Electronic Journal of Mathematical Analysis and Applications
Vol. 13(1) Jan. 2025, No.18.
ISSN: 2090-729X (online)
ISSN: 3009-6731(print)
http://ejmaa.journals.ekb.eg/

APPROXIMATION OF A FUNCTION WITH BOUNDED DERIVATIVES OF FIRST AND SECOND ORDER BY THE EXTENDED SINE-COSINE WAVELET EXPANSION WITH APPLICATIONS

VIVEK KUMAR SHARMA, VIRENDRA SHARMA, SONOO SINGH

Abstract. Wavelets are very powerful tools for solving certain problems in mathematical analysis. Due to their well localized behavior, wavelets are very useful for developing new numerical methods and due to this reason researchers are trying to develop new numerical techniques using different wavelets. Keeping it mind, In this paper, we have introduced the extended sine-cosine wavelet and it is used to find the approximations of a functions having bounded derivatives upto the second order. Next, we have calculated the operational matrix of integration for different values of parameter μ using these approximations. Then, we have applied these approximations and operational matrices to find the solutions of some differential and integral equations. Lastly, the comparison between exact solution and approximate solutions have been discussed to show the usefulness of the method. From the tables 1 and 3, we see that as we increase the value of μ, the approximate solution becomes closer to the exact solution which shows the validity of the proposed method.

1. Introduction

Wavelet analysis has grown to be a crucial component of signal processing, providing strong tools for signal representation and analysis across a range of applications. The extended sine-cosine wavelet is one of the many wavelet families, and because of its special qualities and uses, it is a promising method. An expansion of the sine-cosine wavelet, this wavelet adds new parameters to improve its flexibility and agility in collecting intricate signal aspects.

The extended sine-cosine wavelet inherits its foundation from the sine-cosine wavelet, a popular choice for signal representation due to its frequency localization and orthogonality properties. Through the expansion of this basic wavelet into a

2020 *Mathematics Subject Classification.* 42C40, 65T60, 65L10, 65R20.

Key words and phrases. Extended sine-cosine Wavelet, Wavelet Approximation, functions of bounded derivatives.

Submitted Sep. 28, 2024.

more versatile framework, scholars and professionals are better equipped to customize wavelet functions to the unique properties of signals that arise in a variety of applications. We explore the fundamental principles and properties of the extended sine-cosine wavelet. We look at the mathematical foundations that define its structure and explain how the addition of additional parameters improves its ability to represent signals with greater accuracy. We also talk about the possible uses of the extended sine-cosine wavelet in signal processing, image analysis, and other related areas, while emphasizing its role in meeting the growing demands for complex data representation.

The authors like Mohan and Datta [12] have analyzed identification via Fourier series for a class of lumped and distributed parameter system. Yazdani, et al.[18] gives to solving differential equations by new wavelet transform method based on the quasi-wavelets and differential invariants. Sharma, lal et al.[17] have discussed, approximation of a function having bounded derivatives upto second order by sine cosine wavelet expansion and its applications. Irfan, et al.[9] have discussed sine-cosine wavelets approach in numerical evaluation of Hankel transform for seismology. Kung, et al.[10] gives solution of integral equations using a set of block pulse functions. Azizi, et al.[3] gives applications of sine-cosine wavelets method for solving Drinfel'd–Sokolov–Wilson system. Saeed, et al.[14] discussed sine– cosine wavelets operational matrix method for fractional nonlinear differential equation. Azizi et al.[2] gives applications of sine–cosine wavelets method for solving the generalized Hirota–Satsuma coupled KdV equation. Yilmaz, et al.[19] give numerical solutions of the Fredholm integral equations of the second type. Lal and Abhilasha [11] analyzed Approximation of functions in Holder class by third kind Chebyshev wavelet and its application in solution of Fredholm integro-differential equations. Iqbal, et al.[8] analyzed Approximate solution of fractional differential equations using Shannon wavelet operational matrix method, Saeed, et al.[13] gives Fractional Gegenbauer wavelets operational matrix method for solving nonlinear fractional differential equations. Dincel, et al.[4] gives a sine-cosine wavelet method for the approximation solutions of the fractional Bagley-Torvik equation, Zhu L and Wang[21]. discussed second chebyshev wavelet operational matrix of integration and its application in the calculus of variations. Azin, H., Mohammadi, et.al[1] discussed A piecewise spectral-collocation method for solving fractional Riccati differential equation in large domains. Rahimkhani, et al.[15] discussed Fractional-order Bernoulli functions and their applications in solving fractional Fredholem–Volterra integro-differential equations. Guf, Jin-Sheng, et al.[6] derived "The Haar wavelets operational matrix of integration". Razzaghi, Mohsen, and Samira Yousefi[16] gives the Legendre wavelets operational matrix of integration. Guo, et al.[7] gives a Novel and Optimized sine–cosine Transform Wavelet Threshold Denoising Method Based on the sym4 Basis Function and Adaptive Threshold Related to Noise Intensity. Gabis, et al.[5] analyzed a comprehensive survey of sine cosine algorithm. Yazdani, H.R. and Nadjafikhah[18] gives solving differential equations by new wavelet transform method based on the quasi-wavelets and differential invariants. Yousefi, S. and Banifatemi[20] discussed Numerical solution of Fredholm integral equations by using CAS wavelets.

