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GENERALIZED SOFT b -METRIC SPACES AND SOME OF ITS FIXED SOFT POINT RESULTS

SK. NAZMUL, UTPAL BADYAKAR

Abstract. Our main aim is to develop the fixed point results of b -Metric Spaces in a soft set setting. To accomplish this, we first redefined Wadkar et al.'s soft b -Metric Space, renaming it "Generalized soft b -Metric Space with soft real number $\widetilde{b}$ ($\widetilde{\geq}$ $\overline{1}$)" and demonstrating a non-trivial example. Afterward, some basic concepts in generalized soft b -Metric Spaces such as Cauchy sequence, convergence, completeness etc. are redefined and justified with proper examples. Finally, we established some important fixed soft point results using soft mapping in generalized soft b -Metric Spaces.

1. Introduction

The concept of soft set, introduced by Molodtsov [18] as an alternative way of handling uncertainties adhering to real life situations, unlocked broad areas for the researchers. As a result, researchers have started to formalize different mathematical structures viz. group, ring, field, vector space, metric space, normed linear space, topology etc. in this setting, and it has progressed very fast.

In metric space, the theory of fixed point is a momentous area and is used in different branches of applied mathematics. Naturally, several researchers have tried to develop this theory in metric space [6, 9] and its generalized spaces such as 2-metric [10], D-metric [6], G-metric [19], D*-metric [22], S-metric [23], b-metric [7, 8, 14, 17], Bipolar metric [20] etc.

The concept of soft real numbers, or soft metric, was introduced by Das and Samanta [5] and they also studied some of its properties. After that, some fixed point results on soft metric spaces have been discussed in [1, 2, 3, 11, 12, 13, 26]. Recently, some generalizations of soft metric spaces have been discussed by various researchers.

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In 2017, Wadkar et al. [24, 25] introduced soft b -Metric Spaces and some fixed point results are discussed. It is noted that in condition (4) of the definition of soft b -Metric Space (Definition 2.7), they consider constant soft real number ($\overline{s}$), and this constant is always greater than or equal to $\overline{1}$. So, we have tried to develop the theory of fixed point in soft b -Metric Spaces by replacing an arbitrary soft real number $\widetilde{b} \mathrel{\widetilde{\geq}} \overline{1}$ instead of the constant soft real number $\overline{s} \mathrel{\widetilde{\geq}} \overline{1}$ in the definition of soft b -Metric Space.

In 2019, Abbas et al. [1] proved a result (Theorem 6.3) on Soft b -Metric Spaces by using a finite set of parameters, but in the proof there is no use of the finiteness property, although they showed the importance of the finiteness property by an example.

Accordingly, in the present paper, we have shown the importance of finiteness in the proof. For this reason, we modified the soft b -Metric Space definition, renamed it "Generalized Soft b -Metric Spaces", and verified it with a non-trivial example. Afterwards, in this metric space, some basic characteristics of the Cauchy sequence and its completeness with proper examples are investigated. Finally, using soft mapping, some celebrated fixed soft point results are established in generalized soft b -Metric Spaces. Theorem 6.3 of [1] can be proved by using generalized soft b -metric spaces.

2. Preliminaries

Following [4, 5, 15, 16, 18, 21, 24, 25], we provide certain definitions and preliminary results in our form, which are essential for the main discussions.

Definition 2.1. *Let $\mathscr{F} : \mathscr{A} \to P(\mathscr{X})$ be a mapping, where $P(\mathscr{X})$ be the power set of a set $\mathscr{X}$ $(\neq \phi)$ and $\mathscr{A}$ $(\neq \phi) \subseteq \mathscr{E}$, the set of parameters. Then $(\mathscr{F},\ \mathscr{A})$ is named a soft set over $\mathscr{X}$.*

Definition 2.2. *The soft set $(\mathscr{F},\ \mathscr{A})$ over $\mathscr{X}$, is said to be*

1. *an absolute soft set if $\mathscr{F}(a) = \mathscr{X},\ \forall\ a \in \mathscr{A}$ and it is denoted by $\widetilde{\mathscr{X}}$.*
2. *a soft point if $\mathscr{F}(a) \in \mathscr{X}$ and $\mathscr{F}(b) = \phi,\ \forall\ b \neq a$. If $\mathscr{F}(a) = \{x\}$, then the corresponding soft point is denoted by P_a^x.*

Definition 2.3. *A soft real set is a mapping $\mathscr{F} : \mathscr{A} \to \mathscr{B}(\mathbb{R})$, where $\mathscr{B}(\mathbb{R})$ is the collection of all non-empty bounded subsets of $\mathbb{R}$, the set of all real numbers.*

A soft real set is called soft real number if $\mathscr{F}(a) \in \mathbb{R}$. $\widetilde{r}$ is denoted by a soft real number where $\widetilde{r}(a)$ is an element of $\mathbb{R}$, $\forall\ a \in \mathscr{A}$ and $\overline{r}$ is a special type of soft real number where $\overline{r}(a) = r,\ \forall\ a \in \mathscr{A}$.

Definition 2.4. *Two soft points P_a^x and P_b^y in $\widetilde{\mathscr{X}}$ are said to be unequal if either $x \neq y$ or $a \neq b$.*

Definition 2.5. *For any soft real numbers $\widetilde{r}, \widetilde{s}$ in $(\mathbb{R},\ \mathscr{A})$, we say $\widetilde{r} \mathrel{\widetilde{\leq}} (\widetilde{<})\ \widetilde{s}$ or equivalently $\widetilde{s} \mathrel{\widetilde{\geq}} (\widetilde{>})\ \widetilde{r}$ if $\widetilde{r}(a) \leq (\ <\ )\widetilde{s}(a),\ \forall\ a \in \mathscr{A}$.*

Definition 2.6. *Let $S(\mathscr{X}, \mathscr{A})$ and $S(\mathscr{Y}, \mathscr{B})$ be the families of all soft sets over $\mathscr{X}$ and $\mathscr{Y}$ respectively. The mapping $f_\varphi : S(\mathscr{X}, \mathscr{A}) \to S(\mathscr{Y}, \mathscr{B})$ is called soft mapping from $\mathscr{X}$ to $\mathscr{Y}$; where $f : \mathscr{X} \to \mathscr{Y}$ and $\varphi : \mathscr{A} \to \mathscr{B}$ are two mappings such that the image of a soft set $(\mathscr{F},\ \mathscr{A}) \mathrel{\widetilde{\in}} S(\mathscr{X}, \mathscr{A})$ under the mapping f_φ is denoted by*

$f_\varphi(\mathscr{F},\mathscr{A}) = (f_\varphi(\mathscr{F}),\mathscr{B})$ *and is defined by,*

$$[f_\varphi(\mathscr{F})](\beta) = \begin{cases} \bigcup_{\alpha\in\varphi^{-1}(\beta)}[f(\mathscr{F}(\alpha))], & \text{if } \varphi^{-1}(\beta) \neq \phi \\ \phi, & \text{otherwise.} \end{cases}$$

Definition 2.7. *Let $SP(\widetilde{\mathscr{X}})$ be the collection of all soft points of $\widetilde{\mathscr{X}}$ and $\mathbb{R}(\mathscr{A})^*$ be the set of all non-negative soft real numbers. A mapping $\widetilde{\rho} : SP(\widetilde{\mathscr{X}}) \times SP(\widetilde{\mathscr{X}}) \to \mathbb{R}(\mathscr{A})^*$, satisfying the following conditions, $\forall\ P^x_{\lambda_1^*},\ P^y_{\lambda_2^*},\ P^z_{\lambda_3^*} \widetilde{\in} SP(\widetilde{\mathscr{X}})$ and $\overline{s} \widetilde{\geq} \overline{1}$,*

