Probability Permutation Models

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Abstract

The permutation of a set of independent negative binomial random variables, not identically distributed, will be used for ranking players from the best to the worst. A problem of ranking k players is considered when they score points independently as Bernoulli random variables such that the probability of scoring points by all players are not all the same. Since the players are ranked according to how many trails needed for each player to score the desire number of points, negative binomial distribution function is used to build the model of getting the probability of the permutation of the best order.

1. Introduction

In applied probability, the probability of such a permutation arises in rank order statistics when considering a point-scoring competition among *k* players. It is desirable to calculate the probability of the correct order of scoring specific number of points, say *r*, under some specifications. The correct order is that the best player is the winner and ranked first, the second best player is ranked second and so on till the worst player who takes the last position. The consideration of ending the game might be involved the time or the number of trails required for each player to score *r* points. The methodology, used to rank players, plays an important role to which distribution function is used in constructing a probability permutation model. Some researchers discussed the problem of calculating the probability of a permutation in different ways. Bradley and Terry (1952), Thurstone 1927, and Mosteller 1951 considered a model of scoring points in

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which only two out of k players can play at one time and ranking them at the end of $k(k-1)\backslash 2$ paired comparison games.

To simplify the problem, let us introduce some notations that will be used throughout. Define $\pi = (\pi_1, \pi_2 \ldots, \pi_k)$ to be the correct order and $P(\pi)$ to be the probability of the permutation π of k players where π_i represents the i^{th} ranked player. Suppose that $X_1, X_2, \ldots, X_k$ are set of independent random variables and let the probability of π be $P(\pi)=P(X_{\pi_1} < X_{\pi_2} < \ldots < X_{\pi_k})$.

This paper deals with negative binomial random variables and its application to ranking players in a game. The negative binomial distribution function is given by

$$f(r;N)=\binom{N-1}{r-1}P^r\,(1-p)^{N-r}\,;\quad for\, N=r, r+1, \cdots.$$

It is not interesting to discuss the case where the random variables are identically distributed due to the equivalences of all permutations.

2. GAMMA RANKING MODEL

Stern (1990) proposed the distributions of permutations using the gamma models. Suppose that k players play in a competition, where each player has to score r points to finish. Let X_i stands for the time required for the i^{th} player to score r $(r \geq 1)$ points, where $i = 1, 2, \ldots, k$ and $COV(X_i, X_j)=0$ for all $i \neq j$. Suppose the i^{th} player scores points as a Poisson random variable with mean λ_i. For X_i, the gamma distribution function with parameters (r, λ_i) often arises as the distribution of the amount of time player i has to wait until a total of r points has been scored. The mean and variance of X_i are $r\lambda_i^{-1}$ and $r\lambda_i^{-2}$ respectively where r is assumed to be fixed and known. Note that all the λ_i's are not the same and increasing λ_i will lead to the decreasing of X_i. Thus, the probability of the permutation $\pi = (\pi_1, \pi_2 \ldots, \pi_k)$ is given by:

$$P(\pi) = P(X_{\pi_1} < X_{\pi_2} < \cdots < X_{\pi_k})$$

$$= \int_0^{\infty} \int_0^{(\lambda_{\pi_{k-1}}/\lambda_{\pi_k}) z_{\pi_k}} \cdots \int_0^{(\lambda_{\pi_1}/\lambda_{\pi_3}) z_{\pi_3}} \int_0^{(\lambda_{\pi_1}/\lambda_{\pi_2}) z_{\pi_2}} [\Gamma(r)]^{-k} \prod_{i=1}^{k} (z_{\pi_i}{}^{r-1} \exp(-z_{\pi_i})) dz_{\pi_1} \cdots dz_{\pi_k}$$

$$= f\left(\frac{\lambda_{\pi_1}}{\lambda_{\pi_2}}, \frac{\lambda_{\pi_2}}{\lambda_{\pi_3}}, \cdots, \frac{\lambda_{\pi_{r-1}}}{\lambda_{\pi_k}}\right), \qquad (1)$$

Where $z_i = \lambda_i x_i$, for $(i = 1, 2, \ldots, k)$.

The ratio of λ_i/λ_{i+1}, i=1, 2, . . ., k-1, plays an important role for the determination of the probability of any permutation. λ_i's is selected such that $\sum \lambda_i = 1$. If $\lambda_i = \lambda_j$ for all i and j, the probability of any permutation is $1/k!$. Models that associate with assigning parameter to each player being ranked were also considered by Tallis and Danise(1983), Plackett (1975), Mallows (1957), Feigin and Cohen (1978), Schulman (1979), and Fligner and Verducci (1986).

