Two types of sequential-cum-parallel general optional service in a single server queue

Kailash C. Madan
Department of Statistics, Faculty of Science
Yarmouk University, Irbid, JORDAN.
e-mail: kailashm@yu.edu.jo ; kailashmadan@hotmail.com

Abstract

We analyze a single server queue with two types of optional general heterogeneous service. Under different options for the customers, the server provides sequential-cum-parallel service. Just before his service starts, a customer has the option to choose both services in succession or only one of the two types of services. Steady state probability generating functions for various states of the system have been obtained in explicit and closed form. Some special cases have also been discussed.

Key words: Optional general heterogeneous service, steady state, probability generating function.

AMS primary Classification: 60 k.

1. Introduction

The single server queue has been studied by numerous authors including Lee (1966), Medhi (1982), Cohen (1982), Bhat (1984), Gross and Harris (1985) and Kashyap and Chaudhry (1988), to mention a few. In most of these papers, the server provides only one kind of service to the customers. Madan (1992, 1994) studied single server queues with additional optional service. In the first paper, he considers that the customers requiring second service are served in batches but such customers have to wait until the time a batch of a specified size is ready for second service. In the second paper, he deals with second optional service for a system that has no queueing capacity. Recently Madan (2000) has studied an M/G/1 queue with second optional service in which he considered that the first service is general and the second service is exponential. Each arriving customer must take the first service and if a customer requires second service, it is provided to him just after his first service.

In the present paper, we consider a single server queue in which the server provides two types of general heterogeneous service with different options for the customer. We have assumed that a customer may choose both services, in which case he is provided two services in succession, i.e., the first service followed by the second (sequential service) or else he may choose just one of the two types of services (parallel service). As an ordinary example of such a situation, some of the customers arriving at a barber shop may require both a hair cut and a shave and some others may need just one of the two services. Another example is that some of the jobs arriving at a computer center may require both computing and printing whereas some other jobs may simply need either computing or printing. One may encounter many more such kind of real-life queueing situations and certainly the model may have many wider applications. For convenience, we shall designate our model as a $M/(G_1,G_2),\begin{pmatrix}G_1\\G_2\end{pmatrix}/1$ queue, which is briefly described by the following assumptions.

2. The Mathematical Model

Customers arrive at the system one by one in a Poisson stream with mean arrival rate λ (>0). There is single server who offers two types of heterogeneous service each having a different general distribution with probability density function $b_j(v)$ and the distribution function $B_j(v)$, j=1, 2. Let $\mu(x)$ dx be the first order probability that the service of a customer will complete during the interval (x, x+dx) given that the same was not complete till time x. Therefore,

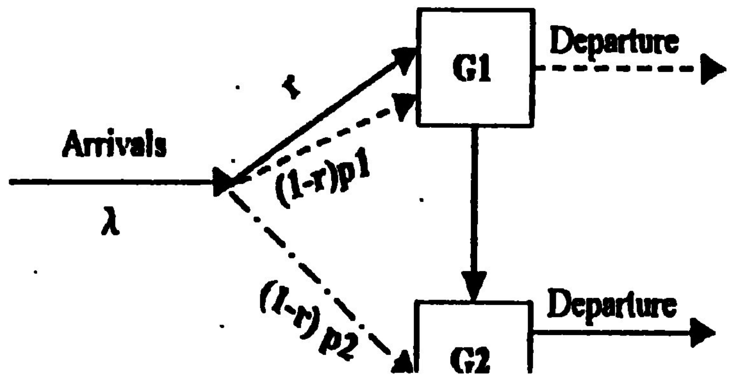
$$\mu_j(x)=\frac{b_j(x)}{1-B_j(x)} \tag{1}$$

so that:

$$b_j(v)=\mu_j(v)\exp\left[-\int_0^v \mu_j(x)dx\right] \quad ,j=1,2. \tag{2}$$

Customers are serviced one by one on a first come, first served basis. However, just before a service starts, a customer has the option to choose either both types of services in succession, in which case the first type of service is followed by the second type of service, or alternatively the customer may select any one of the two types of services. We have assumed that a customer chooses both services with probability r $(0 \le r \le 1)$ or just type j service with probability $(1-r)p_j$, j=1, 2 and $p_1+p_2=1$. The following diagram gives the route of a customer under different options.

Route of a customer under various options



Various stochastic processes involved in the system are independent of each other.