1. $\widetilde{\rho}(P^x_{\lambda_1^*},\ P^y_{\lambda_2^*}) \widetilde{\geq} \overline{0}$,
2. $\widetilde{\rho}(P^x_{\lambda_1^*},\ P^y_{\lambda_2^*}) = \overline{0}$ *if and only if* $P^x_{\lambda_1^*} = P^y_{\lambda_2^*}$,
3. $\widetilde{\rho}(P^x_{\lambda_1^*},\ P^y_{\lambda_2^*}) = \widetilde{\rho}(P^y_{\lambda_2^*},\ P^x_{\lambda_1^*})$,
4. $\widetilde{\rho}(P^x_{\lambda_1^*},\ P^y_{\lambda_2^*}) \widetilde{\leq} \overline{s}\ \{\widetilde{\rho}(P^x_{\lambda_1^*},\ P^z_{\lambda_3^*}) + \widetilde{\rho}(P^z_{\lambda_3^*},\ P^y_{\lambda_2^*})\}$,

is called a soft b -metric on $\widetilde{\mathscr{X}}$ and $(\widetilde{\mathscr{X}},\ \widetilde{\rho},\ \mathscr{A})$ is a soft b -Metric Space.

3. Generalized Soft b -Metric Spaces ($gsbms$)

In this section, we introduced generalized soft b -Metric Spaces and few basic properties are discussed. Also, some fixed point results have been established. Let $\mathbb{R}^*$ be the family of all non-negative soft real numbers over $\mathscr{A}$.

Definition 3.8. *Let $\widetilde{\zeta} : SP(\widetilde{\mathscr{X}}) \times SP(\widetilde{\mathscr{X}}) \to \mathbb{R}^*$, be a mapping. Then $\widetilde{\zeta}$ is called a generalized soft b - metric with soft real number $\widetilde{b}$ on $\widetilde{\mathscr{X}}$ and consequently $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ is a generalized soft b - Metric Space (gsbms) if for any soft points $P^x_{\lambda_1^*},\ P^y_{\lambda_2^*},\ P^z_{\lambda_3^*}$ of $SP(\widetilde{\mathscr{X}})$, the following conditions are satisfied.*

1. $\widetilde{\zeta}(P^x_{\lambda_1^*},\ P^y_{\lambda_2^*}) \widetilde{\geq} \overline{0}$,
2. $\widetilde{\zeta}(P^x_{\lambda_1^*},\ P^y_{\lambda_2^*}) = \overline{0}$ *if and only if* $P^x_{\lambda_1^*} = P^y_{\lambda_2^*}$,
3. $\widetilde{\zeta}(P^x_{\lambda_1^*},\ P^y_{\lambda_2^*}) = \widetilde{\zeta}(P^y_{\lambda_2^*},\ P^x_{\lambda_1^*})$,
4. $\widetilde{\zeta}(P^x_{\lambda_1^*},\ P^y_{\lambda_2^*}) \widetilde{\leq} \widetilde{b}\ \{\widetilde{\zeta}(P^x_{\lambda_1^*},\ P^z_{\lambda_3^*}) + \widetilde{\zeta}(P^z_{\lambda_3^*},\ P^y_{\lambda_2^*})\}$, *for some soft real number* $\widetilde{b} \widetilde{\geq} \overline{1}$.

Example 1. *Let $\mathscr{X} = \mathbb{R}$ and $\mathscr{A} = \{x \in \mathbb{R} : x \geq 1\}$.*
For any $\alpha \in \mathscr{A}$, define $\widetilde{\zeta} : SP(\widetilde{\mathscr{X}}) \times SP(\widetilde{\mathscr{X}}) \to \mathbb{R}^*$ by,

$$[\widetilde{\zeta}(P^x_{\lambda_1^*},\ P^y_{\lambda_2^*})](\alpha) = \big[\,|\,x-y\,| + |\,\lambda_1^*-\lambda_2^*\,|\,\big]^\alpha,\ \forall\ P^x_{\lambda_1^*},\ P^y_{\lambda_2^*} \widetilde{\in} SP(\widetilde{\mathscr{X}}).$$

Then clearly $\widetilde{\zeta}$ satisfies the conditions (1), (2) and (3) of Definition 3.8. Now for condition (4), we have

$$\begin{aligned}
[\widetilde{\zeta}(P^x_{\lambda_1^*},\ P^y_{\lambda_2^*})](\alpha) &= \big[\,|\,x-y\,| + |\,\lambda_1^*-\lambda_2^*\,|\,\big]^\alpha \\
&\leq \big[\,|\,x-z\,| + |\,z-y\,| + |\,\lambda_1^*-\lambda_3^*\,| + |\,\lambda_3^*-\lambda_2^*\,|\,\big]^\alpha \\
&\leq 2^{\alpha-1}\Big[\big[\,|\,x-z\,| + |\,\lambda_1^*-\lambda_3^*\,|\,\big]^\alpha + \big[\,|\,z-y\,| + |\,\lambda_3^*-\lambda_2^*\,|\,\big]^\alpha\Big], \\
&\quad \textit{since } \alpha\in\mathscr{A},\ \alpha\geq 1 \\
&= \widetilde{b}(\alpha)\ \big\{[\widetilde{\zeta}(P^x_{\lambda_1^*},\ P^z_{\lambda_3^*})](\alpha) + [\widetilde{\zeta}(P^z_{\lambda_3^*},\ P^y_{\lambda_2^*})](\alpha)\big\} \\
\Rightarrow\ \widetilde{\zeta}(P^x_{\lambda_1^*},\ P^y_{\lambda_2^*}) &\widetilde{\leq} \widetilde{b}\ \big\{\widetilde{\zeta}(P^x_{\lambda_1^*},\ P^z_{\lambda_3^*}) + \widetilde{\zeta}(P^z_{\lambda_3^*},\ P^y_{\lambda_2^*})\big\},\ \textit{since } \alpha(\in\mathscr{A}) \textit{ is an arbitrary.}
\end{aligned}$$

Therefore, $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ *is a gsbms with soft real number* $\widetilde{b}$, *where* $\widetilde{b}(\alpha) = 2^{\alpha-1},\ \forall\ \alpha \in \mathscr{A}$.

Remark 1. *It is clear that every soft metric is a generalized soft b -metric with soft real number* $\widetilde{b} = \overline{1}$ *and every soft b - metric on* $\widetilde{\mathscr{X}}$ *is a generalized soft b -metric with soft real number* $\widetilde{b} \widetilde{\geq} \overline{b}$, *where* $b \geq 1$.

Also, if the parameter set is finite or $\mathscr{X}$ *is bounded, then for every soft real number* $\widetilde{b}$, $\exists\ r \in \mathbb{R}$, *such that* $\widetilde{b} \widetilde{\leq} \overline{r}$ *and hence in this case every generalized soft b -metric on* $\widetilde{\mathscr{X}}$ *with soft real number* $\widetilde{b}$, *is a soft b -metric on* $\widetilde{\mathscr{X}}$.

Definition 3.9. *Let* $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ *be a gsbms and* $\{\widetilde{x_n} = P^{x_n}_{\lambda^*_{1n}}\}_n$ *be a sequence of soft points in* $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$. *Then*

(i) $\{\widetilde{x_n}\}_n$ *converges to* $\widetilde{x}\ (= P^{x}_{\lambda^*_1})\ \widetilde{\in}\ SP(\widetilde{\mathscr{X}})$, *if for any* $\widetilde{\epsilon} \widetilde{>} \overline{0}$ *and for any* $\alpha \in E,\ \exists\ N_{\epsilon_\alpha} \in \mathbb{N}$, *such that* $[\widetilde{\zeta}\ (\widetilde{x_n},\ \widetilde{x})](\alpha)\ <\ \widetilde{\epsilon}(\alpha),\ \forall\ n \geq\ N_{\epsilon_\alpha}$.