In this present paper, we have discussed the extended sine-cosine wavelet and its operational matrix of integration. This method is widely used in solving different problems in engineering. We used this operational matrix of integration to solve different types of problems of mathematical analysis as other kinds of orthogonal polynomials.

The paper is organized as follows: Section 1 is introductory. In section 2, extended sine-cosine wavelets and some other definitions are introduced. In section 3, two theorems and their proofs are given. In Section 4, theorems are justified with the help of an example. In section 5, operational matrix of integration for $\mu = 2$, 3, and 4 have been calculated. Section 6 contains the applications of the approximations calculated in theorems and finally, in section 7, some conclusions are given.

2. Definitions and Preliminaries

2.1. **Sine-Cosine wavelets.** Sine-Cosine wavelets [17] $\psi_{n,m}(t) = \psi(n, k, m, t)$ have four arguments, n, k, m & t, where $n = 0, 1, 2, 3, ...2^k - 1$, $k = 0, 1, 2, ...$, and m is a non negative integer, and t is normalized time and they are defined as

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k+1}{2}} f_m(2^k t - n), & \dfrac{n}{2^k} \leq \text{t} < \dfrac{n+1}{2^k}; \\ 0, & otherwise. \end{cases}$$

with

$$f_m(t) = \begin{cases} \dfrac{1}{\sqrt{2}}, & m = 0; \\ \cos(2m\pi t), & m = 1, 2, ..., L; \\ \sin 2(m-L)\pi t), & m = L+1, L+2, ..., 2L. \end{cases}$$

where L is a positive integer.

2.2. **Extended Sine-Cosine wavelets.** Extended Sine-Cosine wavelets $\phi_{n,m}(t) = \phi(n, k, m, t)$ have four arguments, n, k, m & t, where
$n = 0, 1, 2, 3, ...\mu^k - 1$, $k = 0, 1, 2, ...$, and m is a non negative integer, and t is normalized time and they are defined as

$$\phi_{n,m}(t) = \begin{cases} \sqrt{2}\mu^{\frac{k}{2}} f_m(\mu^k t - n), & \dfrac{n}{\mu^k} \leq \text{t} < \dfrac{n+1}{\mu^k}; \\ 0, & otherwise. \end{cases}$$

where

$$f_m(t) = \begin{cases} \dfrac{1}{\sqrt{2}}, & m = 0; \\ \cos(2m\pi t), & m = 1, 2, ..., L; \\ \sin(2(m-L)\pi t), & m = L+1, L+2, ..., 2L. \end{cases}$$

Here L is any positive integer. We can easily see that the system of sine-cosine wavelets form an orthonormal set.

Note: If we take $\mu = 2$ in the definition of the extended Sine-Cosine wavelets then we get the classical Sine-Cosine wavelets as defined in (2.1).

2.3. **Function expansion and approximation.** A function $f(t)$ defined over [0 1) can be expanded as follows

$$f(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_{n,m}\phi_{n,m}(t),$$

where

$$c_{n,m}=<f,\phi_{n,m}>. \tag{1}$$

If we cut-off the above infinite series in the form of partial sums $S_{\mu^k,\ 2L+1}$, then we can write,

$$S_{\mu^k,\ 2L+1}(t)=\sum_{n=0}^{\mu^k-1}\sum_{m=0}^{2L}c_{n,m}\phi_{n,m}(t)=C^T\phi(t).$$

Here C and $\phi(t)$ are the column vectors of order $\mu^k(2L+1)\times 1$ which are given by

$$C=[c_{0,0},c_{0,1},......c_{0,2L},c_{1,0},......c_{1,2L},......,c_{2^k-1,0},.....c_{\mu^k-1,2L}]^T,$$

and

$$\phi(t)=[\phi_{0,0}(t),\phi_{0,1}(t),.....\phi_{0,2L}(t),\phi_{1,0}(t),.......\phi_{1,2L}(t),.....,\phi_{\mu^k-1,0}(t),......\phi_{\mu^k-1,2L}(t)]^T.$$

2.4. **Wavelet approximation.** We define, $||f||_2=\left[\int_0^1|f(t)|^2dt\right]^{\frac{1}{2}}$. The Wavelet Approximation $E_{\mu^{k-1},M}(f)$ of f by $S_{\mu^{k-1},M}$ of its sine-cosine expansion under the norm $||.||_2$ is defined by

$$E_{\mu^k,2L+1}(f)=\inf_{S_{\mu^k},2L+1}||f-S_{\mu^k,2L+1}||_2,(Zygmund[13]).$$

If $E_{\mu^k,2L+1}(f)\to 0$ as $k\to\infty$, $L\to\infty$ then $E_{\mu^k,2L+1}(f)$ is called the best wavelet approximation of f (Zygmund[13]).

