on day 7 after receiving a tumor innoculum.

Table (5.2.1): Survival times of Mice

Time (t_i)	17	18	19	20	21	22	23	25	26
Number of mice dying	4	1	3	1	1	5	1	2	1

To illustrate the results presented in this paper, it will be assumed that the minimum sample size required is given by 34 and that the treatment will be considered of value if the survival time of a future observation exceeds 18 with probability $p = 0.70$. An interim analysis is conducted to see whether it is worthwhile to continue the trial with $m = 15$ additional rats for different values of k. Table (5.2.2) displays values of P^* corresponding to several values of k.

Table (5.2.2): P^* for different values of k
$t_0 = 18$, $p = 0.7$ and $m = 15$

K	P^*
2	0.8972
5	0.8801
10	0.8550
15	0.8350

Hence continuation of the trial is likely to achieve the required goal for all values of k considered.

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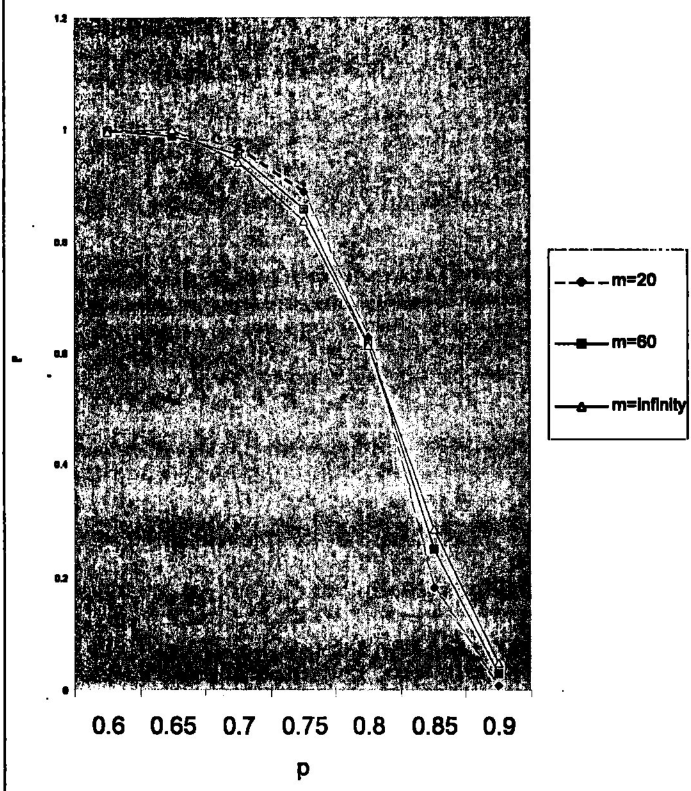
Figure 5.1:Values of P* for different values of p and m when k /m=0.5



APPENDIX

1 : Derivation of equation (3.12)

Suppose that condition (iii) defined in section 3.1 is satisfied. In this case P^* can be expressed as

$P^* = I_1 + I_2$,

where

$$I_1 = \int\int_{(v,u)\in R_1} f(v,u|x_{(1)},s)dvdu$$

and

$$I_2 = \int\int_{(v,u)\in R_2} f(v,u)|x_{(1)},s)dvdu$$

R_1 is the region of ($u > 0$, $v > \ln w$) values satisfying (3.10), R_2 is the region of ($u > 0$, $v < \ln w$) values satisfying (3.11) and $f(v,u \mid x_{(1)},s)$ is defined by equation (3.9). The regions R_1 and R_2 are shown in figures A.1 and A.2 respectively.