3. Definitions and Notations

In order to define notations for the desired probabilities, we shall use two letters A and B, where A indicates that the customer in current service has opted for only one service (of either type) and B indicates that the customer in current service has opted for both services. Accordingly we let $A_n^{(j)}(x)$ (n=0, 1, 2, ... ; j=1, 2) denote the steady state probability that there are n customers in the queue excluding one customer who has opted for type j service and this service is on with elapsed service time x and similarly $B_n^{(j)}(x)$ (n=0, 1, 2, ... ; j=1, 2) denotes the steady state probability that there are n customers in the queue excluding one customer who has opted for both services and this customer is currently in j th service with elapsed service time x. Further, in accordance with the above notations, $A_n^{(j)}=\int_0^\infty A_n^{(j)}(x)dx$ and $B_n^{(j)}=\int_0^\infty B_n^{(j)}(x)dx$ are the corresponding steady state probability generating functions without regard to the elapsed time x. Finally we assume E to be the steady state probability that there is no customer either in queue or in service and the server is idle.

4. Steady state equations of the system

Based on the usual probability reasoning, we have the following set of differential-difference equations for the steady state.

$$\frac{\partial}{\partial x}A_n^{(j)}(x)+(\lambda+\mu_j(x))A_n^{(j)}(x)=\lambda A_{n-1}^{(j)}(x)\,,\ j=1;2;\ n\geq 1 \tag{3}$$

$$\frac{\partial}{\partial x}A_0^{(j)}(x)+(\lambda+\mu_j(x))A_0^{(j)}(x)=0\,,\ j=1,2 \tag{4}$$

$$\lambda E=\int_0^\infty A_0^{(1)}(x)\mu_1(x)dx+\int_0^\infty A_0^{(2)}(x)\mu_2(x)dx+\int_0^\infty B_0^{(2)}(x)\mu_2(x)dx \tag{5}$$

$$\frac{\partial}{\partial x}B_n^{(j)}(x)+(\lambda+\mu_j(x))B_n^{(j)}(x)=\lambda B_{n-1}^{(j)}(x)\,,\ j=1,2:\ n\geq 1 \tag{6}$$

$$\frac{\partial}{\partial x}B^{(j)}(x)+(\lambda+\mu_j(x))B_0^{(j)}(x)=0 \tag{7}$$

The above equations are to be solved subject to the following boundary conditions:

$$A_n^{(1)}(0)=(1-r)p_1\Big[\int_0^\infty A_{n+1}^{(1)}(x)\mu_1(x)dx+\int_0^\infty A_{n+1}^{(2)}(x)\mu_2(x)dx+\int_0^\infty B_{n+1}^{(2)}(x)\mu_2(x)dx\Big],\quad n\geq 1 \tag{8}$$

$$A_0^{(1)}(0)=(1-r)p_1\Big[\int_0^\infty A_1^{(1)}(x)\mu_1(x)dx+\int_0^\infty A_1^{(2)}(x)\mu_2(x)dx+\int_0^\infty B_1^{(2)}(x)\mu_2(x)dx+\lambda E\Big] \tag{9}$$

$$A_n^{(2)}(0)=(1-r)p_2\Big[\int_0^\infty A_{n+1}^{(1)}(x)\mu_1(x)dx+\int_0^\infty A_{n+1}^{(2)}(x)\mu_2(x)dx+\int_0^\infty B_{n+1}^{(2)}(x)\mu_2(x)dx\Big],\ n\geq 1 \tag{10}$$

$$A_0^{(2)}(0)=(1-r)p_2\Big[\int_0^\infty A_1^{(1)}(x)\mu_1(x)dx+\int_0^\infty A_1^{(2)}(x)\mu_2(x)dx+\int_0^\infty B_1^{(2)}(x)\mu_2(x)dx+\lambda E\Big] \tag{11}$$

$$B_n^{(1)}(0)=r\Big[\int_0^\infty A_{n+1}^{(1)}(x)\mu_1(x)dx+\int_0^\infty A_{n+1}^{(2)}(x)\mu_2(x)dx+\int_0^\infty B_{n+1}^{(2)}(x)\mu_2(x)dx\Big],\ n\geq 1 \tag{12}$$

$$B_0^{(1)}(0)=r\Big[\int_0^\infty A_1^{(1)}(x)\mu_1(x)dx+\int_0^\infty A_1^{(2)}(x)\mu_2(x)dx+\int_0^\infty B_1^{(2)}(x)\mu_2(x)dx+\lambda E\Big] \tag{13}$$

$$B_n^{(2)}(0)=\int_0^\infty B_n^{(1)}(x)\mu_1(x)dx\,,\ n\geq 0 \tag{14}$$

5 Steady state probability generating functions

Define the following steady state probability generating functions (pgfs):

$$A_n^{(j)}(x,z)=\sum_{n=0}^{\infty}A_n^{(j)}(x)z^n,\ B_n^{(j)}(x,z)=\sum_{n=0}^{\infty}B_n^{(j)}(x)z^n,\ j=1,2;\ |z|\le 1 \tag{15a}$$

$$A_n^{(j)}(z)=\int_0^{\infty}A_n^{(j)}(x,z)dx,\ B_n^{(j)}(z)=\int_0^{\infty}B_n^{(j)}(x,z)dx,\qquad j=1,2;\ |z|\le 1 \tag{15b}$$