3. Theorems

Theorem 3.1. *Let $f(t)$ be a function belonging to $L^2[0,1]$ such that $f'(t)$ is bounded i.e $|f'(t)|\le N_1$, $\forall\ t\ \in[0,1]$. Let extended sine-cosine wavelet expansion of f is*

$$f(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_{n,m}^{(\mu)}\phi_{n,m}^{(\mu)}(t). \tag{2}$$

Then the extended sine-cosine wavelet approximation error

$$E^{(1)}_{\mu^k,2L+1}(f)=inf||f-S_{(\mu^k),2L+1}||_2=\begin{cases}O\left(\frac{1}{\mu^k}\right), & m=0;\\ O\left(\frac{1}{\mu^k\sqrt{2L+1}}\right), & m>0\end{cases}$$

Theorem 3.2. *Let f be a function belonging to $L^2[0,1]$ such that $f''(t)$ is bounded i.e $|f''(t)|\le N_2$, $\forall\quad t\ \in[0,1]$. Let extended sine-cosine wavelet expansion of f is given by (1).*
Suppose $(\mu^k,2L+1)^{th}$ partial sums of the series (1) is

$$S_{\mu^k,2L+1}=\sum_{n=0}^{\mu^k-1}\sum_{m=0}^{2L}c_{n,m}^{(\mu)}\phi_{n,m}^{(\mu)}. \tag{3}$$

Then the extended sine-cosine wavelet approximation error of f *by* $S_{\mu^k,2L+1}$ *is given by*

$$E^{(2)}_{\mu^k,2L+1}(f) = inf||f - S_{\mu^k,2L+1}||_2 = O\left(\frac{1}{\mu^{2k}}\frac{1}{(2L+1)^{\frac{3}{2}}}\right). \tag{4}$$

Proof of Theorem 3.1

For $m = 0$

$$\phi^{(\mu)}_{n,0}(t) = \begin{cases} \sqrt{2}\mu^{\frac{k}{2}} f_0(\mu^k t - n), & \frac{n}{\mu^k} \leq \text{t} < \frac{n+1}{\mu^k}; \\ 0, & otherwise. \end{cases}$$

$$\phi^{(\mu)}_{n,0}(t) = \begin{cases} \sqrt{2}\mu^{\frac{k}{2}} \frac{1}{\sqrt{2}}, & \frac{n}{\mu^k} \leq \text{t} < \frac{n+1}{\mu^k}; \\ 0, & otherwise. \end{cases}$$

Since

$$\phi^{(\mu)}_{n,0}(t) = \begin{cases} \mu^{\frac{k}{2}}, & \frac{n}{\mu^k} \leq \text{t} < \frac{n+1}{\mu^k}; \\ 0, & otherwise. \end{cases}$$

Let

$$f(t) = \sum_{n=0}^{\infty} c^{(\mu)}_{n,0}\phi^{(\mu)}_{n,0}.$$

and

$$S_{\mu^k,0} = \sum_{n=0}^{\mu^k-1} c^{(\mu)}_{n,0}\phi^{(\mu)}_{n,0}.$$

So,

$$\begin{aligned} c^{(\mu)}_{n,0} &= \int_{\frac{n}{\mu^k}}^{\frac{n+1}{\mu^k}} f(t)\mu^{\frac{k}{2}} dt \\ &= \mu^{\frac{k}{2}} \int_{\frac{n}{\mu^k}}^{\frac{n+1}{\mu^k}} f(t)dt \\ &= \mu^{\frac{k}{2}} \int_0^{\frac{1}{\mu^k}} f\left(u + \frac{n}{\mu^k}\right) du \;\; Here \left(t = u + \frac{n}{\mu^k}\right) \\ &= \mu^{\frac{k}{2}} \int_0^{\frac{1}{\mu^k}} \left[f\left(\frac{n}{\mu^k}\right) + \frac{u}{1!}f'\left(\frac{n}{\mu^k} + \theta u\right)\right] du; \;\; 0 < \theta < 1. \\ &= \mu^{\frac{k}{2}} \left[\frac{1}{\mu^k} f\left(\frac{n}{\mu^k}\right) + \int_0^{\frac{1}{\mu^k}} uf'\left(\frac{n}{\mu^k} + \theta u\right) du\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} c_{n,0}^2 &= \mu^k \left[\frac{1}{\mu^{2k}} f^2\left(\frac{n}{\mu^k}\right) + \left(\int_0^{\frac{1}{\mu^k}} u f'\left(\frac{n}{\mu^k} + \theta u\right) du \right)^2 + \frac{2}{\mu^k} f\left(\frac{n}{\mu^k}\right) \int_0^{\frac{1}{\mu^k}} u f'\left(\frac{n}{\mu^k} + \theta u\right) du \right]. \\ &= \frac{1}{\mu^k} f^2\left(\frac{n}{\mu^k}\right) + \mu^k \left(\int_0^{\frac{1}{\mu^k}} u f'\left(\frac{n}{\mu^k} + \theta u\right) du \right)^2 + 2 f\left(\frac{n}{\mu^k}\right) \int_0^{\frac{1}{\mu^k}} u f'\left(\frac{n}{\mu^k} + \theta u\right) du. \quad (5) \end{aligned}$$