The ranking model using gamma process when r=1 was introduced by Luce (1959). Harville(1973) simplified the model derived by Luce(1959) when applying the conditional probability concept. The ranking model when the number of points gets large is called the Thurstone-Mosteller-Daniels model and was discussed by Daniels (1950) and Stern(1987) for paired comparisons. Stem (1987) noted that $P(X_i < X_j)$ is the probability of at least r successes in 2r-1 binomial trails with probability of success $\lambda_i/(\lambda_i + \lambda_j)$.

3. Poisson Ranking Processes

Our interest here is to identify the process of scores by the k players when considering the counting technique discussed by Henery (1983). The combined process is observed until r points are scored by any player. The i^{th} ranked player scores points as Poisson with probability $\lambda_i \setminus (1 - \sum_{i=1}^{i-1} \lambda_{\pi_i})$, (i=1, 2, 3, . . ., k). Let $\ell_i = N_i - N_{i-1}$ (i = 2, 3, . . ., k) be the number of points required for the i^{th} ranked player to finish scoring the desired points once the $(i-1)^{th}$ ranked player has finished where N_i stands for the total number of points scored by all players when the i^{th} ranked player has finished scoring r points. For example, a total of N_1 points (for all players) are required for the winner to be ranked first where the points are scored by him/her as Poisson with probability λ_{π_1} . The remaining players are then combined together. Then, assume that the total (all players) of N_2 points are required for the second best player to finish next. Points in this combined process are scored by the second best with probability $\lambda_{\pi_2} /(1 - \lambda_{\pi_1})$ where ℓ_2 ($\ell_2 = N_2 - N_1$) is the number of points required for the second ranked player to finish once the first ranked player has finished. Other values of all ℓ_i 's and N_i's can be obtained in the same manner. As soon as the i^{th} player scores r points, the i^{th} Poisson process is ignored. This is repeated until the final player scores r points. Thus, at the end of this procedure each player scored r points, so N_k is always equal to kr. To simplify the notation, let us consider the permutation π^*=(1, 2, . . ., k) i.e; the first player is the winner, the second is ranked second, till the k^{th} player who takes the last position. The probability of the permutation π^* is given by:

$$P(\pi^*) = \frac{\prod_{i=1}^{k} \lambda_i^r}{((r-1)!)^k} \sum_{\ell_k=1}^{r} \frac{(r-1)!}{(r-\ell_k)!\,\lambda_k^{\ell_k}} \sum_{\ell_{k-1}=1}^{2r-1-\ell_k} \frac{(2r-1-\ell_k)!}{(2r-\ell_k-\ell_{k-1})!\,(\lambda_k+\lambda_{k-1})^{\ell_{k-1}}}$$

$$\times \cdots$$

$$\times \sum_{\ell_2=1}^{(k-1)r-1-\sum_{i=3}^{k}\ell_i} \frac{((k-1)r-1-\sum_{i=3}^{k}\ell_i)!\,(kr-1-\sum_{i=2}^{k}\ell_i)!}{((k-1)r-\sum_{i=2}^{k}\ell_i)!\,(\sum_{i=2}^{k}\lambda_i)^{\ell_2}\,(\sum_{i=1}^{k}\lambda_i)^{kr-\sum_{i=2}^{k}\ell_i}}, \qquad (2)$$

Number of different ways that the first (r-1) points are scored by each player plays an important role in the determination of equation (2). Now, for example, for $r = 2$ and $k = 4$, consider the sequence 1, 2, 1, 3, 4, 2, 3, 4, i.e. the first player scores the first point and then the second player scores the second point and then the first player scores the third point etc. For this sequence, N_1=3, N_2=6, N_3=7, N_4=8 ,$\ell_2=3, \ell_3=1, and\ \ell_4=1$. Thus, the probability of the previous sequence is

$$\frac{\lambda_1^2\lambda_2}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^3}\cdot\frac{\lambda_2\lambda_3\lambda_4}{(\lambda_2+\lambda_3+\lambda_4)^3}\cdot\frac{\lambda_3}{(\lambda_3+\lambda_4)}\cdot\frac{\lambda_4}{\lambda_4}.$$

To find the probability that the first player is ranked first, equation (2) is given by

$$\lambda_1^r \sum_{i_2=0}^{r-1} \cdots \sum_{i_k=0}^{r-1} \frac{(r+\sum_{j=2}^{k} i_j - 1)!}{(r-1)!\,i_2!\,i_3!\cdots i_k!} \lambda_2^{i_2}\lambda_3^{i_3}\cdots\lambda_k^{i_k},$$

where i_j represents the number of points scored by player j when the rth point is scored by the first player. Note that the probability is counted over all permutations such that player j is ranked first

Although equation (2) seems to be more complicated, it is the appropriate for the situation where the counting process is of interest.