Multiply equation (3) by z^n, sum over n from 1 to ∞ and add the result to (4), use (15a) and simplify. We thus have for j=1, 2

$$\frac{\partial}{\partial x}A^{(1)}(x,z)+(\lambda+\mu_1(x)-\lambda z)A^{(1)}(x,z)=0, \tag{16}$$

$$\frac{\partial}{\partial x}A^{(2)}(x,z)+(\lambda+\mu_2(x)-\lambda z)A^{(2)}(x,z)=0. \tag{17}$$

A similar operation on (6) and (7) for j=1, 2 yields

$$\frac{\partial}{\partial x}B^{(1)}(x,z)+(\lambda+\mu_1(x)-\lambda z)B^{(1)}(x,z)=0, \tag{18}$$

$$\frac{\partial}{\partial x}B^{(2)}(x,z)+(\lambda+\mu_2(x)-\lambda z)B^{(2)}(x,z)=0. \tag{19}$$

Similarly we multiply (8) by z^{n+1}, sum over n from n=1 to n=∞, and multiply (9) by z, add the two results, use (15a) and simplify. Thus we obtain

$$\begin{aligned}zA^{(1)}(0,z)=&(1-r)p_1\Big[\int_0^{\infty}A^{(1)}(x,z)\mu_1(x)dx+\int_0^{\infty}A^{(2)}(x,z)\mu_2(x)dx+\int_0^{\infty}B^{(2)}(x,z)\mu_2(x)dx\Big]\\&-(1-r)p_1\Big[\int_0^{\infty}A_0^{(1)}(x)\mu_1(x)dx+\int_0^{\infty}A_0^{(2)}(x)\mu_2(x)dx+\int_0^{\infty}B_0^{(2)}(x)\mu_2(x)dx\Big]\\&+(1-r)p_1\lambda zE.\end{aligned} \tag{20}$$

Using (5), (20) simplifies to

$$\begin{aligned}zA^{(1)}(0,z)=&(1-r)p_1\Big[\int_0^{\infty}A^{(1)}(x,z)\mu_1(x)dx+\int_0^{\infty}A^{(2)}(x,z)\mu_2(x)dx+\int_0^{\infty}B^{(2)}(x,z)\mu_2(x)dx\Big]\\&+(1-r)p_1\lambda(z-1)E.\end{aligned} \tag{21}$$

A similar operation on (10) and (11), yields

$$\begin{aligned}zA^{(2)}(0,z)=&(1-r)p_2\Big[\int_0^{\infty}A^{(1)}(x,z)\mu_1(x)dx+\int_0^{\infty}A^{(2)}(x,z)\mu_2(x)dx+\int_0^{\infty}B^{(2)}(x,z)\mu_2(x)dx\Big]\\&-(1-r)p_2\Big[\int_0^{\infty}A_0^{(1)}(x)\mu_1(x)dx+\int_0^{\infty}A_0^{(2)}(x)\mu_2(x)dx+\int_0^{\infty}B_0^{(2)}(x)\mu_2(x)dx\Big]\\&+(1-r)p_2\lambda zE.\end{aligned} \tag{22}$$

Again, using (5), (22) simplifies to

$$zA^{(2)}(0,z)=(1-r)p_2\Big[\int_0^\infty A^{(1)}(x,z)\mu_1(x)dx+\int_0^\infty A^{(2)}(x,z)\mu_2(x)dx+\int_0^\infty B^{(2)}(x,z)\mu_2(x)dx\Big]$$
$$+(1\text{-r})p_2\,\lambda(z-1)E. \tag{23}$$

And yet again we perform a similar operation on (12), (13) and obtain

$$zB^{(1)}(0,z)=r\Big[\int_0^\infty A^{(1)}(x,z)\mu_1(x)dx+\int_0^\infty A^{(2)}(x,z)\mu_2(x)dx+\int_0^\infty B^{(2)}(x,z)\mu_2(x)dx\Big]$$
$$+\,\mathrm{r}\lambda(z-1)E. \tag{24}$$