Next,

$$\begin{aligned} ||f||_2^2 &= \int_{\frac{n}{\mu^k}}^{\frac{n+1}{\mu^k}} |f(t)|^2 dt \\ &= \int_0^{\frac{1}{\mu^k}} \left| f\left(u + \frac{n}{\mu^k}\right) \right|^2 du \\ &= \int_0^{\frac{1}{\mu^k}} f^2\left(u + \frac{n}{\mu^k}\right) du \\ &= \int_0^{\frac{1}{\mu^k}} \left[f\left(\frac{n}{\mu^k}\right) + u f'\left(\frac{n}{\mu^k} + \theta u\right) \right]^2 du \\ &= \int_0^{\frac{1}{\mu^k}} \left[f^2\left(\frac{n}{\mu^k}\right) + u^2 f'^2\left(\frac{n}{\mu^k} + \theta u\right) + 2 f\left(\frac{n}{\mu^k}\right) u f'\left(\frac{n}{\mu^k} + \theta u\right) \right] du \\ &= \frac{1}{\mu^k} f^2\left(\frac{n}{\mu^k}\right) + \int_0^{\frac{1}{\mu^k}} u^2 f'^2\left(\frac{n}{\mu^k} + \theta u\right) du + 2 f\left(\frac{n}{\mu^k}\right) \int_0^{\frac{1}{\mu^k}} u f'\left(\frac{n}{\mu^k} + \theta u\right) du \quad (6) \end{aligned}$$

Now from equations (5) and (6), we have

$$\begin{aligned} ||e_{n,0}||_2^2 = \quad & ||f||^2 - c_{n,0}^2 \\ = \quad & \frac{1}{\mu^k} f^2\left(\frac{n}{\mu^k}\right) + \int_0^{\frac{1}{\mu^k}} u^2 f'^2\left(\frac{n}{\mu^k} + \theta u\right) du + 2 f\left(\frac{n}{\mu^k}\right) \int_0^{\frac{1}{\mu^k}} u f'\left(\frac{n}{\mu^k} + \theta u\right) du \\ & - \frac{1}{\mu^k} f^2\left(\frac{n}{\mu^k}\right) - \mu^k \left(\int_0^{\frac{1}{\mu^k}} u f'\left(\frac{n}{\mu^k} + \theta u\right) du \right)^2 - 2 f\left(\frac{n}{\mu^k}\right) \int_0^{\frac{1}{\mu^k}} u f'\left(\frac{n}{\mu^k} + \theta u\right) du \\ = \quad & \int_0^{\frac{1}{\mu^k}} u^2 f'^2\left(\frac{n}{\mu^k} + \theta u\right) du - \mu^k \left[\int_0^{\frac{1}{\mu^k}} u f'\left(\frac{n}{\mu^k} + \theta u\right) du \right]^2 \\ \leq \quad & \int_0^{\frac{1}{\mu^k}} u^2 \left| f'\left(\frac{n}{\mu^k} + \theta u\right) \right|^2 du + \mu^k \left[\int_0^{\frac{1}{\mu^k}} u \left| f'\left(\frac{n}{\mu^k} + \theta u\right) \right| du \right]^2 \\ \leq \quad & N_1^2 \int_0^{\frac{1}{\mu^k}} u^2 du + \mu^k N_1^2 \left[\int_0^{\frac{1}{\mu^k}} u du \right]^2, \ \left(\because \left| f'\left(\frac{n}{\mu^k} + \theta u\right) \right| \leq N_1 \right) \\ = \quad & \frac{N_1^2}{3} \frac{1}{\mu^{3k}} + \frac{N_1^2 \mu^k}{4} \frac{1}{\mu^{4k}} \\ = \quad & \frac{7N_1^2}{12} \frac{1}{\mu^{3k}}. \end{aligned}$$

Hence,

$$||e_{n,0}^{(\mu)}||_2^2 = ||f||_2^2 - c_{n,0}^2 \leq \frac{7N_1^2}{12}\frac{1}{\mu^{3k}}.$$

So,

$$\begin{aligned} ||f - S_{\mu^k,0}||_2^2 &= \sum_{n=0}^{\mu^k-1} ||e_{n,0}||_2^2 \\ &\leq \sum_{n=0}^{\mu^k-1} \left(\frac{7N_1^2}{12}\right)\frac{1}{\mu^{3k}} \\ &= \frac{7N_1^2}{12}\frac{1}{\mu^{3k}}\mu^k \\ &= \frac{7N_1^2}{12}\frac{1}{\mu^{2k}}, \end{aligned}$$

and so

$$||f - S_{\mu^k,0}||_2 = O\left(\frac{1}{\mu^k}\right).$$

Thus,

$$\begin{aligned} E_{\mu^k,0}^{(1)} &= inf||f - S_{\mu^k,0}||_2. \\ &= O\left(\frac{1}{\mu^k}\right). \end{aligned}$$