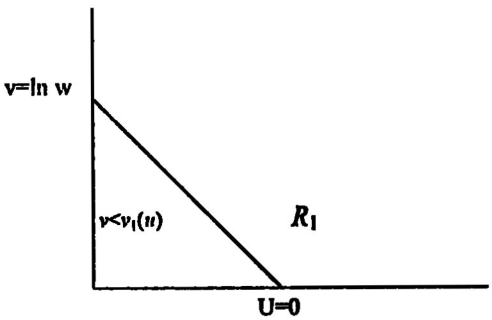


Figure A.1:Region R_1 satisfying $v \geq v_1(u)$

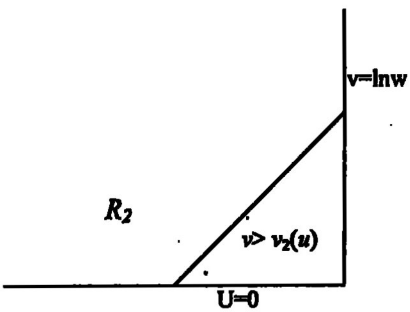


Figure A.2:Region R_2 satisfying $v \leq v_2(u)$

where

$$v_1(u) = \frac{1}{m}\left[\frac{K_1}{K} - u\right]$$

$$v_2(u) = \frac{(Ku + K_2 - \ln(z_0))}{K(a+n)-1}$$

From figure A.1, it can be noted that I_1 can be expressed as

$I_1 = I_{11} + I_{12}$,

where

$$I_{11} = \int_{u_1}^{\infty}\int_{\ln w}^{\infty} f(v,u|x_{(1)},s)dvdu \quad ,$$

and

$$I_{12} = \int_0^{u_1}\int_{v_1(u)}^{\infty} f(v,u|x_{(1)},s)dvdu \quad ,$$

$u_1 = \frac{K_1}{K} - m\ln(w)$.

It can be shown that

$$I_{11} = \frac{(a+n)}{(a+n+m)}\left[1 - \frac{\beta_{C1}(k-1, g+d)}{\beta(k-1, g+d)}\right] \qquad (A.1)$$

and I_{12} can be expressed as

$$I_{12} = P_2^*\ , \qquad (A.2)$$

where c_1 , $\beta_x(a_1,a_2)$ and P_2^* are defined in equation (3.12) .
From figure A.2 , I_2 can be written as

$$I_2 = I_{21} + I_{22}\ ,$$

where

$$I_{21} = \int_{u_0}^{\infty}\int_{-\infty}^{\ln w} f(v,u|x_{(1)},s)\,dv\,du \qquad (A.3)$$

$$I_{22} = \int_0^{u_2}\int_{-\infty}^{v_2(u)} f(v,u|x_{(1)},s)\,dv\,du \qquad (A.4)$$

and

$$u_2 = \frac{(\ln(z_o) - K_2) + \ln(w)(K(a+n)-1)}{K}$$

It can be proved that

$$I_{21} = \frac{m}{(a+n+m)}\left[1 - \frac{\beta_{C1}(k-1, g+d)}{\beta(k-1, g+d)}\right] \qquad (A.5)$$

Applying the following binomial expansion for the case where k-2 is a positive integer

$$(1-x)^{k-2} = \sum_{i=0}^{k-2}\binom{k-2}{i}(-1)^i x^i$$

it can be shown that

$$I_{22} = P_3^*\ , \qquad (A.6)$$

Where P_3^* is defined in equation (3.12) .
Noting that :

$$I_{11} + I_{21} = P_1^*$$

where P_1^* is defined in equation (3.12) , equation (3.12) is obtained .

2 :Derivation of equation (3.13) .

Suppose that condition (iv) in section 3.1 is satisfied . P^* will be given as follows

$$P^* = I_3 + I_4 \quad ,$$

where

$$I_3 = \int\int_{(v,u)\in R_3} f(v,u|x_{(1)},s)dvdu$$

and

$$I_4 = \int\int_{(v,u)\in R_4} f(v,u|x_{(1)},s)dvdu \quad .$$

R_3 is the region of (u>0 , v>ln$w$) values satisfying (3.10) and R_4 is the region of (u>0,v<ln$w$) values satisfying (3.11) .
Note that the inequality $u+mv \geq K_1/K$ defined in (3.10) will be satisfied for all values of u and v > ln w since m ln (w) , which is a lower bound for the left hand side of (3.10) is greater than K_1/K . Hence

$$I_3 = \int_0^\infty \int_{lnw}^\infty f(v,u|x_{(1)},s)\ dvdu$$

It can be shown that

$$I_3 = \frac{(a+n)}{(a+n+m)} \qquad (A.7)$$

R_4 defined in integral I_4 is shown in figure (A.3) .