Similarly (14) yields

$$B^{(2)}(0,z)=\int_0^\infty B^{(1)}(x,z)\mu_1(x)dx\ . \tag{25}$$

Integrating (16), (17), (18) and (19) between the limits 0 and x, we obtain

$$A^{(1)}(x,z)=A^{(1)}(0,z)\exp[-(\lambda-\lambda z)x-\int_0^x\mu_1(t)dt]\ , \tag{26}$$

$$A^{(2)}(x,z)=A^{(2)}(0,z)\exp[-(\lambda-\lambda z)x-\int_0^x\mu_2(t)dt]\ , \tag{27}$$

$$B^{(1)}(x,z)=B^{(1)}(0,z)\exp[-(\lambda-\lambda z)x-\int_0^x\mu_1(t)dt]\ , \tag{28}$$

$$B^{(2)}(x,z)=B^{(2)}(0,z)\exp[-(\lambda-\lambda z)x-\int_0^x\mu_2(t)dt]\ . \tag{29}$$

Again integrating (26), (27), (28) and (29) with respect to x by parts, we obtain

$$A^{(1)}(z)=\frac{1-\overline{b}_1(\lambda-\lambda z)}{\lambda-\lambda z}\ A^{(1)}(0,z)\ , \tag{30}$$

$$A^{(2)}(z)=\frac{1-\overline{b}_2(\lambda-\lambda z)}{\lambda-\lambda z}\ A^{(2)}(0,z)\ , \tag{31}$$

$$B^{(1)}(z)=\frac{1-\overline{b}_1(\lambda-\lambda z)}{\lambda-\lambda z}\ B^{(1)}(0,z)\ , \tag{32}$$

$$B^{(2)}(z)=\frac{1-\overline{b}_2(\lambda-\lambda z)}{\lambda-\lambda z}\ B^{(2)}(0,z), \tag{33}$$

where $\overline{b}_j(\lambda-\lambda z)=\int_0^\infty e^{-(\lambda-\lambda z)x}b_j(x)dx$ is the Laplace transform of $b_j(x)$, j=1, 2.

Next, we multiply (26), (27), (28) and (29) by $\mu_1(x)$, $\mu_2(x)$, $\mu_1(x)$ and $\mu_2(x)$ respectively, and integrate. Thus we obtain

$$\int_0^\infty A^{(1)}(x,z)\mu_1(x)dx = A^{(1)}(0,z)\,\overline{b_1}(\lambda-\lambda z), \tag{34}$$

$$\int_0^\infty A^{(2)}(x,z)\mu_2(x)dx = A^{(2)}(0,z)\,\overline{b_2}(\lambda-\lambda z), \tag{35}$$

$$\int_0^\infty B^{(1)}(x,z)\mu_1(x)dx = B^{(1)}(0,z)\,\overline{b_1}(\lambda-\lambda z), \tag{36}$$

$$\int_0^\infty B^{(2)}(x,z)\mu_2(x)dx = B^{(2)}(0,z)\,\overline{b_2}(\lambda-\lambda z). \tag{37}$$

Now, using (34) to (37) in (21), (23), (24) and (25), we have

$$zA^{(1)}(0,z) = (1-r)p_1\left[A^{(1)}(0,z)\,\overline{b_1}(\lambda-\lambda z) + A^{(2)}(0,z)\,\overline{b_2}(\lambda-\lambda z) + B^{(2)}(0,z)\,\overline{b_2}(\lambda-\lambda z)\right] + (1-r)p_1\lambda(z-1)E, \tag{38}$$

$$zA^{(2)}(0,z) = (1-r)p_2\left[A^{(1)}(0,z)\,\overline{b_1}(\lambda-\lambda z) + A^{(2)}(0,z)\,\overline{b_2}(\lambda-\lambda z) + B^{(2)}(0,z)\,\overline{b_2}(\lambda-\lambda z)\right] + (1-r)p_2\lambda(z-1)E, \tag{39}$$

$$zB^{(1)}(0,z) = r\left[A^{(1)}(0,z)\,\overline{b_1}(\lambda-\lambda z) + A^{(2)}(0,z)\,\overline{b_2}(\lambda-\lambda z) + B^{(2)}(0,z)\,\overline{b_2}(\lambda-\lambda z)\right] + r\lambda(z-1)E \tag{40}$$