(ii) $\{\widetilde{x_n}\}$ *is said to be a Cauchy sequence, if for every* $\widetilde{\epsilon} \widetilde{>} \overline{0}$ *and for every* $\alpha \in E,\ \exists\ N_{\epsilon_\alpha}\ \in\ \mathbb{N}$, *such that* $[\widetilde{\zeta}\ (\widetilde{x}_i,\ \widetilde{x}_j)](\alpha)\ <\ \widetilde{\epsilon}(\alpha),\ \forall\ i,\ j\ \geq\ N_{\epsilon_\alpha}$ *i.e., if* $\widetilde{\zeta}\ (\widetilde{x}_i,\ \widetilde{x}_j)\ \to\ \overline{0}$, *as* $i,\ j\ \to\ \infty$.

Example 2. *Let* $\mathscr{X} = \mathbb{R}$ *and* $\mathscr{A} = \{x \in \mathbb{R} : x \geq 1\}$. *The function* $\widetilde{\zeta}$, *define by* $[\widetilde{\zeta}(P^{x}_{\lambda^*_1},\ P^{y}_{\lambda^*_2})](\alpha)\ =\ \big[\ |\ x - y\ |\ +\ |\ \lambda^*_1 - \lambda^*_2\ |\ \big]^{\alpha},\ \forall\ \alpha \in \mathscr{A}$ *is a generalized soft b -metric with soft real number* $\widetilde{b}$, *where* $\widetilde{b}(\alpha) = 2^{\alpha-1},\ \forall\ \alpha \in \mathscr{A}$.
Let $\left\{\widetilde{x_n} = P^{\frac{1}{n+1}}_{\frac{n+1}{n}}\right\}_n$ *be a sequence in* $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$.
Now for any $\alpha \in \mathscr{A}$,

$$\begin{aligned}
[\widetilde{\zeta}(\widetilde{x_n},\ P^{0}_{1})](\alpha) &= \left[\left|\frac{1}{n+1} - 0\right| + \left|\frac{n+1}{n} - 1\right|\right]^{\alpha} \\
&= \left[\frac{1}{(n+1)} + \frac{1}{n}\right]^{\alpha} \\
&\to 0,\ \text{as } n \to \infty
\end{aligned}$$

Since this is true for all $\alpha \in \mathscr{A}$, *and hence* $\widetilde{\zeta}(\widetilde{x_n},\ P^{0}_{1}) \to \overline{0},\ \text{as } n \to \infty$.
Therefore, $\{\widetilde{x_n}\}_n$ *converges to* P^{0}_{1}.

Again, for any $\alpha \in \mathscr{A}$,

$$\begin{aligned}
[\widetilde{\zeta}(\widetilde{x}_i,\ \widetilde{x}_j)](\alpha) &= \left[\left|\frac{1}{i} - \frac{1}{j}\right| + \left|\frac{i+1}{i} - \frac{j+1}{j}\right|\right]^{\alpha} \\
&= \left[2\ \left|\frac{1}{i}\ -\ \frac{1}{j}\right|\right]^{\alpha} \\
&\to 0,\ \text{as } i,\ j \to \infty
\end{aligned}$$

Since this is true for all $\alpha \in \mathscr{A}$, *and hence* $\widetilde{\zeta}(\widetilde{x}_i,\ \widetilde{x}_j) \to \overline{0}$, *as* $i,\ j \to \infty$.
Therefore, $\{\widetilde{x_n}\}_n$ *is a Cauchy sequence in* $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$.

Definition 3.10. *A gsbms* $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ *is called complete if for each Cauchy sequence in* $\widetilde{\mathscr{X}}$ *is converges to some soft point in* $\widetilde{\mathscr{X}}$.

Example 3. *Let* $\mathscr{X} = \mathbb{R}^+ = \mathscr{A}$ *and for any* $\alpha \in \mathscr{A}$ *define* $\widetilde{\zeta} : SP(\widetilde{\mathscr{X}}) \times SP(\widetilde{\mathscr{X}}) \to \mathbb{R}^*$ *by,*

$$[\widetilde{\zeta}(P^{x}_{\lambda^*_1},\ P^{y}_{\lambda^*_2})](\alpha) = \big[\ |\ x - y\ |\ +\ |\ \lambda^*_1 - \lambda^*_2\ |\ \big]^{\alpha},\ \forall\ P^{x}_{\lambda^*_1},\ P^{y}_{\lambda^*_2} \in SP(\widetilde{\mathscr{X}}).$$

Then $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ *is a gsbms with soft real number* $\widetilde{b}$, *where* $\widetilde{b}(\alpha) = 2^{\alpha-1},\ \forall\ \alpha \in \mathscr{A}$.

Let us consider an arbitrary Cauchy sequence $\{\widetilde{x_n} = P^{x_n}_{\lambda^*_{1n}}\}_n$ *in* $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ *and an arbitrary soft real number* $\widetilde{\epsilon}\ \widetilde{>}\ \overline{0}$.
Since $\{\widetilde{x_n}\}_n$ *be Cauchy sequence, then for each* $\alpha \in \mathscr{A},\ \exists\ N_{\epsilon_\alpha} \in \mathbb{N}$ *such that,*

$$[\widetilde{\zeta}\ (\widetilde{x_n},\ \widetilde{x_m})](\alpha)\ <\ \widetilde{\epsilon}(\alpha),\ \forall\ n,\ m \geq N_{\epsilon_\alpha}.$$

$\Rightarrow \big[\ |\ x_n - x_m\ |\ +\ |\ \lambda^*_{1n} - \lambda^*_{1m}\ |\ \big]^\alpha\ <\ \widetilde{\epsilon}(\alpha),\ \forall\ n,\ m \geq N_{\epsilon_\alpha}$.
Thus, for each $\alpha \in \mathscr{A}$,

$$|\ x_n - x_m\ | <\ \left(\frac{\widetilde{\epsilon}(\alpha)}{2}\right)^{\frac{1}{\alpha}}\ \textit{and}\ \ |\ \lambda^*_{1n} - \lambda^*_{1m}\ | <\ \left(\frac{\widetilde{\epsilon}(\alpha)}{2}\right)^{\frac{1}{\alpha}},\ \forall\ n,\ m \geq N_{\epsilon_\alpha}.$$

Hence, $\{x_n\}_n$ *and* $\{\lambda^*_{1n}\}_n$ *are Cauchy sequence in* $\mathbb{R}^+$.
Since $\mathbb{R}^+$ *is complete,* $\exists\ x, \lambda^*_1 \in \mathbb{R}^+$ *such that* $x_n \to x$ *and* $\lambda^*_{1n} \to \lambda^*_1$.
Choose $\widetilde{x} = P^x_{\lambda^*_1}\ \widetilde{\in}\ SP(\widetilde{\mathscr{X}})$. *Now we have to show that* $\{\widetilde{x_n}\} \to \widetilde{x}$.
For this let $\widetilde{\delta}\ \widetilde{>}\ \overline{0}$ *be any arbitrary soft real number and* $\alpha \in \mathscr{A}$.
Since $x_n \to x$ *and* $\lambda^*_{1n} \to \lambda^*_1$. *Then for* $\widetilde{\delta}(\alpha) > 0$, *we can find a natural number* N_{δ_α} *such that* $|\ x_n - x\ | <\ \left(\frac{\widetilde{\delta}(\alpha)}{2}\right)^{\frac{1}{\alpha}}$ *and* $|\ \lambda^*_{1n} - \lambda^*_1\ | <\ \left(\frac{\widetilde{\delta}(\alpha)}{2}\right)^{\frac{1}{\alpha}},\ \forall\ n > N_{\delta_\alpha}$.
That is for each $\alpha \in \mathscr{A},\ [\widetilde{\zeta}(P^{x_n}_{\lambda^*_{1n}},\ P^x_{\lambda^*_1})](\alpha)\ <\ \widetilde{\delta}(\alpha),\ \forall\ n > N_{\delta_\alpha}$ *and hence,* $[\widetilde{\zeta}(P^{x_n}_{\lambda^*_{1n}},\ P^x_{\lambda^*_1})](\alpha) \to 0$, *as* $n \to \infty$.
Since this is true for all $\alpha \in \mathscr{A}$, *so* $\widetilde{\zeta}(P^{x_n}_{\lambda^*_{1n}},\ P^x_{\lambda^*_1}) \to \overline{0}$, *as* $n \to \infty$.
Hence, $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ *is a complete gsbms.*