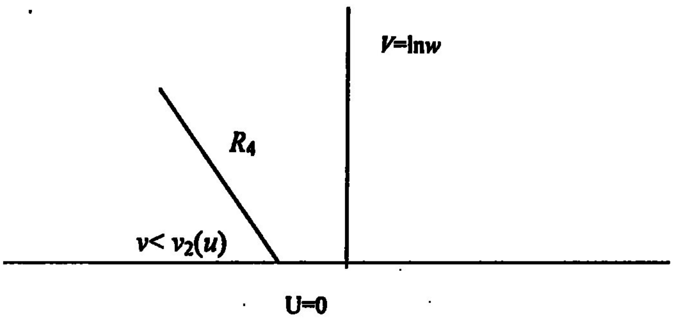


Figure A.3:Region R_4 satisfying $v \geq v_2(u)$

From figure (A.3) , it can be observed that

$$I_4 = \int_0^\infty \int_{v_2(u)}^{lnw} f(v,u|x_{(1)},s)dvdu \quad ,$$

which can be expressed in the form

$$I_4 = \frac{m}{(a+n+m)}\left[1-\left(\frac{H_1}{H_2}\right)^{g+d}(1-(a+n)K)^{g+d+k-1}\right] \qquad (A.8)$$

where H_1 is defined in equation (3.9) and H_2 is defined in equation (3.13).
From equations (A.7) and (A.8), it is noted that P^* is as given by equation (3.13).

3: Derivation of equation (3.14)

Suppose that condition (v) defined in section 3.1 is satisfied.
In this case

$$P^* = I_5 + I_6 \quad ,$$

where

$$I_5 = \iint_{(v,u)\in R_5} f(v,u|x_{(1)},s)\,dv\,du$$

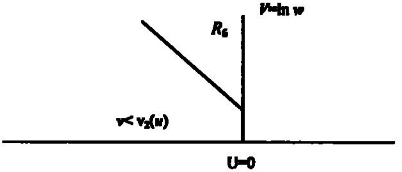
and

$$I_6 = \iint_{(v,u)\in R_6} f(v,u|x_{(1)},s)\,dv\,du \; .$$

R_5 is the region of ($u>0$ and $v>\ln w$) values satisfying inequality (3.10) and R_6 is the region of $u>0$ and $v<\ln w$ values satisfying (3.11).
Since according to condition (v)

$$-[\ln\mu + \sum_{i=1}^{d}\ln x_{(i)} + (n-d)\ln x_{(d)}] + (a+n)\ln w + \frac{1}{K}\ln\left[\frac{z_a}{w}\right] > 0$$

it follows that I_5 will have the same form as I_1 which is the sum of I_{11} and I_{12} defined in equations (A.1) and (A.2) respectively. R_6 defined in I_6 is shown in figure A.4.



FigureA.4: Region R_6 satisfying $v \geq v_2(u)$

Hence I_6 can be expressed as

$$I_6 = \int_{u_2}^{\infty}\int_{v_2(u)}^{\ln w} f(v,u|x_{(1)},s)\,dv\,du \; .$$

It can be shown that the above integral is the sum of two integrals.

$$I_{61} + I_{62} \quad , \qquad (A.9)$$

where, I_{61} has the same form as I_{21}, and I_{62} is given by:

$$I_{62} = -K^{?}\left[1-(a+n)K\right]^{a-1}\int_{u_2}^{\infty}\frac{u^{a-1}}{(H_2+u)^{a-1}}\,du \; . \qquad (A.10)$$

where

$$K_7 = \frac{m(a+n)H_1^{g+d}\Gamma(G)}{\Gamma(k-1)\Gamma(g+d)A}$$

G and A are defined in (3.3). H_1 is defined in (3.9) and H_2 is defined in (3.13).

Note here that according to condition (v), $H_2 > 0$. It can be proved that

$I_{62} = P_4^*$,

where P_4^* is defined in equation (3.14). It thus follows that P^* is as given by (3.14).

4: Derivation of equation (3.19)

According to condition (vi), P^* can be expressed as

$P^* = I_7 + I_8$,

where I_7 has the form as I_5 or I_1 which is the sum of I_{11} and I_{12} defined in equations (A.1) and (A.2) respectively.

$I_8 = I_{81} + I_{82}$.

I_{81} has the same form as I_{21} given by (A.5) and I_{82} is defined by equation (A.10). Noting that $H_2 < 0$ according to condition (vi), It can be shown that:

$I_{82} = P_5^*$,

where P_5^* is defined in equation (3.19).Hence equation (3.19) is obtained.

