The Exponentiated Linear Exponential Geometric Distribution:with Application

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Abstract

This paper provides a new generalization of the Generalized exponential geometric distribution that introduced by Bidram et al. [6]. The new distribution is referred to as exponentiated linear exponential geometric distribution (ELEGD). The new model contains life time distributions as special cases such as new generalization of the Generalized exponential geometric(NGEG)، linear falire rate and exponential distributions, among others.The properties of the new model are discussed and the maximum likelihood estimation is used to evaluate the parameters. Explicit expressions are derived for the moments and examine the order statistics. This model is capable of modeling various shapes of aging and failure criteria.

Keywords: exponentiated, Reliability Function, Order Statistics, Maximum Likelihood Estimation.

1. Introduction

The Linear Exponential distribution LED has many applications in applied statistics and reliability analysis. Broadbent[7],uses the (LED) to describe the service of milk bottles that are filled in a dairy, circulated to customers, and returned empty to the dairy. The Linear exponential model was also used by Carbone et al. [8], to study the survival pattern of patients with plasmacytic myeloma. The linear exponential distribution is also known as the Linear Failure Rate distribution, having exponential and Rayleigh distributions as special cases, is a very well-knowndistribution for modeling lifetime data in reliability and medical studies. It is also models phenomena with increasing failure rate.

Merovci and Elbatal [13], introduced the Kumaraswamy linear exponential distribution. Bidram et al. [6], introducedgeneralized exponential geometric distribution. Ebraheim [8], introduced exponentiated transmuted Weibull distribution.

Louzada et al. [12], introduced the complementary exponential geometric distribution, which is complementary to the exponential geometric model proposed by Adamidis and Loukas [2], based on a complementary risk problem (Basu and Klein[5], in presence of latent risks, in the sense that there is no information about which factor was responsible for the component failure but only the maximum lifetime value among all risks is observed. Louzada et al. [13], introduced complementary exponentiated exponential geometric distribution which considered a generalization to the complementary exponential geometric distribution. Tojeiro et al. [19], introduced the complementary Weibull geometric (CWG) as a complementary distribution to the Weibull geometric (WG) model proposed by Barreto-Souza et al. [4].

A random variable X is said to have the linear exponential distribution with two parameters λ and β, if it has the cumulative distribution function

$$F(x,\lambda,\beta) = 1 - \left(e^{-\left(\lambda x + \frac{\beta}{2}x^2\right)}\right), \quad x > 0, \lambda, \beta > 0. \tag{1}$$

and the corresponding probability density function (pdf) is given by

$$f(x,\lambda,\beta) = (\lambda + \beta x)\left(e^{-\left(\lambda x + \frac{\beta}{2}x^2\right)}\right), \quad x > 0, \lambda, \beta > 0. \tag{2}$$

In this paper we provide mathematical formulation of the exponentiated linear exponential geometric distribution (ELEGD)and some of its properties. The rest of the paper is organized as follows. In Section 2 we demonstrate the subject distribution. In Section 3, we find the reliability function, hazard rate and cumulative hazard rate of the subject model. The Expansion for the pdf and the cdf Functions is derived in Section 4. In section 5, The statistical properties include quantile functions, median , moments and moments generating function are given,. In Section 6, order statisticsare discussed . In Section 7, we introduce the method of likelihood estimation as point estimation and the confidence interval as an interval estimation of the unknown parameters. Finally, we fit the distribution to two real data sets to examine it and to suitability it with nested and non-nested models.

2. Exponentiated Linear Exponential Geometric Distribution (ELEGD)

In this section,we propose the exponentiated linear exponential geometric distribution (ELEGD). The exponentiated linear exponential distribution with parameters λ, β and α if its cumulative distribution function (cdf) is defined as

$$F(x,\lambda,\beta,\alpha) = \left\{1 - \left(e^{-\left(\lambda x + \frac{\beta}{2}x^2\right)}\right)\right\}^{\alpha}, \quad x > 0, \lambda, \beta, \alpha > 0. \tag{3}$$

And the pdf

$$f(x,\lambda,\beta,\alpha) = \alpha(\lambda + \beta x)\left(e^{-\left(\lambda x + \frac{\beta}{2}x^2\right)}\right)\left(1 - e^{-\left(\lambda x + \frac{\beta}{2}x^2\right)}\right)^{\alpha - 1}, x, \lambda, \beta, \alpha > 0. \tag{4}$$

Now, let $X_1, X_2, \dots, X_n$be N iid random random variables from the GE distribution, where N has a geometric distribution with the probability mass function

$$P(N = n) = (1 - p)p^{n-1} \qquad n = 1,2,3, \dots \tag{5}$$

and is independent of $X_i's$. If we define $V = max\{X_i\}_{i=1}^{N}$, then by a conditional argument, the pdf of V is given by,

$$f_V(x,\lambda,\beta,\alpha,p) = \frac{(1-p)\alpha(\lambda+\beta x)\left(e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha-1}}{\left[1-p\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha}\right]^2},\ x>0, \qquad (6)$$

the cdf of is given by

$$F_V(x,\lambda,\beta,\alpha,p) = \frac{(1-p)\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha}}{1-p\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha}},\ x>0. \qquad (7)$$

Where $\lambda,\beta,\alpha > 0$ and $p \in (0,1)$. The random variable V with the density function (6) is said to have a exponentiated linear exponential geometric distribution (ELEGD).

Figures 1 and 2 illustrates some of the possible shapes of the pdf and cdf of the ELEGD distribution for selected values of the parameters λ,β,α and p respectively.

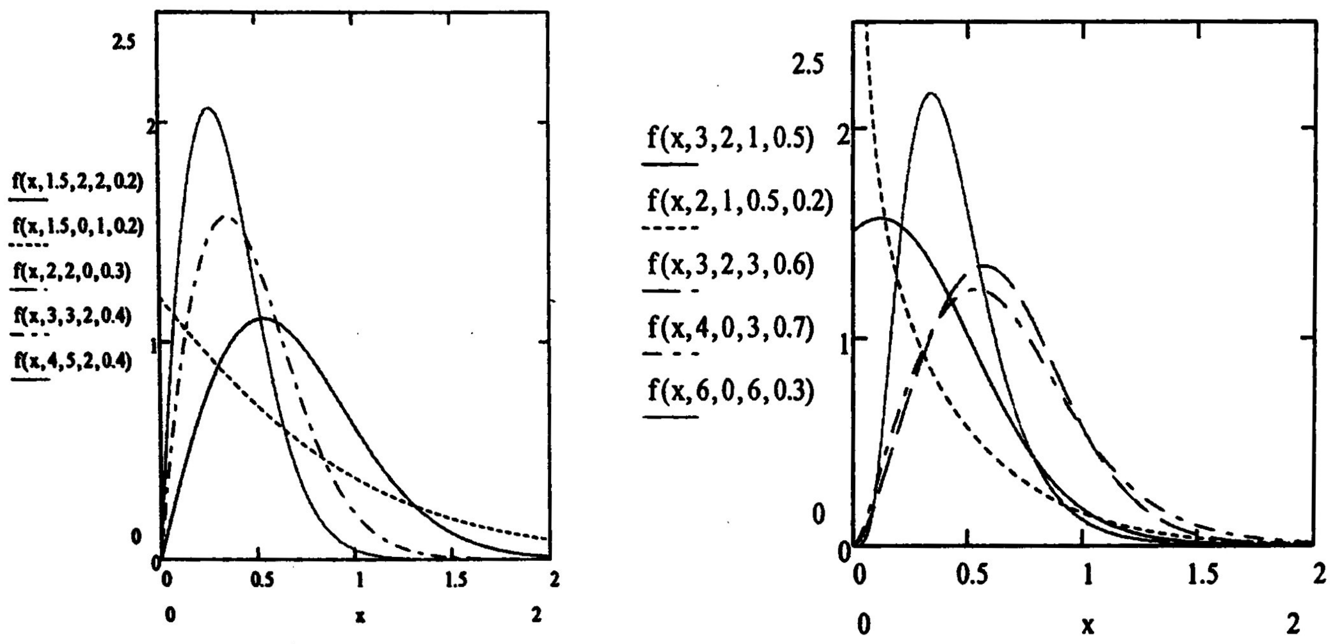

Figure 1: Probability Density Function of the ELEGD.