Now, for $m = 1, 2, ..., L$, we have

$$\begin{aligned} c_{n,m}^{(\mu)} &= \int_{-\infty}^{\infty} f(t)\phi_{n,m}(t)dt \\ &= \int_{\frac{n}{\mu^k}}^{\frac{n+1}{\mu^k}} f(t)\sqrt{2}\mu^{\frac{k}{2}} f_m(\mu^k t - n)dt \\ &= \sqrt{2}\mu^{\frac{k}{2}} \int_{\frac{n}{\mu^k}}^{\frac{n+1}{\mu^k}} f(t)f_m(\mu^k t - n)\, dt \\ &= \sqrt{2}\mu^{\frac{k}{2}} \int_{\frac{n}{\mu^k}}^{\frac{n+1}{\mu^k}} f(t)cos[2m\pi(\mu^k t - n)]\, dt,\ for, m = 1, 2, ...L \\ &= \sqrt{2}\mu^{\frac{k}{2}} \left(f(t)\frac{sin[2m\pi(\mu^k t - n)]}{2m\pi(\mu^k)}\right)_{\frac{n}{\mu^k}}^{\frac{n+1}{\mu^k}} - \int_{\frac{n}{\mu^k}}^{\frac{n+1}{\mu^k}} f'(t)\frac{sin2m\pi(\mu^k t - n)}{2m\pi(\mu^k)}dt \\ &= \sqrt{2}\mu^{\frac{k}{2}} \times 0 - \sqrt{2}\mu^{\frac{k}{2}} \int_{\frac{n}{\mu^k}}^{\frac{n+1}{\mu^k}} f'(t)\frac{sin2m\pi(\mu^k t - n)}{2m\pi(\mu^k)}dt \end{aligned}$$

$$\begin{aligned} &= -\frac{\mu^{\frac{k}{2}}}{\sqrt{2}m\pi(\mu^k)}\int_{\frac{n}{\mu^k}}^{\frac{n+1}{\mu^k}} f'(t)\sin 2m\pi(\mu^k t-n)dt \\ &= -\frac{1}{\sqrt{2}m\pi\mu^{\frac{k}{2}}}\int_{\frac{n}{\mu^k}}^{\frac{n+1}{\mu^k}} f'(t)\sin 2m\pi(\mu^k t-n)dt. \end{aligned}$$

Let $2m\pi(\mu^k t-n)=u$ and simplifying the above equation, we get

$$c_{n,m} = -\frac{1}{2\sqrt{2}m^2\pi^2\mu^{\frac{3k}{2}}}\int_0^{2m\pi} f'\left(\frac{u+2m\pi n}{2m\pi\mu^k}\right)\sin(u)du.$$

Taking modulus on both side of above equation, we get

$$\begin{aligned} |c_{n,m}| &= \left|-\frac{1}{2\sqrt{2}m^2\pi^2\mu^{\frac{3k}{2}}}\int_0^{2m\pi} f'\left(\frac{u+2m\pi n}{2m\pi\mu^k}\right)\sin u du\right| \\ &\leq \frac{1}{2\sqrt{2}m^2\pi^2(\mu^{\frac{3k}{2}})}\int_0^{2m\pi}\left|f'\left(\frac{u+2m\pi n}{2m\pi\mu^k}\right)\right||sinu|du \\ &\leq \frac{N_1}{\sqrt{2}m\pi\mu^{\frac{3k}{2}}}. \end{aligned} \tag{7}$$

Again, for $m = L+1, L+2, ..., 2L$, we have

$$\begin{aligned} c_{n,m}^{(\mu)} &= \int_{\frac{n}{\mu^k}}^{\frac{n+1}{\mu^k}} f(t)\sqrt{2}\mu^{\frac{k}{2}} f_m(\mu^k t-n)dt \\ &= \sqrt{2}\mu^{\frac{k}{2}}\int_{\frac{n}{\mu^k}}^{\frac{n+1}{\mu^k}} f(t)f_m(\mu^k t-n)dt \\ &= \sqrt{2}\mu^{\frac{k}{2}}\int_{\frac{n}{\mu^k}}^{\frac{n+1}{\mu^k}} f(t)sin[2(m-L)\pi(\mu^k t-n)]dt. \end{aligned}$$

Let $2(m-L)(\mu^k t-n)\pi = u$ and simplifying the above equation, we get

$$\begin{aligned} &= \sqrt{2}\mu^{\frac{k}{2}}\int_0^{2(m-L)\pi} f\left[\frac{u+2(m-L)n\pi}{2(m-L)\pi\mu^k}\right] sin(u)\frac{du}{2(m-L)\pi\mu^k}. \\ &= \frac{\sqrt{2}\mu^{\frac{k}{2}}}{2(m-L)\pi\mu^k}\int_0^{2(m-L)\pi} f\left[\frac{u+2(m-L)n\pi}{2(m-L)\pi\mu^k}\right] sin(u)\, du \\ &= \frac{1}{\sqrt{2}(m-L)\pi\mu^k}\int_0^{2(m-L)\pi} f\left[\frac{u+2(m-L)n\pi}{2(m-L)\pi\mu^k}\right] sin(u)\, du \\ &= \frac{1}{\sqrt{2}(m-L)\pi\mu^{\frac{k}{2}}}\left[0+\int_0^{2(m-L)\pi} f'\left(\frac{u+2(m-L)n\pi}{2(m-L)\pi\mu^k}\right)\frac{1}{2(m-L)\pi\mu^k}cos(u)\, du\right] \\ &= \frac{1}{\sqrt{2}(m-L)\pi\mu^{\frac{k}{2}}}\times\frac{1}{2(m-L)\pi\mu^k}\left[\int_0^{2(m-L)\pi} f'\left(\frac{u+2(m-L)n\pi}{2(m-L)\pi\mu^k}\right)cos(u)\, du\right] \\ &= \frac{1}{2^{\frac{3}{2}}(m-L)^2\pi^2\mu^{\frac{3k}{2}}}\left[\int_0^{2(m-L)\pi} f'\left(\frac{u+2(m-L)n\pi}{2(m-L)\pi\mu^k}\right)cos(u)\, du\right]. \end{aligned}$$