Theorem 3.1. *Let us consider two sequences* $\{\widetilde{x_n}\}_n$ *and* $\{\widetilde{y_n}\}_n$ *of soft points in a gsbms* $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ *with soft real number* $\widetilde{b}$. *If* $\{\widetilde{x_n}\}_n$ *and* $\{\widetilde{y_n}\}_n$ *converges to the soft points* $P^x_{\lambda^*_1}$ *and* $P^y_{\lambda^*_2}$ *respectively, then* $\widetilde{\zeta}(\widetilde{x_n},\ \widetilde{y_n})$ *converges to* $\widetilde{b}^2\ \widetilde{\zeta}(P^x_{\lambda^*_1},\ P^y_{\lambda^*_2})$.

Proof. Since $\{\widetilde{x_n}\}_n$ and $\{\widetilde{y_n}\}_n$ converges to the soft points $P^x_{\lambda^*_1}$ and $P^y_{\lambda^*_2}$ respectively, then $\widetilde{\zeta}(\widetilde{x_n},\ P^x_{\lambda^*_1}) \to \overline{0}$ and $\widetilde{\zeta}(\widetilde{y_n},\ P^y_{\lambda^*_2}) \to \overline{0}$, as $n \to \infty$. Now,

$$\begin{aligned}
\widetilde{\zeta}(\widetilde{x_n},\ \widetilde{y_n})\ &\widetilde{\leq}\ \ \widetilde{b}\ [\widetilde{\zeta}(\widetilde{x_n},\ P^x_{\lambda^*_1})\ +\ \widetilde{\zeta}(P^x_{\lambda^*_1},\ \widetilde{y_n})] \\
&\widetilde{\leq}\ \ \widetilde{b}\ \widetilde{\zeta}(\widetilde{x_n},\ P^x_{\lambda^*_1}) \\
&\quad +\ \widetilde{b}^2\ [\widetilde{\zeta}(P^x_{\lambda^*_1},\ P^y_{\lambda^*_2})\ +\ \widetilde{\zeta}(P^y_{\lambda^*_2},\ \widetilde{y_n})] \\
\Rightarrow |\ \widetilde{\zeta}(\widetilde{x_n},\ \widetilde{y_n}) - \widetilde{b}^2\ \widetilde{\zeta}(P^x_{\lambda^*_1},\ P^y_{\lambda^*_2})\ |\ &\widetilde{\leq}\ \ |\ \widetilde{b}\ \widetilde{\zeta}(\widetilde{x_n},\ P^x_{\lambda^*_1})\ | + |\ \widetilde{b}^2\ \widetilde{\zeta}(\widetilde{y_n},\ P^y_{\lambda^*_2})\ | \\
&\quad \to \overline{0},\ \text{as } n \to \infty.
\end{aligned}$$

Therefore, $\widetilde{\zeta}(\widetilde{x_n},\ \widetilde{y_n})$ converges to $\widetilde{b}^2\ \widetilde{\zeta}(P^x_{\lambda^*_1},\ P^y_{\lambda^*_2})$. □

Corollary 3.1. *Let* $\{\widetilde{x_n}\}_n$ *be a sequences of soft points in a gsbms* $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ *with soft real number* $\widetilde{b}$. *If* $P^y_{\lambda^*_2}$ *be a fixed soft point in* $(\widetilde{\mathscr{X}},\ \mathscr{A})$ *and* $\{\widetilde{x_n}\}_n$ *converges to* $P^x_{\lambda^*_1}$, *then* $\widetilde{\zeta}(\widetilde{x_n},\ P^y_{\lambda^*_2})$ *converges to* $\widetilde{b}^2\ \widetilde{\zeta}(P^x_{\lambda^*_1},\ P^y_{\lambda^*_2})$.

Theorem 3.2. *Let $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$, a complete gsbms with soft real number $\widetilde{b}$ and $\mathscr{A}$ be a finite set of parameters. Let f_φ and g_ψ be two soft mapping on $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ satisfying the following conditions,*

$$\widetilde{\zeta}(f_\varphi(P^x_{\lambda_1^*}),\ g_\psi(P^y_{\lambda_2^*})) \ \widetilde{\leq}\ \widetilde{r}\left[\widetilde{\zeta}(P^x_{\lambda_1^*},\ f_\varphi(P^x_{\lambda_1^*}))\right] + \widetilde{s}\left[\widetilde{\zeta}(P^y_{\lambda_2^*},\ g_\psi(P^y_{\lambda_2^*}))\right], \quad \forall\ P^x_{\lambda_1^*},\ P^y_{\lambda_2^*}\ \widetilde{\in}\ SP(\widetilde{\mathscr{X}}),$$

where $\overline{0}\ \widetilde{\leq}\ \widetilde{r},\ \widetilde{s}\ \widetilde{<}\ \overline{\frac{1}{2}}$. Then f_φ and g_ψ have unique common fixed soft point in $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$.

Proof. Let $\widetilde{x_0} = P^x_{\lambda_1^*}$ be any arbitrary member of $SP(\widetilde{\mathscr{X}})$.
Define a sequence $\{\widetilde{x_n}\}$ in $SP(\widetilde{\mathscr{X}})$ by
$\widetilde{x_1} = f_\varphi(\widetilde{x_0}),\ \widetilde{x_2} = g_\psi(\widetilde{x_1}),\ \ldots, \widetilde{x_{2k+1}} = f_\varphi(\widetilde{x_{2k}}),\ \widetilde{x_{2k+2}} = g_\psi(\widetilde{x_{2k+1}}),\ k = 0, 1, 2, \ldots$

$$\begin{aligned}
\text{Now, } \widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}}) &= \widetilde{\zeta}(f_\varphi(\widetilde{x_{2k}}),\ g_\psi(\widetilde{x_{2k+1}}))\\
&\widetilde{\leq}\ \widetilde{r}\left[\widetilde{\zeta}(\widetilde{x_{2k}},\ \widetilde{x_{2k+1}})\right] + \widetilde{s}\left[\widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}})\right]\\
&\widetilde{\leq}\ \overline{a_1}\left[\widetilde{\zeta}(\widetilde{x_{2k}},\ \widetilde{x_{2k+1}})\right] + \overline{a_2}\left[\widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}})\right] (\text{since } \mathscr{A} \text{ is finite}\\
&\quad \exists\ a_1\ , a_2 \in \mathbb{R} \text{ such that, } \overline{\frac{1}{2}}\ \widetilde{>}\ \overline{a_1}\ \widetilde{\geq}\ \widetilde{r},\ \overline{\frac{1}{2}}\ \widetilde{>}\ \overline{a_2}\ \widetilde{\geq}\ \widetilde{s}.)\\
&\widetilde{\leq}\ \overline{a_1}\left[\widetilde{\zeta}(\widetilde{x_{2k}},\ \widetilde{x_{2k+1}})\right] + \overline{a_1}\left[\widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}})\right] (\overline{a_1}\ \widetilde{\geq}\ \overline{a_2} \text{ (say)})\\
\Rightarrow \widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}}) &\widetilde{\leq}\ \frac{\overline{a_1}}{\overline{1} - \overline{a_1}}\ \widetilde{\zeta}(\widetilde{x_{2k}},\ \widetilde{x_{2k+1}}) \ (1)
\end{aligned}$$