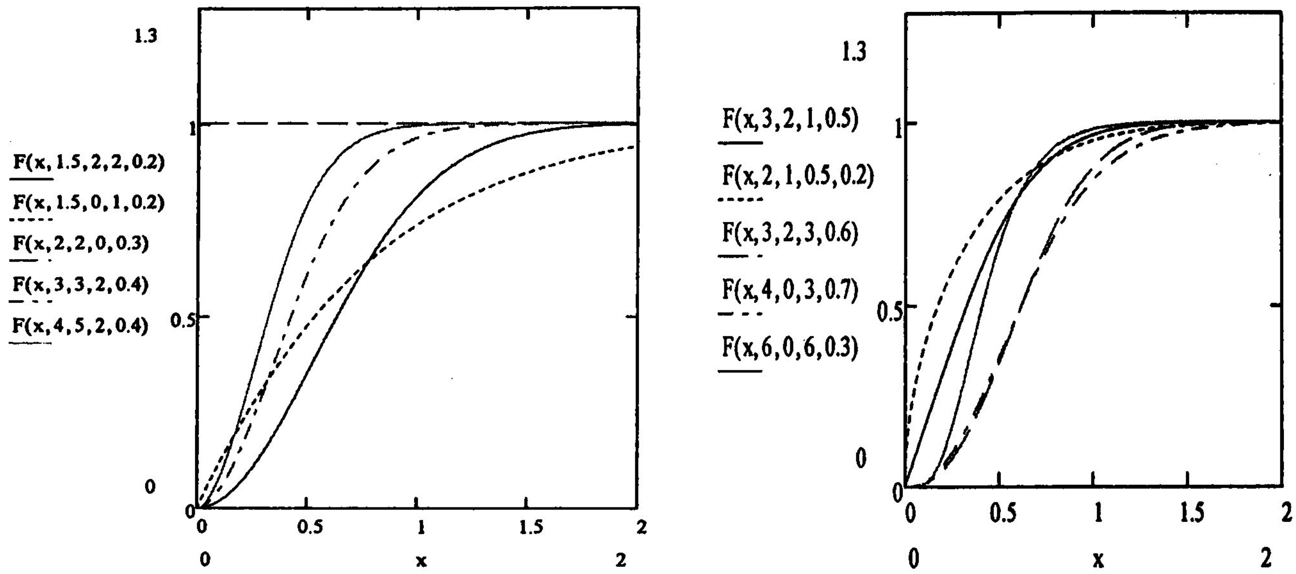

Figure 2: Distribution Function of the ELEGD.

3. Reliability Analysis

The characteristics in reliability analysis which are the reliability function (RF), the hazard rate function (HF) and the cumulative hazard rate function (CHF) for the ELEGD are introduced in this section.

3.1 Reliability Function

The reliability function (RF) also known as the survival function, which is the probability of an item not failing prior to some time t, is defined by $R(x) = 1 - F(x)$. The reliability function of the ELEGD denoted by $R_{\text{ELEGD}}(x, \lambda, \beta, \alpha, p)$, can be a useful characterization of life time data analysis. It can be defined as

$$R_{\text{ELEGD}}(\mathrm{x}, \lambda, \beta, \alpha, p) = 1 - F_{\text{ELEGD}}(\mathrm{x}, \lambda, \beta, \alpha, p),$$

the survival functionis given by

$$R_{\text{ELEGD}}(\mathrm{x}, \lambda, \beta, \alpha, p) = \frac{1-\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha}}{1-p\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha}}, \; x > 0. \qquad (8)$$

Figure 3 illustrates the pattern of the exponentiated linear exponential geometric distribution (ELEGD) reliability function with different choices of parameters λ, β, α and p.

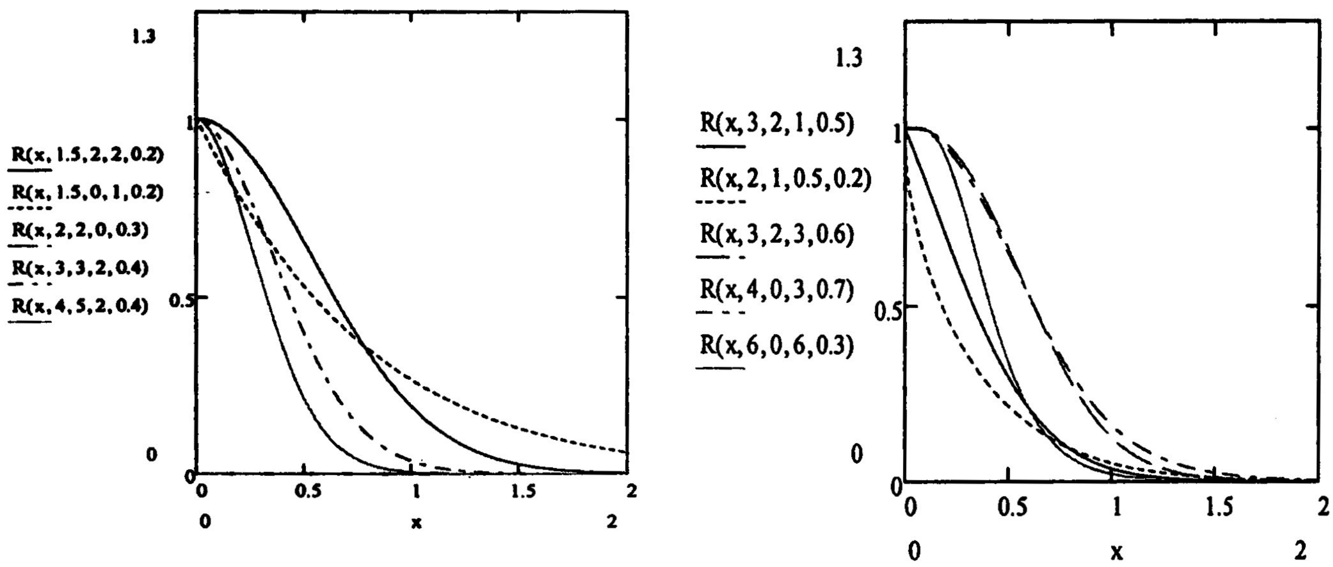

Figure 3: Reliability Function of theELEGD.