5: Derivation of equation (3.20)

Suppose that condition (i) in section 3.2 is satisfied. For the case $u > 0$ and $-\infty < v < \ln z_0$, the probability defined by (3.2) is given by (3.11). Since $(a+n) > 1/K$, $z_0 < w$ and $H_1 > 0$, it follows that $-\ln z_0 (a+n-1/K)$ which is a lower bound for the left hand side of (3.11) will always be greater than the right hand side of the same inequality. Hence (3.11) is always satisfied. For values of $u > 0$ and $\ln z_0 < v < \infty$, the probability given by (3.2) is also always satisfied when $p \le (A/(A+1))$.

6:Derivation of equation (3.21)

Suppose that $p \le (A/(A+1))$ and $(a+n-(1/K)) < 0$ as described in condition (ii) in section 3.2. In this case P^* can be written as follows:

$$P^* = I_9 + I_{10} + I_{13} \quad , \qquad \text{(A.11)}$$

where

$$I_9 = \int\int_{(v,u)\in R_9} f(v,u|x_{(1)},s)\,dv\,du$$

$$I_{10} = \int\int_{(v,u)\in R_{10}} f(v,u|x_{(1)},s)\,dv\,du$$

and

$$I_{13} = \int\int_{(v,u)\in R_{11}} f(v,u|x_{(l)},s)\,dv\,du$$

R_9 is the region of ($u>0$, $-\infty<v<\ln z_0$) values satisfying (3.2), R_{10} is the region of ($u>0$, $\ln z_0<v<\ln w$) values satisfying (3.2). R_{11} is the region of ($u>0$, $v>\ln w$) values satisfying (3.2). For values of ($u>0$, $-\infty<v<\ln z_0$) satisfying (3.2) requires satisfying the inequality given by (3.11). In this case the region R_9 will be as given in figure A.9.

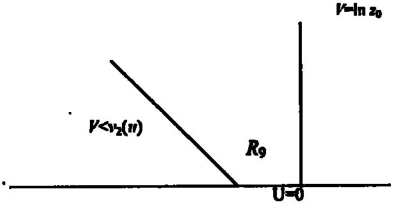


Figure A.5 : Region R_9 satisfying $v \geq v_2(u)$

$$I_9 = \int_0^\infty \int_{v_2(u)}^{\ln z_0} f(v|x_{(l)},s)\,dv\,du$$

It can be shown that

$$I_9 = \frac{m}{(a+n+m)}\left(\frac{H_1}{H_2}\right)^{g+l}\left[1-[1-K(a+n)]^{g+d+k-1}\right] \tag{A.12}$$

For values of $\ln z_0 < v < \ln w$, since $p \leq (A/(A+1))$, (3.2) will always be satisfied. Hence I_{10} can be expressed as

$$I_{10} = \int_0^\infty \int_{\ln z_0}^{\ln w} f(v,u|x_{(l)},s)\,dv\,du$$

which is given by

$$I_{10} = \frac{m}{(a+n+m)}\left[1-\left(\frac{H_1}{H_2}\right)^{g+l}\right] \tag{A.13}$$

For values of $v > \ln w$

$$I_{11} = \int_0^\infty \int_{\ln w}^\infty f(v,u|x_{(l)},s)\,dv\,du.$$

$$I_{13} = \frac{n+a}{(a+n+m)} \tag{A.14}$$

Substituting (A.12), (A.13) and (A.14) in (A.11), equation (3.21) is obtained.

7: Derivation of equation (3.22)

Suppose that condition (iii) is satisfied. Since $p > (A/(A+1))$, then for $-\infty < v < \ln z_0$ (3.2) is not satisfied. For $\ln z_0 < v < \ln w$, (3.2) is given by equation (3.6) which can be put in the form

$$u - K_6 v \leq -\frac{(a+n+m)\ln z_0}{K_5} - \left[\ln\mu + \sum_{i=1}^{d}\ln x_i + (n-d)\ln x_d\right] \qquad \text{(A.15)}$$

where K_5 and K_6 are defined in equations (3.22) and (3.23) respectively. A lower bound for the left hand side of (A.15) is given by $-K_6 \ln w$. Note that this lower bound is always greater than the right hand side of (A.15), hence (A.15) will not be satisfied. Applying a similar argument it can be shown that for $v > \ln w$, (3.2) is also not satisfied. It thus follows that under condition (iii) of section 3.2. equation (3.22) is obtained.