$$\begin{aligned}
\text{Again, } \widetilde{\zeta}(\widetilde{x_{2k+2}},\ \widetilde{x_{2k+3}}) &= \widetilde{\zeta}(f_\varphi(\widetilde{x_{2k+1}}),\ g_\psi(\widetilde{x_{2k+2}}))\\
&\widetilde{\leq}\ \overline{a_1}[\widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}}) + \widetilde{\zeta}(\widetilde{x_{2k+2}},\ \widetilde{x_{2k+3}})]\\
\Rightarrow \widetilde{\zeta}(\widetilde{x_{2k+2}},\ \widetilde{x_{2k+3}}) &\widetilde{\leq}\ \frac{\overline{a_1}}{\overline{1} - \overline{a_1}}\ \widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}}) \ (2)
\end{aligned}$$

Let $l = \frac{a_1}{1-a_1}$. Then $\bar{l} = \frac{\overline{a_1}}{\overline{1}-\overline{a_1}}$ and clearly $\overline{0}\ \widetilde{\leq}\ \bar{l}\ \widetilde{<}\ \overline{1}$, as $\overline{0}\ \widetilde{\leq}\ \overline{a_1}\ \widetilde{<}\ \overline{\frac{1}{2}}$.
So, from (1) and (2) we have,

$$\begin{aligned}
\widetilde{\zeta}(\widetilde{x_{2k+2}},\ \widetilde{x_{2k+3}}) &\widetilde{\leq}\ \bar{l}\widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}})\\
&\widetilde{\leq}\ \bar{l}^2\widetilde{\zeta}(\widetilde{x_{2k}}\ , \widetilde{x_{2k+1}});\ k = 0, 1, 2, \cdots
\end{aligned}$$

$$\begin{aligned}
\text{And hence for any } n \in \mathbb{N},\ \widetilde{\zeta}(\widetilde{x_n},\ \widetilde{x_{n+1}}) &\widetilde{\leq}\ \bar{l}\widetilde{\zeta}(\widetilde{x_{n-1}},\ \widetilde{x_n})\\
&\widetilde{\leq}\ \bar{l}^2\widetilde{\zeta}(\widetilde{x_{n-2}},\ \widetilde{x_{n-1}})\\
&\ \vdots\\
&\widetilde{\leq}\ \bar{l}^n\widetilde{\zeta}(\widetilde{x_0},\ \widetilde{x_1}) \ (3)
\end{aligned}$$

Since $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ be a gsbms with soft constant $\widetilde{b}$ and $\mathscr{A}$ be finite, $\exists\ c \in \mathbb{R}$ such that $\widetilde{\zeta}(\widetilde{x_u},\ \widetilde{x_v})\ \widetilde{\leq}\ \overline{c}\ [\widetilde{\zeta}(\widetilde{x_u},\ \widetilde{x_w}) + \widetilde{\zeta}(\widetilde{x_w},\ \widetilde{x_v})]$, where $\overline{c}\ \widetilde{\geq}\ \widetilde{b}$ and $(\overline{1} - \overline{c}\bar{l})(\alpha) \neq \overline{0}(\alpha),\ \forall\ \alpha\ \in\ \mathscr{A}$.

Thus, for any $m,\ n \in \mathbb{N}$ with $m \geq n$, we have

$$\begin{aligned}
\widetilde{\zeta}(\widetilde{x_n},\ \widetilde{x_m}) &\widetilde{\leq} \overline{c}\ \widetilde{\zeta}(\widetilde{x_n},\ \widetilde{x_{n+1}}) + \overline{c}^2\ \widetilde{\zeta}(\widetilde{x_{n+1}},\ \widetilde{x_{n+2}}) + \\
&\quad \overline{c}^3\ \widetilde{\zeta}(\widetilde{x_{n+2}},\ \widetilde{x_{n+3}}) + \cdots + \overline{c}^{m-n}\ \widetilde{\zeta}(\widetilde{x_{m-1}},\ \widetilde{x_m}) \\
&\widetilde{\leq} \overline{c}\ \widetilde{\zeta}(\widetilde{x_n},\ \widetilde{x_{n+1}}) + \overline{c}^2\ \widetilde{\zeta}(\widetilde{x_{n+1}},\ \widetilde{x_{n+2}}) + \overline{c}^3\ \widetilde{\zeta}(\widetilde{x_{n+2}},\ \widetilde{x_{n+3}}) + \cdots \\
&\widetilde{\leq} \overline{c}\ \overline{l}^n\ \widetilde{\zeta}(\widetilde{x_0},\ \widetilde{x_1}) + \overline{c}^2\ \overline{l}^{n+1}\ \widetilde{\zeta}(\widetilde{x_0},\ \widetilde{x_1}) + \overline{c}^3\ \overline{l}^{n+2}\ \widetilde{\zeta}(\widetilde{x_0},\ \widetilde{x_1}) + \cdots \\
&= \overline{c}\overline{l}^n(\overline{1} + \overline{c}\overline{l} + \overline{c}^2\overline{l}^2 + \cdots)\widetilde{\zeta}(\widetilde{x_0},\ \widetilde{x_1}) \\
&= \frac{\overline{c}\overline{l}^n}{\overline{1} - \overline{c}\overline{l}}\ \widetilde{\zeta}(\widetilde{x_0},\ \widetilde{x_1}) \\
\Rightarrow\ \widetilde{\zeta}(\widetilde{x_n},\ \widetilde{x_m}) &\to \overline{0},\ \text{as } n \to \infty\ [\text{since, } \overline{0}\ \widetilde{\leq}\ \overline{l}\ \widetilde{<}\ \overline{1} \text{ and } (\overline{1} - \overline{c}\overline{l})(\alpha) \neq \overline{0}(\alpha),\ \forall\ \alpha\ \in\ \mathscr{A}]
\end{aligned}$$

So, $\{\widetilde{x_n}\}$, a Cauchy sequence in $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$. As $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ is complete, $\exists\ P^y_{\lambda_2^*}\ \widetilde{\in}\ SP(\widetilde{\mathscr{X}})$ such that $\lim\limits_{n\to\infty} \widetilde{x_n} = P^y_{\lambda_2^*}$.

$$\begin{aligned}
\text{Now, } \widetilde{\zeta}(f_\varphi(P^y_{\lambda_2^*}),\ \widetilde{x_{2n}}) &= \widetilde{\zeta}(f_\varphi(P^y_{\lambda_2^*}),\ g_\psi(\widetilde{x_{2n-1}})) \\
&\widetilde{\leq} \overline{a_1}\ \left[\widetilde{\zeta}(P^y_{\lambda_2^*},\ f_\varphi(P^y_{\lambda_2^*}))\ +\ \widetilde{\zeta}(\widetilde{x_{2n-1}},\ g_\psi\ (\widetilde{x_{2n-1}})\right] \\
&= \overline{a_1}\ \left[\widetilde{\zeta}(P^y_{\lambda_2^*},\ f_\varphi(P^y_{\lambda_2^*})) + \widetilde{\zeta}(\widetilde{x_{2n-1}},\ \widetilde{x_{2n}})\right]
\end{aligned}$$