3.2 Hazard Rate Function

The other characteristic of interest of a random variable is the hazard rate function (HF).The exponentiated linear exponential geometric distribution also known as instantaneous failure rate denoted by $h_{\text{ELEGD}}(x)$, is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to the time t. The HF of the exponentiated linear exponential geometric distribution is defined by $h_{\text{ELEGD}}(x, \lambda, \beta, \alpha, p) = f_{\text{ELEGD}}(x, \lambda, \beta, \alpha, p)/R_{\text{ELEGD}}(x, \lambda, \beta, \alpha, p)$,

$$h_{\text{ELEGD}}(x, \lambda, \beta, \alpha, p) = \frac{(1-p)\alpha(\lambda+\beta x)\left(e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha-1}}{\left[1-p\left\{\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha}\right\}\right]\left\{1-\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha}\right\}}, \quad x > 0, \quad (9)$$

Figure 4 illustrates some of the possible shapes of the hazard rate function of the exponentiated linear exponential geometric distribution for different values of the parametersλ, β, α and p.

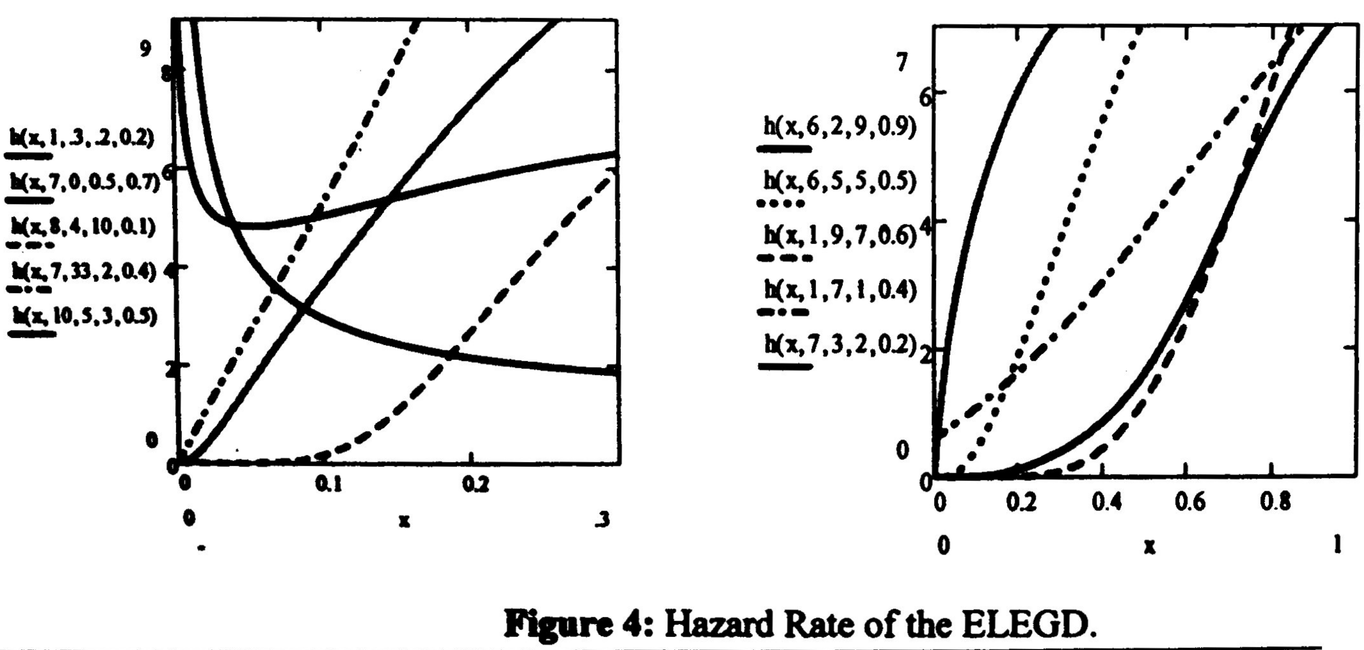

Figure 4: Hazard Rate of the ELEGD.

3.3 Cumulative Hazard Rate Function

The Cumulative hazard function (CHF) of exponentiated linear exponential geometric, denoted by $H_{\text{ELEGD}}(\mathrm{x},\lambda,\beta,\alpha,p)$, is defined as $H_{\text{ELEGD}}(\mathrm{x},\lambda,\beta,\alpha,p) = \int_0^x h_{\text{ELEGD}}(\mathrm{x},\lambda,\beta,\alpha,p)dx = -\ln R_{\text{ELEGD}}(\mathrm{x},\lambda,\beta,\alpha,p)$,

$$H_{\text{ELEGD}}(\mathrm{x},\lambda,\beta,\alpha,p) = -\ln R_{\text{ELEGD}}(\mathrm{x},\lambda,\beta,\alpha,p) = \ln\left[\frac{1-p\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha}}{1-\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha}}\right],$$

$$x>0. \qquad (10)$$

Figure 5. illustrates some of the possible shapes of the cumulative hazard rate of the exponentiated linear exponential geometric for different values of the parametersλ,β,α and p.

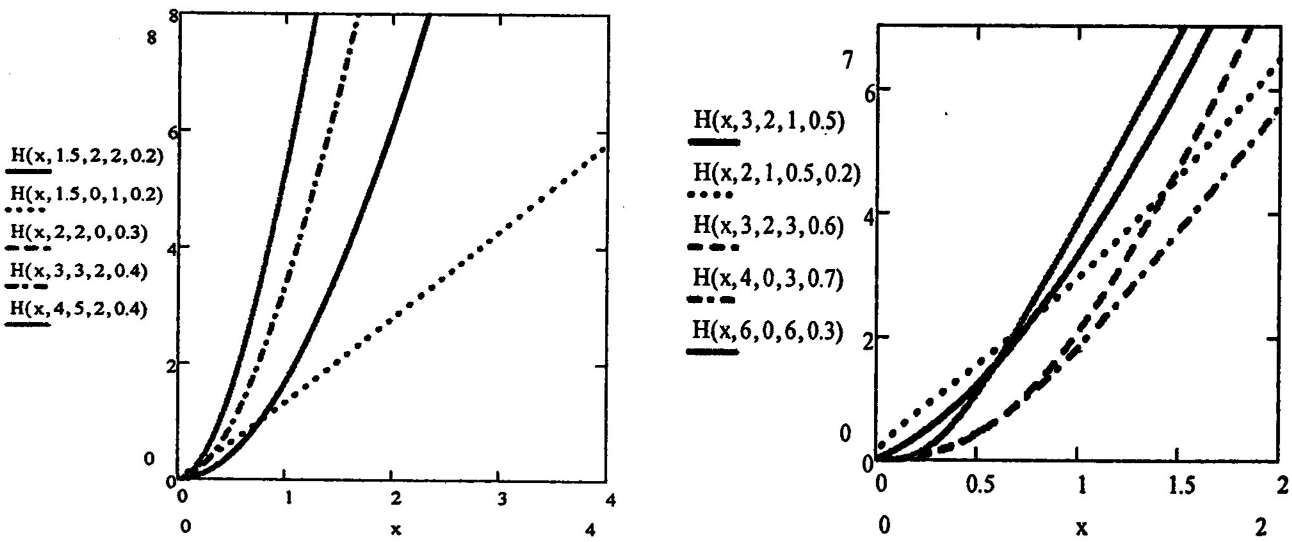

Figure 5: Cumulative Hazard Rate of the ELEGD.

4. Expansion for the pdf and the cdf Functions

In this section we introduced another expressions for the pdf and the cdf functions using.

The Maclaurin expansion to simplifying the pdf and the cdf forms.