Taking the modulus on both sides of above, we get

$$|c_{n,m}| \qquad \leq \frac{N_1}{\sqrt{2}(m-L)\pi\mu^{\frac{3k}{2}}}. \tag{8}$$

By using (7) and (8), we have

$$\begin{aligned}
||f - S_{\mu^k,2L+1}||_2^2 &= \sum_{n=0}^{\mu^k-1} \sum_{m=2L+1}^{\infty} |c_{n,m}^{(\mu)}|^2 \\
&= \sum_{n=0}^{\mu^k-1} \left[\sum_{m=2L+1}^{3L} |c_{n,m}^{(\mu)}|^2 + \sum_{m=3L+1}^{4L} |c_{n,m}^{(\mu)}|^2 + \sum_{m=4L+1}^{5L} |c_{n,m}^{(\mu)}|^2 + \sum_{m=5L+1}^{6L} |c_{n,m}^{(\mu)}|^2 + ... \right] \\
&= \leq \sum_{n=0}^{\mu^k-1} \left[\frac{1}{2L+1} + \frac{1}{2L+1} \right] \frac{N_1^2}{2\pi^2} \frac{1}{\mu^{3k}} \\
&= \frac{N_1^2}{2\pi^2} \frac{2}{2L+1} \frac{1}{\mu^{2k}}.
\end{aligned}$$

Thus,

$$\|f - S_{\mu^k,2L+1}\| = O\left(\frac{1}{\mu^k} \frac{1}{\sqrt{2L+1}}\right),$$

and so

$$E_{\mu^k,2L+1}^{(1)} = inf||f - S_{\mu^k,2L+1}||_2 = O\left(\frac{1}{\mu^k} \frac{1}{\sqrt{2L+1}}\right).$$

Proof of Theorem 3.2. By equation (1), it follows that

$$\begin{aligned}
c_{n,m}^{(\mu)} &= \frac{-1}{2\sqrt{2}m^2\pi^2\mu^{\frac{3k}{2}}} \int_0^{2m\pi} f'\left(\frac{u+2mn\pi}{2m\pi\mu^k}\right) \sin(u)\ du \\
&= \frac{-1}{2\sqrt{2}m^2\pi^2\mu^{\frac{3k}{2}}} \left[0 - \int_0^{2m\pi} f''\left(\frac{u+2mn\pi}{2m\pi\mu^k}\right) \times \frac{1}{2m\pi\mu^k}(-cos(u))\ du\right] \\
&= \frac{-1}{4\sqrt{2}m^3\pi^3\mu^{\frac{5k}{2}}} \int_0^{2m\pi} f''\left(\frac{u+2mn\pi}{2m\pi\mu^k}\right) cos(u)du.
\end{aligned}$$

So,

$$\begin{aligned}
|c_{n,m}^{(\mu)}| &\leq \frac{1}{4\sqrt{2}m^3\pi^3\mu^{\frac{5k}{2}}} \int_0^{2m\pi} \left|f''\left(\frac{u+2mn\pi}{2m\pi\mu^k}\right)\right| |cos(u)|\, du \\
&= \frac{1}{4\sqrt{2}m^3\pi^3\mu^{\frac{5k}{2}}} N_2.2m\pi \\
&= \frac{N_2}{2\sqrt{2}m^2\pi^2\mu^{\frac{5k}{2}}}.
\end{aligned} \tag{9}$$

Now for $m = L+1, L+2, ...2L$, we have

$$\begin{aligned}
c_{n,m}^{(\mu)} &= \frac{1}{2\sqrt{2}(m-L)^2\pi^2\mu^{\frac{3k}{2}}}\int_0^{2(m-L)\pi} f'\left(\frac{u+2(m-L)n\pi}{2(m-L)\pi\mu^k}\right) cosu\, du \\
&= \frac{1}{2\sqrt{2}(m-L)^2\pi^2\mu^{\frac{3k}{2}}}\left[0-\int_0^{2(m-L)\pi} f''\left(\frac{u+2(m-L)n\pi}{2(m-L)\pi\mu^k}\right) sinu \times \frac{1}{2(m-L)\pi\mu^k}\, du\right] \\
&= \frac{1}{4\sqrt{2}(m-L)^3\pi^3\mu^{\frac{5k}{2}}}\int_0^{2(m-L)\pi} \left|f''\left(\frac{u+2(m-L)n\pi}{2(m-L)\pi\mu^k}\right)\right| |sinu|\, du. \\
&\leq \frac{N_2}{4\sqrt{2}(m-L)^3\pi^3\mu^{\frac{5k}{2}}} \times 2(m-L)\pi \\
&= \frac{N_2}{2\sqrt{2}(m-L)^2\pi^2\mu^{\frac{5k}{2}}}
\end{aligned}$$