4. Negative Binomial Ranking Models

While we give models for ranking players based on the time and counting techniques in the previous sections, the idea might be extended to include models concerning the number of trials required for each player to accumulate a fixed number of points to end the game. Suppose that k players score points according to k independent Bernoulli random variables. If the i^{th} player scores points independently with probability P_i ($0<P_i<1$), then the number of trials until the player's r^{th} point is scored has a negative binomial distribution with a mean of r/P_i and a variance of $r(1-P_i)/P^2_i$. The players are then ranked according to how many trials are required to achieve their targets when all players start with no credits. The player who reaches his goal with the fewest number of trials is the winner, the player to reach his goal with the next fewest number of trials among the rest of the (k-1) players is second and so on.

For now, r is a parameter defining the number of points required of the game; it is assumed to be fixed and known. Suppose X_i is the number of trials required for the i^{th} player to score r points, where $i=1, 2, \ldots, k$. In terms of the number of trials which all have negative binomial distributions with the same value of r but different probabilities of scoring points, it is assumed that $\Sigma P_i=1$. The probability of the permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ is given by:

$$P(\pi) = P(X_{\pi_1} < X_{\pi_2} < \cdots < X_{\pi_k})$$

$$= \sum_{i_{\pi_k}=r+(k-1)}^{\infty} \sum_{i_{\pi_{k-1}}=r+(k-2)}^{i_{\pi_k}-1} \cdots \sum_{i_{\pi_2}=r+1}^{i_{\pi_3}-1} \sum_{i_{\pi_1}=r}^{i_{\pi_2}-1} P(X_{\pi_1}=i_{\pi_1}, X_{\pi_2}=i_{\pi_2}, \cdots, X_{\pi_k}=i_{\pi_k})$$

$$= \sum_{i_{\pi_k}=r+(k-1)}^{\infty} \sum_{i_{\pi_{k-1}}=r+(k-2)}^{i_{\pi_k}-1} \cdots \sum_{i_{\pi_2}=r+1}^{i_{\pi_3}-1} \sum_{i_{\pi_1}=r}^{i_{\pi_2}-1} \{ P(X_{\pi_1}=i_{\pi_1}) \cdot P(X_{\pi_2}=i_{\pi_2}) \cdots P(X_{\pi_{k-1}}=i_{\pi_{k-1}}) \cdot P(X_{\pi_k}=i_{\pi_k}) \}$$

$$= \sum_{i_{\pi_k}=r+(k-1)}^{\infty} \sum_{i_{\pi_{k-1}}=r+(k-2)}^{i_{\pi_k}-1} \cdots \sum_{i_{\pi_2}=r+1}^{i_{\pi_3}-1} \sum_{i_{\pi_1}=r}^{i_{\pi_2}-1} \prod_{j=1}^{k} [\binom{i_{\pi_j}-1}{r-1} \cdot P_{\pi_j}^{r} \cdot (1-p_{\pi_j})^{i_{\pi_j}-r}]\}$$

$$= \sum_{i_{\pi_k}=r+(k-1)}^{\infty} \sum_{i_{\pi_{k-1}}=r+(k-2)}^{i_{\pi_k}-1} \cdots \sum_{i_{\pi_2}=r+1}^{i_{\pi_3}-1} \sum_{i_{\pi_1}=r}^{i_{\pi_2}-1} \{[(r-1)!]^{-k} \prod_{j=1}^{k} [\frac{(i_{\pi_j}-1)!}{(i_{\pi_j}-r)!} \cdot P_{\pi_j}^{r} (1-p_{\pi_j})^{i_{\pi_j}-r}]\}, \quad (3)$$

where it depends on how many trials are needed for the last (k-1) ranked players to finish. Once the i^{th} ranked player finished scoring the r^{th} point, the $(i+1)^{th}$ ranked player needs at least $r+(i+1)$ trails to score his/her r^{th} points to finish the game. If $P_i = P_j$ for all i, j, then the k random variables are independent and identically distributed and it is expected that the probability of any permutation is $1/k!$. The i^{th} player will have low rank in the permutation among the k players as long as P_i gets large. Since r is a number of points, equation (3) is valid for the integer values of r ($r \geq 1$).