٤.١ Expansion for the pdf Function

From equation (6) and using the expansion

$$(1-z)^{-k} = \sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{\Gamma(k)j!}z^j. \qquad (11)$$

Which holds for $|z|<1$ and $k>0$. Using (11) in Equation.(6) yields,

$$f_V(x,\lambda,\beta,\alpha,p) = \sum_{j,i,m=0}^{\infty} \frac{(-1)^{i+m}\Gamma(j+2)!\,\Gamma(\alpha)}{m!\,i!\,\Gamma((j-i+1))\Gamma(\alpha-m)}$$

$$*w_j\,\alpha(\lambda+\beta x)\left(e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)(i\alpha+m+1)}\right), x>0,.(12)$$

Therefore

$$f_V(x,\lambda,\beta,\alpha,p) = \sum_{j,i,m,k=0}^{\infty} \frac{(-1)^{i+m+k-1}\Gamma(j+2)\Gamma(\alpha)}{m!\,i!\,k!\,\Gamma((j-i+1))\Gamma(\alpha-m)} w_j\,\alpha(\lambda+\beta x)$$

$$*\left[\left(\lambda x+\frac{\beta}{2}x^2\right)(i\alpha+m+1)\right]^k, x>0.(13)$$

Where $w_j = (1-p)p^j$.

4.2 Expansion for the cdf Function

Using expansion (11) to Equation (7) ,then the cdf function of the exponentiated linear exponential geometric can be written as:

$$F_V(x,\lambda,\beta,\alpha,p) = \sum_{j,i,m=0}^{\infty} \frac{(-1)^{i+m}\Gamma(j+1)\Gamma(\alpha+1)}{\Gamma(j+1-i)\Gamma(\alpha-m+1)} w_j\,\alpha\left(e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)(i\alpha+m)}\right)$$

Therefore

$$F_V(x,\lambda,\beta,\alpha,p) = \sum_{j,i,m,k=0}^{\infty} \frac{(-1)^{i+m+k-1}\Gamma(j+1)\Gamma(\alpha+1)}{i!\,m!\,k!\,\Gamma(j+1-i)\Gamma(\alpha-m+1)} w_j\,\alpha$$

$$*\left[\left(\lambda x+\frac{\beta}{2}x^2\right)(i\alpha+m)\right]^k, x>0.(14)$$

5. Statistical properties

In this section we discuss few statistical properties of the ELEGD distribution.

5.1 Quantiles and median

The quantile function is obtained by inverting the cumulative distribution (14), where the p-th quantile x_p of the ELEGD model is the real solution of the following equation:

$$\sum_{j,i,m,k=0}^{\infty} \frac{(-1)^{i+m+k-1}\Gamma(j+1)\Gamma(\alpha+1)}{i!m!k!\Gamma(j+1-i)\Gamma(\alpha-m+1)} w_j\,\alpha(i\alpha+m)^k\left[\left(\lambda t_p+\frac{\beta}{2}{t_p}^2\right)\right]^k - p = 0$$

$$,\quad 0\le q\le 1$$

An expansion for the median M follows by taking $p = 0.5$.

5.2 Central and Non-Central Moments

The r^{th}non-central moments or (moments about the origin) of the ELEGD under using equation (12) is given by:

$$\mu_r' = E(X^r) = \sum_{j,i,m,k=0}^{\infty} \Delta_{j:k} \times \left[\frac{\lambda\Gamma(2k+r+1)}{(\lambda(ia+m+1))^{2k+r+1}} + \frac{\beta\Gamma(2k+r+2)}{(\lambda(ia+m+1))^{2k+r+2}}\right] \quad (15)$$

where $\Delta_{j:m}$ is a constant term given by

$$\Delta_{j:k} = \frac{(-1)^{i+m+k}\Gamma(j+2)\Gamma(\alpha)(i\alpha+m+1)^k\beta^k}{m!\,i!\,k!\,2^k(i+1)!\,\Gamma((j-i+1))\Gamma(\alpha-m)} w_j\,\alpha.$$

The n^{th}central moments or (moments about the mean) can be obtained easily from therth non-central moments throw the relation:

$$m_u = E(X-\mu)^n = \sum_{r=0}^{n} (-\mu)^{n-r} E(X^r).$$

Then the n^{th}central moments of the ELEGD is given by:

$$m_u = \sum_{j,i,m,k,r=0}^{\infty} (-\mu)^{n-r}\Delta_{j:k} \times \left[\frac{\lambda\Gamma(2k+r+1)}{(\lambda(ia+m+1))^{2k+r+1}} + \frac{\beta\Gamma(2k+r+2)}{(\lambda(ia+m+1))^{2k+r+2}}\right]. \quad (16)$$

5.3The Moment Generating Function

The moment generating function,$M_x(t)$ can be easily obtained from the r^{th} non-central moment through the relation

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu'_r.$$

Then, the moment generating function of the ELEG distribution is given by,

$$M_x(t) = \sum_{j,i,m,k,r=0}^{\infty} \frac{t^r}{r!}\Delta_{j:k} \times \left[\frac{\lambda\Gamma(2k+r+1)}{(\lambda(ia+m+1))^{2k+r+1}} + \frac{\beta\Gamma(2k+r+2)}{(\lambda(ia+m+1))^{2k+r+2}}\right]$$

6. Order Statistics

The order statistics and their moments have great importance in many statistical problems and they have many applications in reliability analysis and life testing. The order statistics arise in the study of reliability of a system. The order statistics can represent the lifetimes of units or components of a reliability system. Let

$Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from the ELEGD($(y,\lambda,\beta,\alpha,p)$) with cumulative distribution function (cdf), and the corresponding probability density function (pdf), as in (6) and (7), respectively. Let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ be the corresponding order statistics. Then the pdf of $Y_{(r:n)}$, $1 \le r \le n$, denoted by $f_{r:n}(y)$, is given by,

$$f_{r:n}(y) = C_{r:n} f_{TCWG}(y,\alpha,\beta,\gamma,\delta)[F_{TCWG}(y,\alpha,\beta,\gamma,\delta)]^{r-1}[R_{TCWG}(y,\alpha,\beta,\gamma,\delta)]^{n-r},$$

$$f_{r:n}(y) = C_{r:n} \frac{(1-p)\alpha(\lambda+\beta y)\left(e^{-\left(\lambda y+\frac{\beta}{2}y^2\right)}\right)\left(1-e^{-\left(\lambda y+\frac{\beta}{2}y^2\right)}\right)^{\alpha-1}}{\left(1-p\left\{\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha}\right\}\right)^2} \times$$

$$\left(\frac{(1-p)\left(1-e^{-\left(\lambda y+\frac{\beta}{2}y^2\right)}\right)^{\alpha}}{1-p\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha}}\right)^{r-1} \times$$

$$\left(\frac{1-\left(1-e^{-\left(\lambda y+\frac{\beta}{2}y^2\right)}\right)^{\alpha}}{1-p\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha}}\right)^{n-r}, \quad (17)$$

where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$.

Therefore, the pdf of the largest order statistic Y_n is given by:

$$f_{y_n}(y) = \frac{n(1-p)\alpha(\lambda+\beta y)\left(e^{-\left(\lambda y+\frac{\beta}{2}y^2\right)}\right)\left(1-e^{-\left(\lambda y+\frac{\beta}{2}y^2\right)}\right)^{\alpha-1}}{\left(1-p\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha}\right)^2} \times$$

$$\left(\frac{(1-p)\left(1-e^{-\left(\lambda y+\frac{\beta}{2}y^2\right)}\right)^{\alpha}}{1-p\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha}}\right)^{n-1}.$$

While, the pdf of the smallest order statistic Y_1 is given by:

$$f_{y_1}(y) = \frac{n(1-p)\alpha(\lambda+\beta y)\left(e^{-\left(\lambda y+\frac{\beta}{2}y^2\right)}\right)\left(1-e^{-\left(\lambda y+\frac{\beta}{2}y^2\right)}\right)^{\alpha-1}}{\left(1-p\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha}\right)^2} \times$$

$$\left(\frac{1-\left(1-e^{-\left(\lambda y+\frac{\beta}{2}y^2\right)}\right)^{\alpha}}{1-p\left(1-e^{-\left(\lambda x+\frac{\beta}{2}x^2\right)}\right)^{\alpha}}\right)^{n-1}.$$

7. Estimation of the Parameters

In this section, we introduce the method of likelihood to estimate the parameters involved and use them to create confidence intervals for the unknown parameters.