$i.e$

$$\begin{aligned}
|c_{n,m}^{(\mu)}| &\leq \frac{N_2}{2\sqrt{2}(m-L)^2\pi^2\mu^{\frac{5k}{2}}} \\
&= \frac{N_2}{2\sqrt{2}(m-L)^2\pi^2\mu^{\frac{5k}{2}}}. \qquad (10)
\end{aligned}$$

By using (9) and (10), we have

$$\begin{aligned}
\|f - S_{\mu^k,2L+1}\|_2^2 &= \sum_{n=0}^{\mu^k-1}\sum_{m=2L+1}^{\infty} |c_{n,m}^{(\mu)}|^2 \\
&= \sum_{n=0}^{\mu^k-1}\left[\sum_{m=2L+1}^{3L}|c_{n,m}^{(\mu)}|^2 + \sum_{m=3L+1}^{4L}|c_{n,m}^{(\mu)}|^2 + \sum_{m=4L+1}^{5L}|c_{n,m}^{(\mu)}|^2 + \sum_{m=5L+1}^{6L}|c_{n,m}^{(\mu)}|^2 + ...\right] \\
&\leq \frac{N_2^2}{8\pi^4\ \mu^{5k}}\mu^k\left[\frac{2}{(2L+1)^3}\right] \\
&= \frac{N_2^2}{4\pi^4\ \mu^{4k}}\left[\frac{1}{(2L+1)^3}\right].
\end{aligned}$$

Hence,

$$\|f - S_{\mu^k,2L+1}\|_2 = O\left(\frac{1}{\mu^{2k}}\frac{1}{(2L+1)^{\frac{3}{2}}}\right).$$

Thus

$$E_{\mu^k,2L+1}^{(2)} = inf\|f - S_{\mu^k,2L+1}\|_2 = O\left(\frac{1}{\mu^{2k}}\frac{1}{(2L+1)^{\frac{3}{2}}}\right).$$

Remark: If we take $\mu = 2$ in the theorems proved in section 3, then we can obtain the theorems proved in [17]. This shows that the theorems of this paper generalizes the theorems proved in [17].

4. Numerical justification of calculated approximation

In this section, approximation of the function $f(t) = e^{-t^2}t^{\frac{1}{2}}$ over interval [0,1) for $k = 1$, $L = 3$, $\mu = 2$; $k = 1$, $L = 3$, $\mu = 3$; and also for $k = 1$, $L = 3$, $\mu = 4$

have been calculated and it is shown in the figures [1], [2], and [3].

$$S_{2^1,7}(t)=\begin{cases} 0.300295\times\sqrt{2}-0.0291881\times 2\,cos(4\pi t)-0.0100289\times 2\,cos(8\pi t)-0.00543327 \\ \times 2\,cos(12\pi t)-0.0610448\times 2\,sin(4\pi t)-0.0610448\times 2\,sin(8\pi t)-0.0610448 \\ \times 2\,sin(12\pi t), \qquad\qquad 0<t\leq\frac{1}{2}.; \\ \\ 0.340898\times\sqrt{2}-0.00690776\times 2\,cos(2\pi(2t-1))-0.0017418\times 2\,cos((4\pi(2t-1)) \\ -0.000775441\times 2\,cos(6\pi(2t-1))+0.0312898\times 2\,sin(2\pi(2t-1))+0.0148156 \\ \times 2\,sin(4\pi(2t-1))+0.00977767\times 2\,sin(6\pi(2t-1)). \qquad \frac{1}{2}\leq t<1. \end{cases}$$

$$S_{3^1,7}(t)=\begin{cases} 0.212004\times\sqrt{3}-0.0160893\times\sqrt{6}\,cos(6\pi t)-0.00591381\times\sqrt{6}\,cos(12\pi t)-0.00328554 \\ \times\sqrt{6}\,cos(18\pi t)-0.0492571\times\sqrt{6}\,sin(6\pi t)-0.0270556\times\sqrt{6}\,sin(12\pi t)-0.0188054 \\ \times\sqrt{6}\,sin(18\pi t), \qquad\qquad 0<t\leq\frac{1}{3}.; \\ \\ 0.312054\times\sqrt{3}-0.00506428\times\sqrt{6}\,cos(2\pi(3t-1))-0.00126709\times\sqrt{6}\,cos(4\pi(3t-1)) \\ -0.000563429\times\sqrt{6}\,cos(6\pi(3t-1))-0.000348594\times\sqrt{6}\,sin(2\pi(3t-1))-0.000377697 \\ \times\sqrt{6}\,sin(4\pi(3t-1))-0.000277499\times\sqrt{6}\,sin(6\pi(3t-1)), \qquad \frac{1}{3}\leq t<\frac{2}{3}. \\ \\ 0.261239\times\sqrt{3}-0.0016738\times\sqrt{6}\,cos(2\pi(3t-2))-0.00042319\times\sqrt{6}\,cos(4\pi(3t-2)) \\ -0.000188465\times\sqrt{6}\,cos(6\pi(3t-2))-0.00207328\times\sqrt{6}\,sin(2\pi(3t-2))-0.0101755 \\ \times\sqrt{6}\,sin(4\pi(3t-2))-0.0067604\times\sqrt{6}\,sin(6\pi(3t-2)). \qquad \frac{2}{3}\leq t<1. \end{cases}$$