Taking $n \to \infty$ we get,

$$\begin{aligned}
\widetilde{b}^2\ \widetilde{\zeta}(f_\varphi(P^y_{\lambda_2^*}),\ P^y_{\lambda_2^*}) &\widetilde{\leq} \overline{a_1}\ \left[\widetilde{\zeta}(P^y_{\lambda_2^*}, f_\varphi(P^y_{\lambda_2^*}))\ +\ \widetilde{b}^2\ \widetilde{\zeta}(P^y_{\lambda_2^*}, P^y_{\lambda_2^*})\right],\ \text{from Theorem 3.1} \\
\Rightarrow\ \widetilde{b}^2\ \widetilde{\zeta}(f_\varphi(P^y_{\lambda_2^*}), P^y_{\lambda_2^*}) &\widetilde{\leq} \overline{a_1}\ \widetilde{\zeta}(P^y_{\lambda_2^*}, f_\varphi(P^y_{\lambda_2^*})) \\
\Rightarrow\ f_\varphi(P^y_{\lambda_2^*}) &= P^y_{\lambda_2^*},\ \text{since } \overline{0}\ \widetilde{\leq}\ \overline{a_1}\ \widetilde{<}\ \overline{\frac{1}{2}} \text{ and } \widetilde{b}\ \widetilde{>}\ \overline{0}.
\end{aligned}$$

Similarly we can prove that $g_\psi(P^y_{\lambda_2^*}) = P^y_{\lambda_2^*}$.

Therefore, in $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$, f_φ and g_ψ have common fixed soft point.

To prove uniqueness, let $P^z_{\lambda_3^*}\ \widetilde{\in}\ SP(\widetilde{\mathscr{X}})$ be another fixed soft point in $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$.

$$\begin{aligned}
\text{Then, } \widetilde{\zeta}(P^y_{\lambda_2^*}, P^z_{\lambda_3^*}) &= \widetilde{\zeta}(f_\varphi(P^y_{\lambda_2^*}),\ g_\psi(P^z_{\lambda_3^*})) \\
&\widetilde{\leq} \overline{a_1}\ \left[\widetilde{\zeta}(P^y_{\lambda_2^*}, f_\varphi(P^y_{\lambda_2^*})) + \widetilde{\zeta}(P^z_{\lambda_3^*}, g_\psi(P^z_{\lambda_3^*}))\right] \\
&= \overline{a_1}\ \left[\widetilde{\zeta}(P^y_{\lambda_2^*},\ P^y_{\lambda_2^*}) + \widetilde{\zeta}(P^z_{\lambda_3^*},\ P^z_{\lambda_3^*})\right] = \overline{0} \\
\Rightarrow\ \widetilde{\zeta}(P^y_{\lambda_2^*},\ P^z_{\lambda_3^*}) &= \overline{0} \text{ and hence } P^y_{\lambda_2^*} = P^z_{\lambda_3^*}.
\end{aligned}$$

Therefore, in $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$, f_φ and g_ψ have unique common fixed soft point. □

Theorem 3.3. *Let $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$, a complete gsbms with soft real number $\widetilde{b}$ and $\mathscr{A}$ be a finite set of parameters. Let us consider a soft mapping h_γ on $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ satisfying the following conditions,*

$$\begin{aligned}
\widetilde{\zeta}(h_\gamma(P^x_{\lambda_1^*}),\ h_\gamma(P^y_{\lambda_2^*})) &\widetilde{\leq} \widetilde{r}\ \left[\widetilde{\zeta}(P^x_{\lambda_1^*},\ h_\gamma(P^x_{\lambda_1^*})\right] + \widetilde{s}\ \left[\widetilde{\zeta}(P^y_{\lambda_2^*},\ h_\gamma(P^y_{\lambda_2^*}))\right], \\
&\forall\ P^x_{\lambda_1^*},\ P^y_{\lambda_2^*}\ \widetilde{\in}\ SP(\widetilde{\mathscr{X}}),
\end{aligned}$$

where $\overline{0}\ \widetilde{\leq}\ \widetilde{r},\ \widetilde{s}\ \widetilde{<}\ \overline{\frac{1}{2}}$. Then h_γ has unique fixed soft point in $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$.

Proof. The results follows from Theorem 3.2 by setting $f_\varphi = h_\gamma = g_\psi$. $\square$

Theorem 3.4. *Let $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ be a complete gsbms with soft real number $\widetilde{b}$ and $\mathscr{A}$ be a finite set of parameters. Let f_φ and g_ψ be two soft mapping on $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ satisfying the following conditions,*

$$\widetilde{\zeta}(f_\varphi(P^x_{\lambda_1^*}),\ g_\psi(P^y_{\lambda_2^*})) \ \widetilde{\leqq}\ \widetilde{r}\,\widetilde{\zeta}(P^x_{\lambda_1^*},\ P^y_{\lambda_2^*}) + \widetilde{s}\left[\widetilde{\zeta}(P^y_{\lambda_2^*},\ f_\varphi(P^x_{\lambda_1^*})) + \widetilde{\zeta}(P^x_{\lambda_1^*},\ g_\psi(P^y_{\lambda_2^*}))\right],$$
$$\forall\ P^x_{\lambda_1^*},\ P^y_{\lambda_2^*} \widetilde{\in} SP(\widetilde{\mathscr{X}}),$$

where $\widetilde{r} \ \widetilde{\geqq}\ \overline{0},\ \widetilde{s}\ \widetilde{\geqq}\ \overline{0}$ and $\widetilde{r} + \overline{2}\ \widetilde{s}\ \widetilde{b}\ \widetilde{<}\ \overline{1}$.
Then f_φ and g_ψ have unique common fixed soft point in $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$.

Proof. Let $\widetilde{x_0} = P^x_{\lambda_1^*}$ be any arbitrary member of $SP(\widetilde{\mathscr{X}})$.
Define a sequence $\{\widetilde{x_n}\}$ in $SP(\widetilde{\mathscr{X}})$ by $\widetilde{x_1} = f_\varphi(\widetilde{x_0}),\ \widetilde{x_2} = g_\psi(\widetilde{x_1}),\ \cdots,$
$\widetilde{x_{2k+1}} = f_\varphi(\widetilde{x_{2k}}),\ \widetilde{x_{2k+2}} = g_\psi(\widetilde{x_{2k+1}}),\ k = 0, 1, 2, \cdots$