The maximum likelihood estimators (MLEs) for the parameters of the exponentiated linear exponential geometric distribution (ELEGD)$(x,\lambda,\beta,\alpha,p)$ is discussed in this section. Consider the random sample $x_1,x_2,\dots,x_n$ of size n fromELEGD$(x,\lambda,\beta,\alpha,p)$ with probability density function in (6), then the likelihood function can be expressed as follows

$$L(x_1,x_2,\dots,x_n,\lambda,\beta,\alpha,p)=\prod_{i=1}^{n} f_{\text{ELEGD}}(x_i,\lambda,\beta,\alpha,p),$$

$$L=\frac{((1-p)\alpha)^n\prod_{i=1}^{n}(\lambda+\beta x_i)\left(e^{-\left(\lambda x_i+\frac{\beta}{2}x_i^2\right)}\right)\left(1-e^{-\left(\lambda x_i+\frac{\beta}{2}x_i^2\right)}\right)^{\alpha-1}}{\prod_{i=1}^{n}\left(1-p\left(1-e^{-\left(\lambda x_i+\frac{\beta}{2}x_i^2\right)}\right)^{\alpha}\right)^2}.$$

Then, the log-likelihood function $\Psi=\ln L$ becomes:

$$\Psi=n(\ln(1-p)+\ln\alpha)+\sum_{i=1}^{n}\ln(\lambda+\beta x_i)-\sum_{i=1}^{n}\ln\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)$$
$$+(\alpha-1)\sum_{i=1}^{n}\ln\left(1-e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}\right)-2\sum_{i=1}^{n}\ln\left(1-p\left(1-e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}\right)^{\alpha}\right). \quad (18)$$

Differentiating Equation (18) with respect to λ,β,α and p then equating it to zero, we obtain the MLEs of λ,β,α and p as follows,

$$\frac{\partial\Psi}{\partial\lambda}=\sum_{i=1}^{n}\frac{1}{\lambda+\beta x_i}-\sum_{i=1}^{n}x_i+(\alpha-1)\sum_{i=1}^{n}\frac{x_i e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}}{\left(1-e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}\right)}$$
$$+2\alpha p\sum_{i=1}^{n}\frac{x_i e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}\left(1-e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}\right)^{\alpha-1}}{\left(1-p\left(1-e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}\right)^{\alpha}\right)}=0(19)$$

$$\frac{\partial\Psi}{\partial B}=\sum_{i=1}^{n}\frac{x_i}{\lambda+\beta x_i}-\sum_{i=1}^{n}\frac{x^2{}_i}{2}+\frac{(\alpha-1)}{2}\sum_{i=1}^{n}\frac{x^2{}_i e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}}{\left(1-e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}\right)}$$

$$+\alpha p\sum_{i=1}^{n}\frac{x^2{}_i e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}\left(1-e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}\right)^{\alpha-1}}{\left(1-p\left(1-e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}\right)^{\alpha}\right)}=0, (20)$$

$$\frac{\partial\Psi}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\ln\left(1-e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}\right)$$

$$+2p\sum_{i=1}^{n}\frac{\left(1-e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}\right)^{\alpha}\log\left(1-e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}\right)}{\left(1-p\left(1-e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}\right)^{\alpha}\right)}=0, (21)$$

$$\frac{\partial\Psi}{\partial p}=\frac{n}{1-p}+2\sum_{i=1}^{n}\frac{\left(1-e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}\right)^{\alpha}}{\left(1-p\left(1-e^{-\left(\lambda x_i+\frac{\beta}{2}x^2{}_i\right)}\right)^{\alpha}\right)}=0 \quad (22)$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above nonlinear system of Equations (19) - (22) to zero and solve them simultaneously. These solutions will yield the ML estimators $\hat{\lambda}, \hat{\beta}, \hat{\alpha}$, and $\hat{p}$. For the four parameters exponentiated linear exponential geometric distribution ELEGD$(x,\lambda,\beta,\alpha,p)pdf$ all the second order derivatives exist. Thus we have the inverse dispersion matrix is given by

$$\begin{pmatrix}\hat{\lambda}\\ \hat{\beta}\\ \hat{\alpha}\\ \hat{p}\end{pmatrix}\sim N\left[\begin{pmatrix}\lambda\\ \beta\\ \alpha\\ p\end{pmatrix},\begin{pmatrix}\hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} & \hat{V}_{14}\\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} & \hat{V}_{24}\\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} & \hat{V}_{34}\\ \hat{V}_{41} & \hat{V}_{42} & \hat{V}_{43} & \hat{V}_{44}\end{pmatrix}\right],$$

$$V^{-1}=-E\begin{pmatrix}V_{11} & V_{12} & V_{13} & V_{14}\\ V_{21} & V_{22} & V_{23} & V_{24}\\ V_{31} & V_{32} & V_{33} & V_{34}\\ V_{41} & V_{42} & V_{43} & V_{44}\end{pmatrix}. (23)$$

Equation (23) is the variance covariance matrix of theELEGD$(x,\lambda,\beta,\alpha,p)$where

$$V_{11}=\frac{\partial^2\Psi}{\partial\lambda^2}\quad V_{12}=\frac{\partial^2\Psi}{\partial\lambda\,\partial\beta}\quad V_{13}=\frac{\partial^2\Psi}{\partial\lambda\,\partial\alpha}\quad V_{14}=\frac{\partial^2\Psi}{\partial\lambda\,\partial p}$$

$$V_{22}=\frac{\partial^2\Psi}{\partial\beta^2}V_{23}=\frac{\partial^2\Psi}{\partial\beta\,\partial\alpha}V_{24}=\frac{\partial^2\Psi}{\partial\beta\,\partial p}$$

$$V_{33} = \frac{\partial^2 \Psi}{\partial \alpha^2} V_{34} = \frac{\partial^2 \Psi}{\partial \alpha \, \partial p}$$

$$V_{44} = \frac{\partial^2 \Psi}{\partial p^2}$$

By solving this inverse dispersion matrix, these solutions will yield the asymptotic variance and covariances of these MLEs for $\hat{\lambda}, \hat{\beta}, \hat{\alpha}$, and $\hat{p}$. Approximate $100(1-\phi)\%$ confidence intervals for λ, β, α and p can be determined as:

$$\hat{\lambda} \pm Z_{\frac{\phi}{2}}\sqrt{\hat{V}_{11}}, \hat{\beta} \pm Z_{\frac{\phi}{2}}\sqrt{\hat{V}_{22}}, \hat{\alpha} \pm Z_{\frac{\phi}{2}}\sqrt{\hat{V}_{33}} \text{and } \hat{p} \pm Z_{\frac{\phi}{2}}\sqrt{\hat{V}_{44}},$$

where $Z_{\frac{\phi}{2}}$ is the upper ϕth percentile of the standard normal distribution.