$$B^{(2)}(0,z) = B^{(1)}(0,z)\,\overline{b_1}(\lambda-\lambda z). \tag{41}$$

Using (41), equations (38), (39), (40) can be further simplified to

$$[z-(1-r)p_1\overline{b_1}(\lambda-\lambda z)]A^{(1)}(0,z) = (1-r)p_1\,\overline{b_2}(\lambda-\lambda z)\left[A^{(2)}(0,z)+\overline{b_1}(\lambda-\lambda z)\,B^{(1)}(0,z)\right] + (1-r)p_1\lambda(z-1)E \tag{42}$$

$$[z-(1-r)p_2\overline{b_2}(\lambda-\lambda z)]A^{(2)}(0,z) = (1-r)p_2\,\overline{b_1}(\lambda-\lambda z)\left[A^{(1)}(0,z)+\overline{b_2}(\lambda-\lambda z)\,B^{(1)}(0,z)\right] + (1-r)p_1\lambda(z-1)E \tag{43}$$

$$[z - r\overline{b_1}(\lambda - \lambda z)\overline{b_2}(\lambda - \lambda z)]B^{(1)}(0,z) = r\left[\overline{b_1}(\lambda - \lambda z)\, A^{(1)}(0,z) + \overline{b_2}(\lambda - \lambda z)\, A^{(2)}(0,z)\right] + r\lambda(z-1)E \qquad (44)$$

We shall now solve equations (42), (43) and (44) simultaneously for $A^{(1)}(0,z)$, $A^{(2)}(0,z)$ and $B^{(1)}(0,z)$. In matrix form these can be written as

$$\begin{bmatrix} g_1 & -(1-r)p_1\bar{b}_2(\lambda-\lambda z) & -(1-r)p_1\bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z) \\ -(1-r)p_2\bar{b}_1(\lambda-\lambda z) & g_2 & -(1-r)p_2\bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z) \\ -r\bar{b}_1(\lambda-\lambda z) & -r\bar{b}_2(\lambda-\lambda z) & g_3 \end{bmatrix}\begin{bmatrix} A^{(1)}(0,z) \\ A^{(2)}(0,z) \\ B^{(1)}(0,z) \end{bmatrix}$$

$$= \begin{bmatrix} (1-r)p_1\lambda(z-1)E \\ (1-r)p_2\lambda(z-1)E \\ r\lambda(z-1)E \end{bmatrix}, \qquad (45)$$

where $g_1 = z - (1-r)p_1\overline{b_1}(\lambda - \lambda z)$

$g_2 = z - (1-r)p_2\overline{b_2}(\lambda - \lambda z)$

$g_3 = z - r\overline{b_1}(\lambda - \lambda z)\overline{b_2}(\lambda - \lambda z)$.

Then using Cramer's Rule equation (45) yields

$$A^{(1)}(0,z) = \frac{N^{(1)}(z)}{D(z)}, \qquad (46)$$

$$A^{(2)}(0,z) = \frac{N^{(2)}(z)}{D(z)}, \qquad (47)$$

$$B^{(1)}(0,z) = \frac{N^{(3)}(z)}{D(z)}, \qquad (48)$$

where

$$\begin{aligned} N^{(1)}(z) = \lambda(z-1)E \Big\{ & (1-r)p_1\left[g_2g_3 - r(1-r)p_2\bar{b}_1(\lambda-\lambda z)(\bar{b}_2(\lambda-\lambda z))^2\right] \\ & + (1-r)p_2\left[g_3(1-r)p_1\bar{b}_2(\lambda-\lambda z) + r(1-r)p_1\bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z)\right] \\ & + r\left[(1-r)^2 p_1p_2\bar{b}_1(\lambda-\lambda z)\bar{b}_2^2(\lambda-\lambda z) + g_2(1-r)p_1\bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z)\right]\Big\}, \end{aligned} \qquad (49)$$

$$\begin{aligned} N^{(2)}(z) = \; & \\ \lambda(z-1)E \Big\{ & (1-r)p_1\left|g_3(1-r)p_2\bar{b}_1(\lambda-\lambda z) + r(1-r)p_2\bar{b}_2(\lambda-\lambda z)(\bar{b}_1(\lambda-\lambda z))^2\right| \\ & + (1-r)p_2\left|g_1g_3 - r(1-r)p_1(\bar{b}_1(\lambda-\lambda z))^2\bar{b}_2(\lambda-\lambda z)\right| \\ & + r\left|g_1(1-r)p_2\bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z) + (1-r)^2 p_1p_2(\bar{b}_1(\lambda-\lambda z))^2\bar{b}_2(\lambda-\lambda z)\right|\Big\}, \end{aligned} \qquad (50)$$