8: Derivation of equation (3.23)

Suppose that condition (iv) in section 3.2 is satisfied. For $-\infty < v < \ln z_0$ (3.2) is not satisfied. For $\ln z_0 < v < \ln w$, calculating (3.7) requires finding the values of (u,v) satisfying (A.15). It can be shown that in this case the probability given by the following integral is obtained.

$$J_{11} = \int_0^{u_3}\int_{v_3(u)}^{\ln w} f(v,u|x_{(l)},s)\,dv\,du$$

where

$$u_3 = K_6 \ln w - \frac{(a+n+m)\ln z_0}{K_5} - \left[\ln\mu + \sum_{i=1}^{d}\ln x_i + (n-d)\ln x_d\right]$$

and

$$v_3(u) = \frac{1}{K_6}\left[u + \frac{(a+n+m)\ln z_0}{K_5} + \ln\mu + \sum_{i=1}^{d}\ln x_i + (n-d)\ln x_d\right].$$

J_{14} will be given by

$$J_{14} = \frac{m}{(a+n+m)}\left(\frac{\beta_{c_3}(k-1,g+d)}{\beta(k-1,g+d)}\right) + P_7^* \qquad \text{(A.16)}$$

where P_7^* and c_3 are defined in (3.23).

For $v > \ln w$, it can be shown that (3.7) is given by the following probability.

$$J_{15} = \frac{(a+n)}{(a+n+m)}\left(\frac{\beta_{c_3}(k-1,g+d)}{\beta(k-1,g+d)}\right) - P_8^* \qquad \text{(A.17)}$$

where P_8^* is defined in (3.23).

Combining (A.16) and (A.17), equation (3.23) is obtained.

REFERENCES

AL – Hussaini, E. K. (1999). Predicting observables from a general class of distributions . *J. Statist. Plann. Inf.* ,79, 79-91.

AL – Hussaini, E.K. (2001). Prediction advances and new research . Presented as an invited topical paper in The International Mathematical Conference (Cairo, Jan.15-20,2000) .Proceedings of [Mathematics and the 21 st Century], 233-245, World Scientific, Singapore.

Arnold, B.C.(1983). *Pareto Distributions* .International Co-operative publishing House, Maryland.

Arnold, B.C. and Press S J. (1983). Bayesian inference for Pareto populations. *Econometrics,* 21, 287-306.

Arnold, B.C. and Press S.J.(1989). Bayesian estimation and prediction for Pareto data. *J. Amer. Statist. Assoc.*, 84, 1079-1084.

Dunsmore, I.R.(1974). The Bayesian predictive distribution in life testing models. *Technometrics,* 16, 455-460.

Dunsmore, I.R .and Amin, Z.H.(1997). Bayesian prediction and tolerance regions in Pareto populations. The ninth annual conference on Statistics and Computer Modeling in Human and Social Studies, Faculty of Economics and Political Science, Cairo University , 1-12.

Epstein, B. and Sobel, M. (1954). Some theorems relevant to life testing from an exponential distribution. *Ann. Math. Statist.* ,25, 373-381.

Evans, I .G. and Nigm, A.H. (1980). Bayesian Prediction for the left truncated
exponential distribution. *Technometrics,* 22,201-204.

Geisser, S. (1984) Predicting Pareto and exponential observables. *Canad . J. Statist.* ,12, 143-152.

Geisser, S. (1985). Interval prediction for Pareto and exponential observables. J. *Econometrics,* 29, 173-185.

Geisser, S. (1992). On the curtailment of sampling. *Canad. J. Statist.*, 20, 297-309.

Geisser, S. (1993). Bayesian interim analysis of censored exponential observations. *Statist. Probab. Lett.* , 18, 163-168.

Johnson, N., Kotz, S.and Balakrishnan, N.(1994).*ContinuousUnivariate Distributions* Vol.1, Second Edition, Wiley, New York.

Johnson, R. E. Zelen, M. and Kemp, M. N. (1965) Chemotherapeutic effects on mammalian tumour cells: I Modification of leukemia L1210 growth kinetics and Karyotype with alkylating agent.J. *Nat .Cancer Inst.*, 34, 277-290.

Lwin, T. (1972). Estimation of the tail of the Paretian Law. *Skand. Aktuar-ietidsker.* 55, 170-178.

Nigm, A. M. and Hamdy, H. I. (1987). Bayesian Prediction bounds for the Pareto lifetime model. *Comm. Statist.* A,. 16, 1761-1772.

Papandonatos, G. D. and Geisser, S. (1999) . Bayesian interim analysis of life-time data. *Canad. J. Statist,* 27, 1-21.