$$\begin{aligned}
\text{Now, } \widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}}) &= \widetilde{\zeta}(f_\varphi(\widetilde{x_{2k}}), g_\psi(\widetilde{x_{2k+1}}))\\
&\widetilde{\leqq}\ \widetilde{r}\,\widetilde{\zeta}(\widetilde{x_{2k}},\ \widetilde{x_{2k+1}}) + \widetilde{s}\left[\widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+1}}) + \widetilde{\zeta}(\widetilde{x_{2k}},\ \widetilde{x_{2k+2}})\right]\\
\Rightarrow \widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}}) &\widetilde{\leqq}\ \widetilde{r}\,\widetilde{\zeta}(\widetilde{x_{2k}},\ \widetilde{x_{2k+1}}) + \widetilde{s}\widetilde{b}\left[\widetilde{\zeta}(\widetilde{x_{2k}},\ \widetilde{x_{2k+1}}) + \widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}})\right]\\
\Rightarrow \widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}}) &\widetilde{\leqq}\ \overline{a_1}\,\widetilde{\zeta}(\widetilde{x_{2k}},\ \widetilde{x_{2k+1}}) + \overline{a_2}\ \overline{a_3}\left[\widetilde{\zeta}(\widetilde{x_{2k}},\ \widetilde{x_{2k+1}}) + \widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}})\right],\\
&\quad (\text{since } \mathscr{A} \text{ is finite } \exists\ a_1,\ a_2,\ a_3 \in \mathbb{R} \text{ such that, } \overline{0}\ \widetilde{\leqq}\ \widetilde{r}\ \widetilde{\leqq}\ \overline{a_1},\\
&\quad \overline{0}\ \widetilde{\leqq}\ \widetilde{s}\ \widetilde{\leqq}\ \overline{a_2},\ \overline{0}\ \widetilde{\leqq}\ \widetilde{b}\ \widetilde{\leqq}\ \overline{a_3}, \text{ and } \overline{a_1} + \overline{2}\ \overline{a_2}\ \overline{a_3}\ \widetilde{<}\ \overline{1})\\
\Rightarrow \widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}}) &\widetilde{\leqq}\ \frac{\overline{a_1} + \overline{a_2}\ \overline{a_3}}{\overline{1} - \overline{a_2}\ \overline{a_3}}\ \widetilde{\zeta}(\widetilde{x_{2k}},\ \widetilde{x_{2k+1}}) \ (1)
\end{aligned}$$

$$\begin{aligned}
\text{Similarly, } \widetilde{\zeta}(\widetilde{x_{2k+2}},\ \widetilde{x_{2k+3}}) &= \widetilde{\zeta}(f_\varphi(\widetilde{x_{2k+1}}), g_\psi(\widetilde{x_{2k+2}}))\\
&\widetilde{\leqq}\ \overline{a_1}\,\widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}})\\
&\quad + \overline{a_2}\left[\widetilde{\zeta}(\widetilde{x_{2k+2}},\ \widetilde{x_{2k+2}}) + \widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+3}})\right]\\
\Rightarrow \widetilde{\zeta}(\widetilde{x_{2k+2}}, \widetilde{x_{2k+3}}) &\widetilde{\leqq}\ \overline{a_1}\,\widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}})\\
&\quad + \overline{a_2}\ \overline{a_3}\left[\widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}}) + \widetilde{\zeta}(\widetilde{x_{2k+2}},\ \widetilde{x_{2k+3}})\right]\\
\Rightarrow \widetilde{\zeta}(\widetilde{x_{2k+2}},\ \widetilde{x_{2k+3}}) &\widetilde{\leqq}\ \frac{\overline{a_1} + \overline{a_2}\ \overline{a_3}}{\overline{1} - \overline{a_2}\ \overline{a_3}}\ \widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}}) \ (2)
\end{aligned}$$

Let $\overline{l} = \frac{\overline{a_1} + \overline{a_2}\ \overline{a_3}}{\overline{1} - \overline{a_2}\ \overline{a_3}}$. Then $\overline{0}\ \widetilde{\leqq}\ \overline{l}\ \widetilde{<}\ \overline{1}$, as $\overline{a_1}\ \widetilde{\geqq}\ \overline{0},\ \overline{a_2}\ \widetilde{\geqq}\ \overline{0}$ and $\overline{a_1} + \overline{2}\ \overline{a_2}\ \overline{a_3}\ \widetilde{<}\ \overline{1}$.
So, from (1) and (2) we have,

$$\begin{aligned}
\widetilde{\zeta}(\widetilde{x_{2k+2}},\ \widetilde{x_{2k+3}}) &\widetilde{\leqq}\ \overline{l}\,\widetilde{\zeta}(\widetilde{x_{2k+1}},\ \widetilde{x_{2k+2}})\\
&\widetilde{\leqq}\ \overline{l}^2\widetilde{\zeta}(\widetilde{x_{2k}}\ , \widetilde{x_{2k+1}});\ k = 0, 1, 2, \cdots
\end{aligned}$$

$$\begin{aligned}
\text{And hence for any } n \in \mathbb{N},\ \widetilde{\zeta}(\widetilde{x_n},\ \widetilde{x_{n+1}}) &\widetilde{\leqq}\ \overline{l}\,\widetilde{\zeta}(\widetilde{x_{n-1}},\ \widetilde{x_n})\\
&\widetilde{\leqq}\ \overline{l}^2\widetilde{\zeta}(\widetilde{x_{n-2}},\ \widetilde{x_{n-1}})\\
&\ \vdots\\
&\widetilde{\leqq}\ \overline{l}^n\widetilde{\zeta}(\widetilde{x_0},\ \widetilde{x_1}) \ (3)
\end{aligned}$$

Since $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ is a gsbms with soft constant $\widetilde{b}$ and $\mathscr{A}$ is finite, so $\exists\ c \in \mathbb{R}$ such that $\widetilde{\zeta}(\widetilde{x_u},\ \widetilde{x_v}) \mathrel{\widetilde{\leq}} \overline{c}\ [\widetilde{\zeta}(\widetilde{x_u},\ \widetilde{x_w})\ +\ \widetilde{\zeta}(\widetilde{x_w},\ \widetilde{x_v})]$, where $\overline{c} \mathrel{\widetilde{\geq}} \widetilde{b}$ and $(\overline{1}-\overline{c}\overline{l})(\alpha) \neq \overline{0}(\alpha),\ \forall\ \alpha\ \in\ \mathscr{A}$.

Thus, for any $m,\ n \in \mathbb{N}$ with $m \geq n$, we have

$$\begin{aligned}
\widetilde{\zeta}(\widetilde{x_n},\ \widetilde{x_m}) &\mathrel{\widetilde{\leq}} \overline{c}\ \widetilde{\zeta}(\widetilde{x_n},\ \widetilde{x_{n+1}}) + \overline{c}^2\ \widetilde{\zeta}(\widetilde{x_{n+1}},\ \widetilde{x_{n+2}}) + \\
&\quad \overline{c}^3\ \widetilde{\zeta}(\widetilde{x_{n+2}},\ \widetilde{x_{n+3}}) + \cdots + \overline{c}^{m-n}\ \widetilde{\zeta}(\widetilde{x_{m-1}},\ \widetilde{x_m}) \\
&\mathrel{\widetilde{\leq}} \overline{c}\ \widetilde{\zeta}(\widetilde{x_n},\ \widetilde{x_{n+1}}) + \overline{c}^2\ \widetilde{\zeta}(\widetilde{x_{n+1}},\ \widetilde{x_{n+2}}) + \overline{c}^3\ \widetilde{\zeta}(\widetilde{x_{n+2}},\ \widetilde{x_{n+3}}) + \cdots \\
&\mathrel{\widetilde{\leq}} \overline{c}\ \overline{l}^n\ \widetilde{\zeta}(\widetilde{x_0},\ \widetilde{x_1}) + \overline{c}^2\ \overline{l}^{n+1}\ \widetilde{\zeta}(\widetilde{x_0},\ \widetilde{x_1}) + \overline{c}^3\ \overline{l}^{n+2}\ \widetilde{\zeta}(\widetilde{x_0},\ \widetilde{x_1}) + \cdots \\
&= \overline{c}\ \overline{l}^n(\overline{1} + \overline{c}\ \overline{l} + \overline{c}^2\overline{l}^2 + \cdots)\ \widetilde{\zeta}(\widetilde{x_0},\ \widetilde{x_1}) \\
&= \frac{\overline{c}\ \overline{l}^n}{\overline{1} - \overline{c}\ \overline{l}}\ \widetilde{\zeta}(\widetilde{x_0},\ \widetilde{x_1})
\end{aligned}$$

$$\Rightarrow\ \widetilde{\zeta}(\widetilde{x_n},\ \widetilde{x_m}) \ \to\ \overline{0},\ \text{as } n \to \infty\ [\text{since, } \overline{0} \mathrel{\widetilde{\leq}} \overline{l} \mathrel{\widetilde{<}} \overline{1} \text{ and } (\overline{1}-\overline{c}\overline{l})(\alpha) \neq \overline{0}(\alpha),\ \forall\ \alpha\ \in\ \mathscr{A}]$$