$$N^{(3)}(z) = \lambda(z-1)E\left\{(1-r)p_1\left[g_2 r\bar{b}_1(\lambda-\lambda z)+r(1-r)p_2\bar{b}_1(\lambda-\lambda z)(\bar{b}_2(\lambda-\lambda z)\right]\right.$$
$$+(1-r)p_2\left[g_1 r\bar{b}_2(\lambda-\lambda z)+r(1-r)p_1\bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z)\right]$$
$$\left.+r\left[g_1 g_2-(1-r)^2 p_1 p_2\bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z)\right]\right\}, \tag{51}$$

$$D(z)= g_1 g_2 g_3 - g_1 r(1-r)p_2\bar{b}_1(\lambda-\lambda z)(\bar{b}_1(\lambda-\lambda z))^2$$
$$- g_2 r(1-r)p_1(\bar{b}_1(\lambda-\lambda z))^2\bar{b}_1(\lambda-\lambda z)$$
$$- g_3(1-r)^2 p_1 p_2\bar{b}_1(\lambda-\lambda z))\bar{b}_2(\lambda-\lambda z)$$
$$-2r(1-r)^2 p_1 p_2(\bar{b}_1(\lambda-\lambda z))^2(\bar{b}_2(\lambda-\lambda z))^2, \tag{52}$$

Then using (48) in (41), we obtain

$$B^{(2)}(0,z) = \bar{b}_1(\lambda-\lambda z)\,\frac{N^{(3)}(z)}{D(z)}. \tag{53}$$

Now, substituting for $A^{(1)}(0,z)$, $A^{(2)}(0,z)$, $B^{(1)}(0,z)$ and $B^{(2)}(0,z)$ from equations (46), (47), (48) and (53) into equations (30) to (33) respectively, we obtain

$$A^{(1)}(z) = \left(\frac{1-\bar{b}_1(\lambda-\lambda z)}{\lambda-\lambda z}\right)\left(\frac{N^{(1)}(z)}{D(z)}\right), \tag{54}$$

$$A^{(2)}(z) = \left(\frac{1-\bar{b}_2(\lambda-\lambda z)}{\lambda-\lambda z}\right)\left(\frac{N^{(2)}(z)}{D(z)}\right), \tag{55}$$

$$B^{(1)}(z) = \left(\frac{1-\bar{b}_1(\lambda-\lambda z)}{\lambda-\lambda z}\right)\left(\frac{N^{(3)}(z)}{D(z)}\right), \tag{56}$$

$$B^{(2)}(z) = \bar{b}_1(\lambda-\lambda z)\left(\frac{1-\bar{b}_2(\lambda-\lambda z)}{\lambda-\lambda z}\right)\left(\frac{N^{(3)}(z)}{D(z)}\right). \tag{57}$$

In order to completely determine all the pgfs, we need to determine the only unknown constant E which appears in the numerators of the right hand sides of each of the equations (54) to (56). For that purpose, we shall make use of the normalizing condition

$$A^{(1)}(1)+A^{(2)}+B^{(1)}(1)+B^{(2)}(1)+E=1 \tag{58}$$

However, we note that at z=1, each of the above pgf is indeterminate of the 0/0 form. Therefore, using L'Hopital's rule and simplifying a lot of algebra, we obtain

$$A^{(1)}(1)=\lim_{z\to 1}A^{(1)}(z)=\lim_{z\to 1}\left(\frac{1-\bar{b}_1(\lambda-\lambda z)}{\lambda-\lambda z}\right)\left(\frac{N^{(1)}(z)}{D(z)}\right)$$

$$=\frac{\lambda(1-r)p_1E(V_1)E}{1-\lambda E(V_1)[r^2+(1-r)(r+p_1)]-E(V_2)[r^2+(1-r)(r+p_2)]}, \tag{59}$$

which is steady state probability that type 1 service is being provided to a customer who opted for only that type of service.

$$A^{(2)}(1)=\lim_{z\to 1}A^{(2)}(z)=\lim_{z\to 1}\left(\frac{1-\bar{b}_2(\lambda-\lambda z)}{\lambda-\lambda z}\right)\left(\frac{N^{(2)}(z)}{D(z)}\right)$$

$$=\frac{\lambda(1-r)p_2E(V_2)E}{1-\lambda E(V_1)[r^2+(1-r)(r+p_1)]-E(V_2)[r^2+(1-r)(r+p_2)]}, \tag{60}$$

which is steady state probability that type 2 service is being provided to a customer who opted for only that type of service.