These non-linear can be routinely solved using Newton's method or fixed-point iteration techniques. The subroutines to solve non-linear optimization problem are available in R 18,software namely optim ( ), nlm ( ) and bbmle ( ) etc. We used nlm ( ) package for optimizing (18).

8. Applications

In this section, we use two real data sets to to see how the new model works in practice.compare the fits of the ELEG distribution with others models. In each case, the parameters are estimated by maximum likelihood as described in Section 7, using the R code.

8.1 Data Set 1

The first data set represents the ages for 155 patients of breast tumors taken from (June-November 2014), whose entered in (Breast Tumors Early Detection Unit, Benda Hospital University, Egypt).

In order to compare the two distribution models, we consider criteria like KS (Kolmogorov Smirnov),$-2\mathcal{L}$ AIC (Akaike information criterion), AICc (corrected Akaike information criterion), and BIC (Bayesian information criterion) for the data set. The better distribution corresponds to smaller KS,$-2\mathcal{L}$ AIC and AICc values:

$$\text{AIC} = -2\mathcal{L} + 2k,$$

$$\text{AIC}_C = -2\mathcal{L} + \left(\frac{2kn}{n-k-1}\right),$$

and

$$\text{BIC} = -2\mathcal{L} + k\log(n),$$

where $\mathcal{L}$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters, and n is the sample size.

Also, for calculating the values of KS we use the sample estimates of λ, β, α and p. Table 2 shows the parameter estimation based on the maximum likelihood and gives the values of the criteria AIC, AICc, BIC, and KS test. The values in Table 2 indicate that the ELEG distribution leads to a better fit over all the other models.

Table 1: the ages for 155 patients of breast tumors

46	32	50	46	44	42	69	31	25	29	40	42	24	17	35
48	49	50	60	26	36	56	65	48	66	44	45	30	28	40
40	50	41	39	36	63	40	42	45	31	48	36	18	24	35

30	40	48	50	60	52	47	50	49	38	30	52	52	12	48
50	45	50	50	50	53	55	38	40	42	42	32	40	50	58
48	32	45	42	36	30	28	38	54	90	80	60	45	40	50
50	40	50	50	50	60	39	34	28	18	60	50	20	40	50
38	38	42	50	40	36	38	38	50	50	31	59	40	42	38
40	38	50	50	50	40	65	38	40	38	58	35	60	90	48
58	45	35	38	32	35	38	34	43	40	35	54	60	33	35
36	43	40	45	56										

Table 2. MLEs the measures AIC, AIC_C and BIC, and KS test to 155 patients of breast tumors data for the models

Model	Parameter Estimates	$-logL$	AIC	AIC_C	BIC	KS
ELEG	$\hat{\lambda} = 0.0243$	603.68986	1215.38	1215.539	1227.553	0.09722351
	$\hat{\beta} = 0.00160$					
	$\hat{\alpha} = 8.006694$					
	$\hat{p} = 0.08999$					
TEE	$\hat{\lambda} = -0.779$	606.38793	1218.776	1218.935	1227.906	0.1010722
	$\hat{\beta} = 21.8864$					
	$\hat{\alpha} = 0.09525$					
EE	$\hat{\beta} = 0.0865$	611.244	1226.489	1226.568	1232.576	0.1134637
	$\hat{\alpha} = 25.598$					
W	$\hat{\beta} = 3.68710$	610.29668	1224.593	1224.672	1230.68	0.1351548
	$\hat{\lambda} = 0.02078$					
E	$\hat{\beta} = 0.02290$	740.3172	1482.634	1482.661	1485.678	0.4089441
LE	$\hat{\lambda} = 6.210.10^{-5}$	650.626	1305.252	1305.331	1311.339	0.2696026
	$\hat{\beta} = 9.732.10^{-4}$					

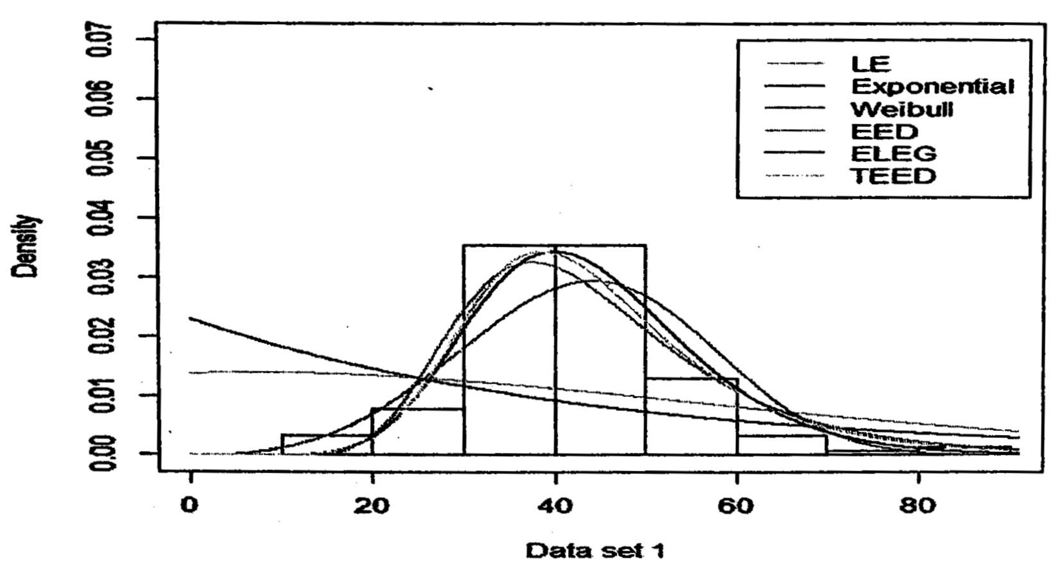

Figure 6: Estimated densities of data set 1.

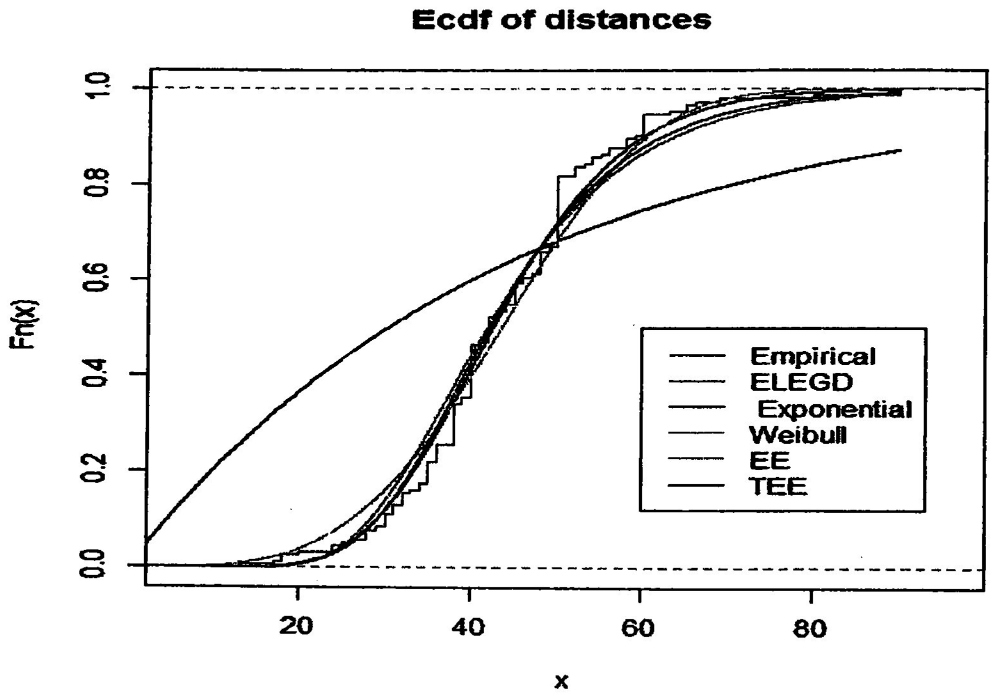

Figure 7:Empirical, fitted ELEG, Exponential, Weibull, Exponentiated exponential, Linear exponential and Transmuted Exponentiated exponential distributions of data set 1.