For example, suppose that player i scores points as Bernoulli trials with probability P_i, and player j scores as Bernoulli trials with probability P_j such that $P_i + P_j = 1$. The two Bernoulli trials are assumed to be independent. We say that player i defeats player j if player i scores a point before player j. The number of trials for the first point to be obtained for player i is a geometric random variable X_i with mean p_i^{-1}. The same model can also be used for the other random variables. The probability that player i defeats player j can be computed as the probability that that X_i is less than X_j and given by

$$P(X_i < X_j) = \sum_{i_j=1}^{\infty} P(X_i < i_j, X_j = i_j)$$

$$= \sum_{i_j=2}^{\infty} \sum_{i_i=1}^{i_j-1} P(X_i = i_i) \cdot P(X_j = i_j)$$

$$= \sum_{i_j=2}^{\infty} \sum_{i_i=1}^{i_j-1} p_i(1-p_i)^{i_i-1} \cdot p_j(1-p_j)^{i_j-1}$$

$$= p_i p_j \sum_{i_j=2}^{\infty} (1-p_j)^{i_j-1} \left[\frac{1-(1-p_i)^{i_j-1}}{p_i}\right]$$

$$= p_i - \left[\frac{p_i p_j^2}{1-p_i p_j}\right].$$

The probability that the i^{th} player will be ranked first increases whenever P_i increases. The probability that the j^{th} player will win can be obtained, in the same manner, is $p_j - p_j p_i^2/(1-p_i p_j)$, where as the two players will be tied with the probability of $p_j p_i/(1-p_i p_j)$. If $P_i = P_j = 0.5$, $P(X_i > X_j) = P(X_i < X_j) = P(X_i = X_j) = 1/3$.

Suppose that the required number of points for each player to finish the game is large, then the negative binomial random variable, as the sum of large number of independent geometric random variables, tends to be a Gaussian random variables. From the central limit theorem as r gets large, the standardized negative binomial random variable tends to the standard Gaussian random variable with mean zero and variance one. Let X_i be the total number of trails required for player i to score very large number of points r. The random variable X_i has a negative binomial distribution with a mean of r/p_i and a variance of $r(1-p_i)/p_i^2$. For large value of r, $\frac{X_i - (r/p_i)}{\sqrt{r(1-p_i)/p_i^2}}$ is a random variable with at standard normal distribution function.

Thus, the probability that the i^{th} player beats the j^{th} player when they are trying to score r (r is very large) points is:

$$P(X_i < X_j) \cong P\left(\frac{(X_i - X_j) - r(p_i^{-1} - p_j^{-1})}{\sqrt{r[(1-p_i)p_i^{-2} + (1-p_j)p_j^{-2}]}} < -\frac{r(p_i^{-1} - p_j^{-1})}{\sqrt{r[(1-p_i)p_i^{-2} + (1-p_j)P_j^{-2}]}}\right)$$

$$= \Phi\left(-\frac{\sqrt{r}(p_i^{-1} - p_j^{-1})}{\sqrt{[(1-p_i)p_i^{-2} + (1-p_j)p_j^{-2}]}}\right), \qquad (3)$$

where $\Phi(-a) = P(Z < -a)$ such that Z is a standard normal random variable.

In other cases where k players ($k>2$) are ranked according to the number of trails required to score r points where r is very large, the terminology here will be slightly different from comparing just two players. Since $X_i \sim N(r/p_i, r(1-p_i)/p_i^2)$ for all i = $1, 2, \ldots, k$, the probability of the permutation $\pi = (\pi_1, \pi_2 \ldots, \pi_k)$ can be computed as follow

$$P(X_{\pi_1} < X_{\pi_2} < \cdots < X_{\pi_k})$$

$$= \sum_{i_{\pi_k}=r+(k-1)}^{\infty} \; \sum_{i_{\pi_{k-1}}=r+(k-2)}^{i_{\pi_k}-1} \cdots \sum_{i_{\pi_2}=r+1}^{i_{\pi_3}-1} \; \sum_{i_{\pi_1}=r}^{i_{\pi_2}-1} \prod_{i=1}^{k} \left\{ \frac{p_i}{\sqrt{2\pi r(1-p_i)}} \exp\left[-\frac{1}{2}\left(\frac{x_{\pi_i} - rp_{\pi_i}^{-1}}{\sqrt{r(1-p_i)/p_i^2}}\right)^2\right] \right\}, \qquad (4)$$

where, again, it depends on how many trials are needed for the last (k-1) ranked players to finish. As long the i^{th} ranked player finished scoring his/her r^{th} point, the $(i+1)^{th}$ ranked player needs at least $r+(i+1)$ trials to score r points to finish the game.

5. Summary

Probability permutation models of ranking k players are considered in this paper. Number of trails needed for each player to achieve the target of scoring r points was the criterion used to rank the players from the best to the worst. Each trial would result either on a fail or a success, and that would suggest the use of probability distribution function of the discrete random variable. Specifically, negative binomial distribution function was used to propose the model.

From the probability theory point view, the computation of the probability of the proposed model gets more complicated due to the possibility of $X_i = X_j$ for some i and j. This is not the case when considering the gamma ranking model. A conditional probability concept can be applied to simplify the probability of the best order as it is considered by Harville (1973) when considering the gamma ranking model.

6. References

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