So, $\{\widetilde{x_n}\}$, a Cauchy sequence in $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$. As $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ is complete, $\exists\ P^y_{\lambda_2^*} \mathrel{\widetilde{\in}} SP(\widetilde{\mathscr{X}})$ such that $\lim\limits_{n\to\infty} \widetilde{x_n} = P^y_{\lambda_2^*}$.

$$\begin{aligned}
\text{Now, } \widetilde{\zeta}(f_\varphi(P^y_{\lambda_2^*}),\ \widetilde{x_{2n}}) &= \widetilde{\zeta}(f_\varphi(P^y_{\lambda_2^*}),\ g_\psi(\widetilde{x_{2n-1}})) \\
&\mathrel{\widetilde{\leq}} \overline{a_1}\ \widetilde{\zeta}(P^y_{\lambda_2^*},\ \widetilde{x_{2n-1}}))\ +\ \overline{a_2}\ [\widetilde{\zeta}(\widetilde{x_{2n-1}},\ f_\varphi(P^y_{\lambda_2^*})) + \widetilde{\zeta}(P^y_{\lambda_2^*},\ \widetilde{x_{2n}})]
\end{aligned}$$

Taking $n \to \infty$ we get,

$$\begin{aligned}
\widetilde{b}^2\ \widetilde{\zeta}(f_\varphi(P^y_{\lambda_2^*}),\ P^y_{\lambda_2^*}) &\mathrel{\widetilde{\leq}} \overline{a_1}\ \widetilde{b}^2\ \widetilde{\zeta}(P^y_{\lambda_2^*},\ P^y_{\lambda_2^*})) \\
&\quad + \overline{a_2}\ \widetilde{b}^2\ [\widetilde{\zeta}(P^y_{\lambda_2^*},\ f_\varphi(P^y_{\lambda_2^*}))\ +\ \widetilde{\zeta}(P^y_{\lambda_2^*},\ P^y_{\lambda_2^*})], \\
&\quad \text{from Theorem 3.1} \\
\Rightarrow\ \widetilde{b}^2\ \widetilde{\zeta}(f_\varphi(P^y_{\lambda_2^*}),\ P^y_{\lambda_2^*}) &\mathrel{\widetilde{\leq}} \overline{a_2}\ \widetilde{b}^2\ \widetilde{\zeta}(P^y_{\lambda_2^*},\ f_\varphi(P^y_{\lambda_2^*})) \\
\Rightarrow\ \widetilde{b}^2\ (\overline{1} - \overline{a_2})\ \widetilde{\zeta}(f_\varphi(P^y_{\lambda_2^*}),\ P^y_{\lambda_2^*}) &\mathrel{\widetilde{\leq}} \overline{0} \\
\Rightarrow\ f_\varphi(P^y_{\lambda_2^*}) &= P^y_{\lambda_2^*},\ \text{since } \overline{a_2} \mathrel{\widetilde{\geq}} \overline{0} \text{ and } \widetilde{b} \mathrel{\widetilde{>}} \overline{0}.
\end{aligned}$$

Similarly, we can prove that $g_\psi(P^y_{\lambda_2^*}) = P^y_{\lambda_2^*}$.

Therefore, f_φ and g_ψ have common fixed soft point in $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$.

To prove uniqueness, let $P^z_{\lambda_3^*} \mathrel{\widetilde{\in}} SP(\widetilde{\mathscr{X}})$ be another fixed soft point in $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$.

$$\begin{aligned}
\text{Then, } \widetilde{\zeta}(P^y_{\lambda_2^*}, P^z_{\lambda_3^*}) &= \widetilde{\zeta}(f_\varphi(P^y_{\lambda_2^*}),\ g_\psi(P^z_{\lambda_3^*})) \\
&\mathrel{\widetilde{\leq}} \overline{a_1}\ \widetilde{\zeta}(P^y_{\lambda_2^*},\ P^z_{\lambda_3^*})) + \overline{a_2}\ [\widetilde{\zeta}(P^z_{\lambda_3^*},\ f_\varphi(P^y_{\lambda_2^*})) + \widetilde{\zeta}(P^y_{\lambda_2^*},\ g_\psi(P^z_{\lambda_3^*}))] \\
\Rightarrow\ \widetilde{\zeta}(P^y_{\lambda_2^*}, P^z_{\lambda_3^*}) &\mathrel{\widetilde{\leq}} \overline{a_1}\ \widetilde{\zeta}(P^y_{\lambda_2^*},\ P^z_{\lambda_3^*})) + \overline{a_2}\ [\widetilde{\zeta}(P^z_{\lambda_3^*},\ P^y_{\lambda_2^*}) + \widetilde{\zeta}(P^y_{\lambda_2^*},\ P^z_{\lambda_3^*})] \\
\Rightarrow\ \widetilde{\zeta}(P^y_{\lambda_2^*},\ P^z_{\lambda_3^*}) &= \overline{0} \text{ and hence } P^y_{\lambda_2^*} = P^z_{\lambda_3^*}.
\end{aligned}$$

Therefore, f_φ and g_ψ have unique common fixed soft point in $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$. □

Theorem 3.5. *Let $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$ be a complete gsbms with soft real number $\widetilde{b}$ and $\mathscr{A}$ be a finite set of parameters. Let us consider h_γ, a soft mapping on $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$*

satisfying the following conditions,

$$\widetilde{\zeta}(h_\gamma(P^x_{\lambda_1^*}),\ h_\gamma(P^y_{\lambda_2^*})) \quad \widetilde{\leq} \quad \widetilde{r}\,\widetilde{\zeta}(P^x_{\lambda_1^*},\ P^y_{\lambda_2^*}) + \widetilde{s}\,\left[\,\widetilde{\zeta}(P^y_{\lambda_2^*},\ h_\gamma(P^x_{\lambda_1^*})) + \widetilde{\zeta}(P^x_{\lambda_1^*},\ h_\gamma(P^y_{\lambda_2^*}))\right],$$
$$\forall\ P^x_{\lambda_1^*},\ P^y_{\lambda_2^*} \widetilde{\in}\ SP(\widetilde{\mathscr{X}}),$$

where $\widetilde{r} \ \widetilde{\geq}\ \overline{0},\ \widetilde{s}\ \widetilde{\geq}\ \overline{0}$ *and* $\widetilde{r} + \overline{2}\ \widetilde{s}\ \widetilde{b}\ \widetilde{<}\ \overline{1}$.
Then h_γ *has unique fixed soft point in* $(\widetilde{\mathscr{X}},\ \widetilde{\zeta},\ \mathscr{A})$.

Proof. The results follows from Theorem 3.4 by setting $f_\varphi = h_\gamma = g_\psi$. □

4. Conclusion

In the present paper, we have introduced the generalized soft b -Metric Space and some of its basic properties are discussed. Also, we have established some significant fixed point results in this setting. Our prospect is that this investigation has great weight and will support the researchers in cultivating new concepts in the field of soft fixed point results.

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Sk. Nazmul
Department of Mathematics, Kazi Nazrul University, Asansol, Paschim Bardwan-713340, West Bengal, India
Email address: sk.nazmul_math@yahoo.in

Utpal Badyakar (Corresponding Author)
Department of Mathematics, Bankura Sammilani College, Bankura-722102, West Bengal, India
Email address: ubmath16@gmail.com