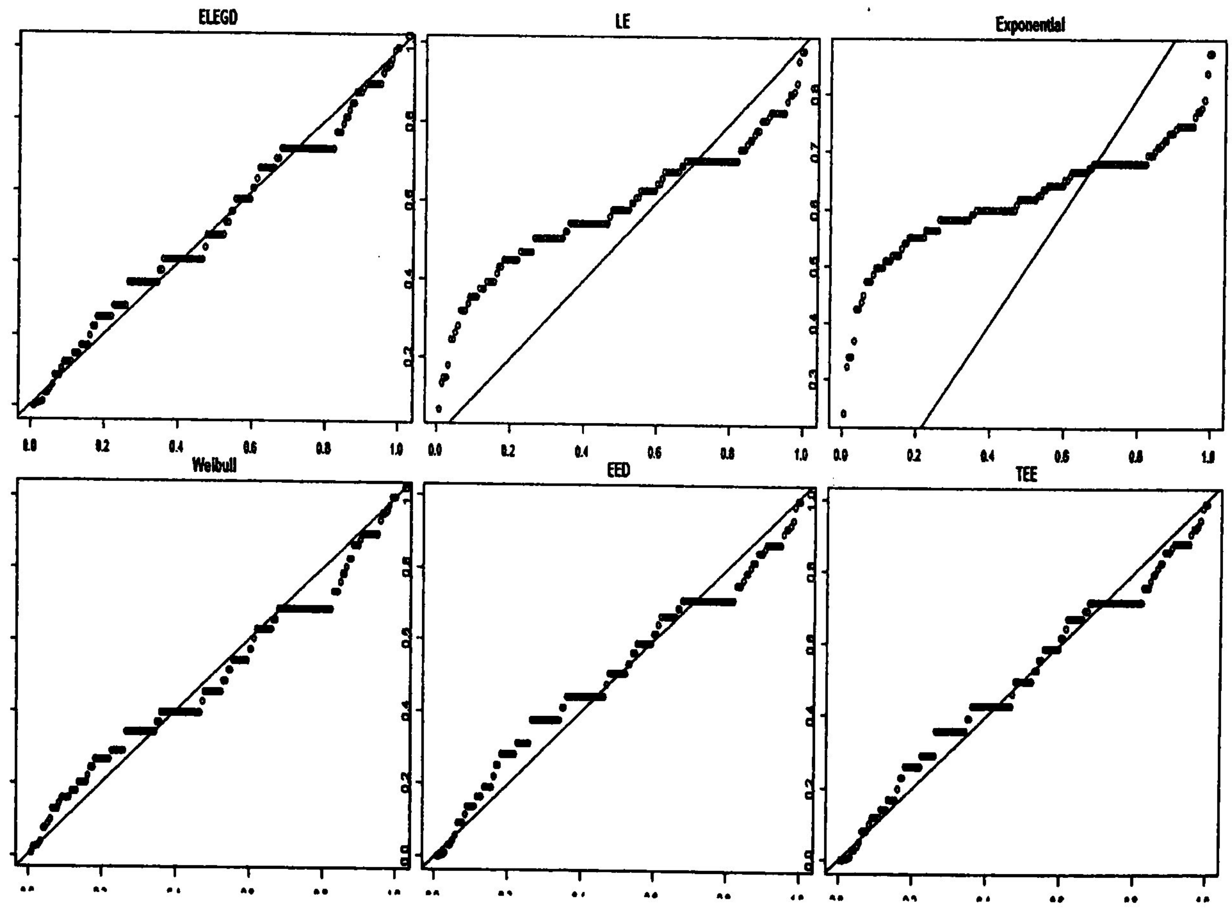

Figure 8: Probability plots for the fits of the ELEG, Exponential, Weibull, exponentiated exponential, Linear exponential and Transmuted Exponentiated exponential distributions of data set 1.

8.2 Data Set 2

The second data set represents failure time of 50 items reported in Aarset [1].

Some summary statistics for the failure time data are as follows:

Min.	1^{st} Qu.	Median	Mean	3^{rd} Qu.	Max.
0.10	13.50	48.50	45.67	81.25	86.00

Table 3. MLEs the measures AIC, AIC_C and BIC, and KS test to failure time data for the models

Model	Parameter Estimates	$-logL$	AIC	AIC_C	BIC	KS
ELEG	$\hat{\lambda}=0.84*10^{-5}$	206.4957	420.9914	421.5132	428.6395	0.1543141
	$\hat{\beta}=1.69*10^{-4}$					
	$\hat{\alpha}=0.30966$					
	$\hat{p}=0.09240$					
TEE	$\hat{\lambda}=-3.21*10^{-5}$	238.6896	483.3793	483.9011	489.1154	0.1662423
	$\hat{\beta}=6.14*10^{-5}$					
	$\hat{\alpha}=2.23*10^{-3}$					
EE	$\hat{\beta}=0.01870912$	239.9733	483.9467	484.2021	487.7708	0.1843086
	$\hat{\alpha}=0.77984131$					
W	$\hat{\beta}=0.94895018$	240.9795	485.9592	486.2145	489.7832	0.1729689
	$\hat{\lambda}=0.02227629$					
E	$\hat{\beta}=0.02189828$	241.0677	484.1354	484.2187	486.0474	0.1712294
LE	$\hat{\lambda}=0.01364528$	238.04883 6	480.0977	480.353	483.9217	0.1570382
	$\hat{\beta}=0.00023990$					

These results indicate that the ELEG model has the lowest AIC and AIC_C,KS and BIC values among the fitted models. The values of these statistics indicate that the ELEG model provides the best fit to all of the data.

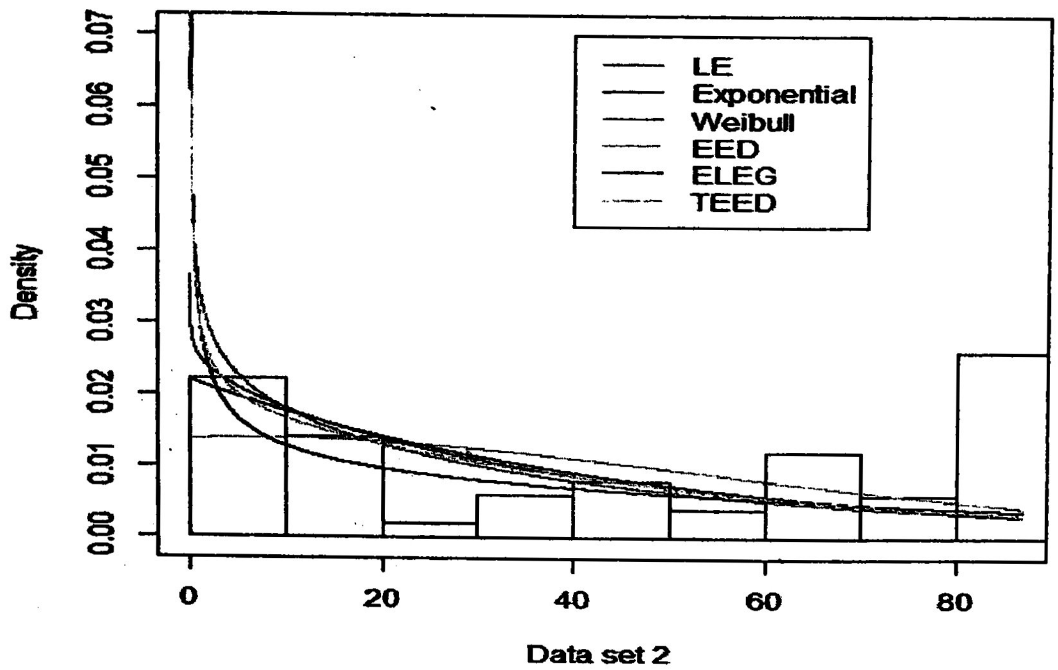

Figure 9: Estimated densities of data set 2.