$$B^{(1)}(1)=\lim_{z\to 1}B^{(1)}(z)=\lim_{z\to 1}\left(\frac{1-\bar{b}_1(\lambda-\lambda z)}{\lambda-\lambda z}\right)\left(\frac{N^{(3)}(z)}{D(z)}\right)$$

$$=\frac{\lambda rE(V_1)E}{1-\lambda E(V_1)[r^2+(1-r)(r+p_1)]-E(V_2)[r^2+(1-r)(r+p_2)]}, \tag{61}$$

which is steady state probability that type 1 service is being provided to a customer who opted for sequential service.

$$B^{(2)}(1) = \lim_{z\to 1} B^{(2)}(z) = \lim_{z\to 1} \bar{b}_1(\lambda - \lambda z)\left(\frac{1-\bar{b}_2(\lambda - \lambda z)}{\lambda - \lambda z}\right)\left(\frac{N^{(3)}(z)}{D(z)}\right)$$

$$= \frac{\lambda r E(V_2)E}{1-\lambda E(V_1)[r^2+(1-r)(r+p_1)] - E(V_2)[r^2+(1-r)(r+p_2)]}, \qquad (62)$$

which is steady state probability that type 2 service is being provided to a customer who opted for sequential service.

We note that $E(V_1)$ and $E(V_2)$ appearing in the numerators of the right hand sides of equations (59), (60), (61) and (62) are the mean service times of the first and second service respectively.

Then using (59) to (62) into the normalizing condition (58) and simplifying we obtain

$$E = 1-\lambda E(V_1)[r^2+(1-r)(r+p_1)] - E(V_2)[r^2+(1-r)(r+p_2)] \qquad (63)$$

We note that the stability condition under which the steady state shall exist has emerged from (63). This condition is

$$\lambda E(V_1)[r^2+(1-r)(r+p_1)] + E(V_2)[r^2+(1-r)(r+p_2)] < 1 \qquad (64)$$

Finally, substituting the value of E from (63) into $N^{(1)}(z), N^{(2)}(z)$ and $N^{(3)}(z)$, all the pgfs $A^{(1)}(z)$, $A^{(2)}(z)$, $B^{(1)}(z)$ and $B^{(2)}(z)$, found in (54) to (57), are now completely and explicitly determined.

6. Special cases

Case 1 : All customers opt for the sequential service ($M/(G_1,G_2)/1$ queue)

In this case, we let r=1 in the main results. Consequently, $g_1 = g_2 = z$ and $g_3 = z - \bar{b}_1(\lambda - \lambda z)\bar{b}_2(\lambda - \lambda z)$ and hence with these substitutions, we have from equations (49), (50), (51) and (52)

$$N^{(1)}(z) = N^{(2)}(z) = 0, \tag{65}$$

$$N^{(3)}(z) = \lambda(z-1)Eg_1g_2 = \lambda(z-1)z^2E, \tag{66}$$

$$D(z) = g_1g_2g_3 = z^2\big((z - \bar{b}_1(\lambda - \lambda z)\bar{b}_2(\lambda - \lambda z)\big). \tag{67}$$

And hence from (54), (55), (56) and (57) we obtain

$$A^{(1)}(z) = A^{(2)}(z) = 0, \tag{68}$$

$$B^{(1)}(z) = \left(\frac{1 - \bar{b}_1(\lambda - \lambda z)}{\lambda - \lambda z}\right)\left(\frac{\lambda(z-1)z^2E}{z^2(z - \bar{b}_1(\lambda - \lambda z)\bar{b}_2(\lambda - \lambda z)}\right)$$

$$= \frac{(\bar{b}_1(\lambda - \lambda z) - 1)E}{z^2\big(z - \bar{b}_1(\lambda - \lambda z)\bar{b}_2(\lambda - \lambda z)\big)}, \tag{69}$$

$$B^{(2)}(z) = \bar{b}_1(\lambda - \lambda z)\left(\frac{\bar{b}_2(\lambda - \lambda z) - 1}{z - \bar{b}_1(\lambda - \lambda z)\bar{b}_2(\lambda - \lambda z)}\right)E, \tag{70}$$

where we have from (63)

$$E = 1 - \lambda E(V_1) - \lambda E(V_2). \tag{71}$$

Case 2. Each customer opts for only one of the two types of service $\left(M/\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}/1\right.$ queue)