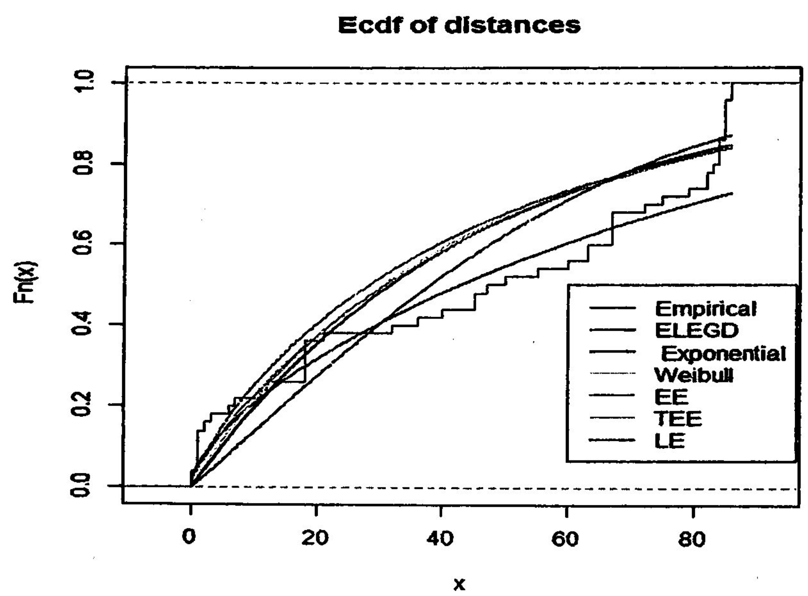

Figure 10: Empirical, fitted ELEG, Exponential, Weibull, Exponentiated exponential, Linear exponential and Transmuted Exponentiated exponential distributions of data set 2.

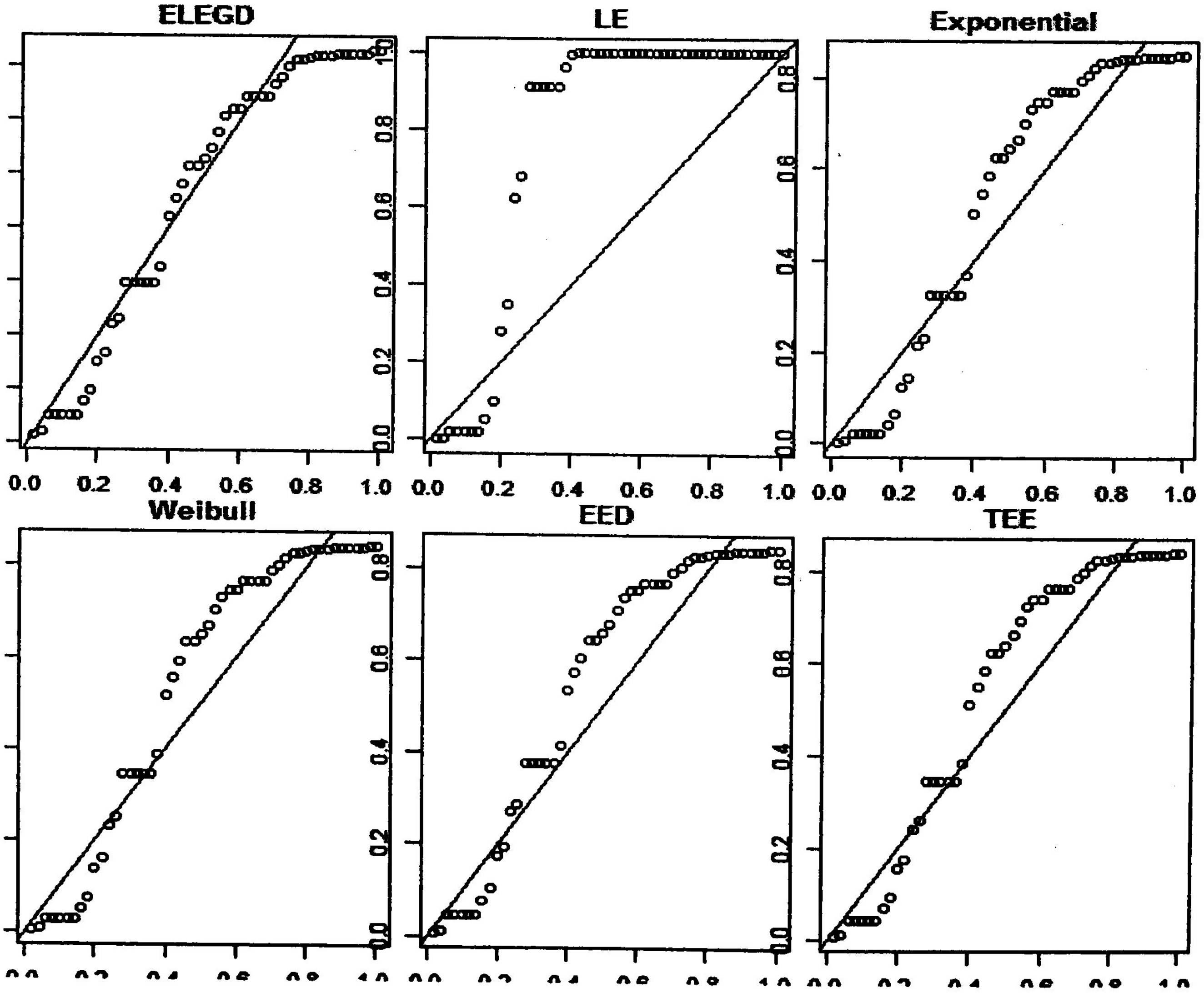

Figure 11: Probability plots for the fits of the ELEG, Exponential, Weibull, exponential, Linear exponential and Transmuted Exponentiated exponential distributions of data set 2.

8. Conclusions

There has been a great interest among statisticians and applied researchers in constructing flexible lifetime models to facilitate better modeling of survival data. Consequently, a significant progress has been made towards the generalization of some well-known lifetime models and their successful application to problems in several areas. In this paper, a new lifetime distribution is provided and discussed. We refer to the new model as the ELEGD distribution and study some of its mathematical and statistical properties. We provide the pdf, the cdf and the hazard rate function of the new model, explicit expressions for the moments. The model parameters are estimated by maximum likelihood. The new model is compared with some of models and provides consistently better fit than other classical lifetime models. We hope that the proposed life time distribution attract wider applications in statistics.

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