In this case we let r=0 in the main results. Consequently,

$$N^{(1)}(z) = \lambda(z-1)E\big(p_1g_2g_3 + p_1p_2g_3\bar{b}_2(\lambda - \lambda z)\big), \tag{72}$$

$$N^{(2)}(z) = \lambda(z-1)E\left(p_2 g_1 g_3 + p_1 p_2 g_3 \bar{b}_1(\lambda-\lambda z)\right), \tag{73}$$

$$N^{(3)}(z) = 0, \tag{74}$$

$$D(z) = g_1 g_2 g_3 - g_3 p_1 p_2 \bar{b}_1(\lambda - \lambda z)\bar{b}_2(\lambda - \lambda z), \tag{75}$$

where $g_1 = z - p_1\bar{b}_1(\lambda - \lambda z)$

$g_2 = z - p_2\bar{b}_2(\lambda - \lambda z)$

$g_3 = z$.

Then with these substitutions equations (54), (55), (56) and (57) yield

$$A^{(1)}(z) = \frac{\left(\bar{b}_1(\lambda-\lambda z)-1\right)\left(p_1 g_2 g_3 + p_1 p_2 g_3 \bar{b}_2(\lambda-\lambda z)\right)E}{g_1 g_2 g_3 - g_3 p_1 p_2 \bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z)}$$

$$= \frac{p_1\left(\bar{b}_1(\lambda-\lambda z)-1\right)\left\{\left(z-p_2\bar{b}_2(\lambda-\lambda z)\right)+p_2\bar{b}_2(\lambda-\lambda z)\right\}E}{\left(z-p_1\bar{b}_1(\lambda-\lambda z)\right)\left(z-p_2\bar{b}_2(\lambda-\lambda z)\right)-p_1p_2\bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z)}, \tag{76}$$

$$A^{(2)}(z) = \frac{\left(\bar{b}_2(\lambda-\lambda z)-1\right)\left(p_2 g_1 g_3 + p_1 p_2 g_3 \bar{b}_1(\lambda-\lambda z)\right)E}{g_1 g_2 g_3 - g_3 p_1 p_2 \bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z)},$$

$$= \frac{p_2\left(\bar{b}_2(\lambda-\lambda z)-1\right)\left\{\left(z-p_1\bar{b}_1(\lambda-\lambda z)\right)+p_1\bar{b}_1(\lambda-\lambda z)\right\}E}{\left(z-p_1\bar{b}_1(\lambda-\lambda z)\right)\left(z-p_2\bar{b}_2(\lambda-\lambda z)\right)-p_1p_2\bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z)}, \tag{77}$$

$$B^{(1)}(z) = 0 = B^{(2)}(z), \tag{78}$$

where from (63), now we have

$$E = 1 - \lambda p_1 E(V_1) - \lambda p_2 E(V_2)\,. \tag{79}$$

Case 3. Each customer opts for only the first service (M/ G/1 queue)

In this case, we put r=0, $p_1 = 1$ and $p_2 = 0$ in the main results. Consequently, we have $g_1 = z - \bar{b}_1(\lambda - \lambda z)$, $g_2 = g_3 = z$. Then with these substitutions, we obtain

$$A^{(2)}(z) = B^{(1)}(z) = B^{(2)} = 0 \tag{80}$$

$$A^{(1)}(z) = \frac{\left(\bar{b}_1(\lambda - \lambda z) - 1\right)E}{z - \bar{b}_1(\lambda - \lambda z)}, \tag{81}$$

where $E = 1 - \lambda E(V_1)$.

The result (81) is known result for the M/G/1 queue.

References

Bhat, U.N. (1984). *Elements of Applied Stochastic Processes*, John Wiley and Sons, New York.

Cohen, J.W. (1982). *The Single Server Queue*, 2nd ed., Elsevier, North Holland, New York.

Gross, C. and Harris, C.M. (1985). *The Fundamentals of Queueing Theory*, John Wiley & Sons, New York.

Kashyap, B.R.K. and Chaudhry, M.L. (1988). *An Introduction to Queueing Theory*, A & A publications, Ontario, Canada.

Lee, A.M. (1966). *Applied Queueing Theory*, Macmillan, London.

Madan, K.C. (1992). An a M/M/1 queueing system with additional optional service in batches', *IAPQR Transactions, Journal of the Indian Association of Productivity, Quality and Reliability*, 17(1), 23-33.

Madan, K.C. (1994). An M/G/1 queueing system with additional optional service and no waiting capacity, *Microelectronics and Reliability*, 34(3), 521-527.

Madan, Kailash. C. (2000). An M/G/1 queue with second optional service, *Queueing Systems*, 34, 37-46.

Medhi, J., 1982, *Stochastic Processes*, Wiley Eastern.