$$S_{4^1,7}(t)=\begin{cases} 0.16229\times 2-0.0110593\times 2\sqrt{2}\,cos(8\pi t)-0.00419668\times 2\sqrt{2}\,cos(16\pi t)-0.00235942 \\ \times 2\sqrt{2}\,cos(24\pi t)-0.0393367\times 2\sqrt{2}\,sin(8\pi t)-0.0215295\times 2\sqrt{2}\,sin(16\pi t)-0.014935 \\ \times 2\sqrt{2}\,sin(24\pi t), \qquad , \qquad 0<t\leq\frac{1}{4}.; \\ \\ 0.262391\times 2-0.00312367\times 2\sqrt{2}\,cos(2\pi(4t-1))-0.000786262\times 2\sqrt{2}\,cos(4\pi(4t-1)) \\ -0.000350031\times 2\sqrt{2}\,Cos(6\pi(4t-1))-0.00887901\times 2\sqrt{2}\,sin(2\pi(4t-1))-0.00452601 \\ \times 2\sqrt{2}\,sin(4\pi(4t-1))-0.00302877\times 2\sqrt{2}\,sin(6\pi(4t-1)), \qquad \frac{1}{4}\leq t<\frac{2}{4}. \\ \\ 0.26532\times 2-0.00184218\times 2\sqrt{2}\,cos(2\pi(4t-2))-0.000460353\times 2\sqrt{2}\,cos(4\pi(4t-2)) \\ -0.000204593\times 2\sqrt{2}\,cos(6\pi(4t-2))+0.00664311\times 2\sqrt{2}\,sin(2\pi(4t-2))+0.00324634 \\ \times 2\sqrt{2}\,sin(4\pi(4t-2))+0.00215498\times 2\sqrt{2}\,sin(6\pi(4t-2)), \qquad \frac{2}{4}\leq t<\frac{3}{4}. \\ \\ 0.216783\times 2-0.000621097\times 2\sqrt{2}\,cos(2\pi(4t-3))-0.000156884\times 2\sqrt{2}\,cos(4\pi(4t-3)) \\ -0.0000698564\times 2\sqrt{2}\,cos(6\pi(4t-3))+0.0143093\times 2\sqrt{2}\,sin(2\pi(4t-3))+0.00708774 \\ \times 2\sqrt{2}\,sin(4\pi(4t-3))+0.00471696\times 2\sqrt{2}\,sin(6\pi(4t-3)). \qquad \frac{3}{4}\leq t<1. \end{cases}$$

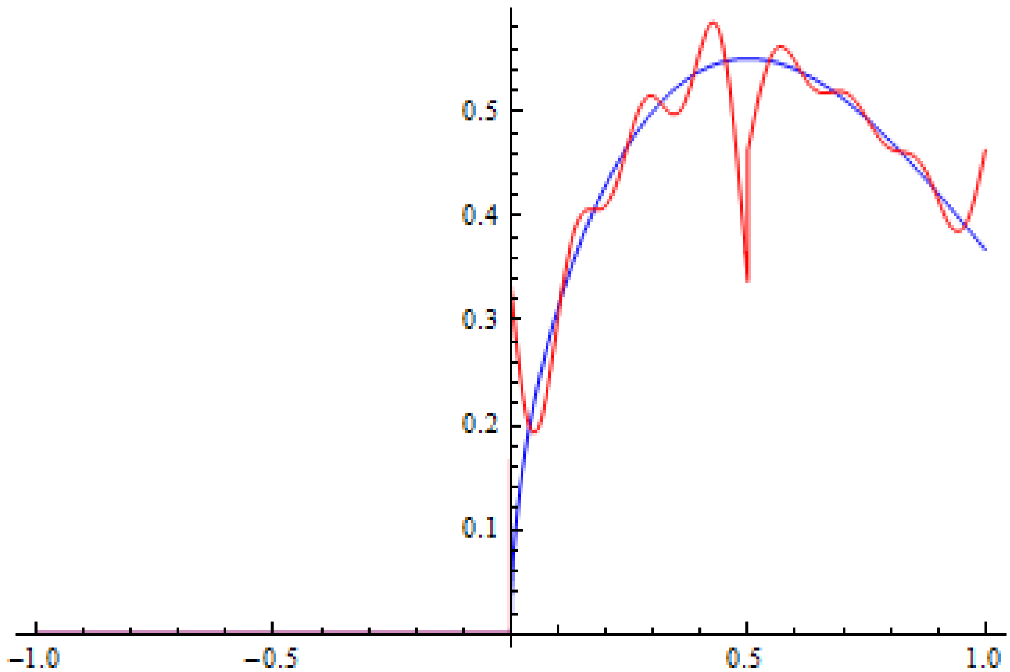

FIGURE 1. Graph of $S_{2^1,7}(t)$ and function $f(t)$

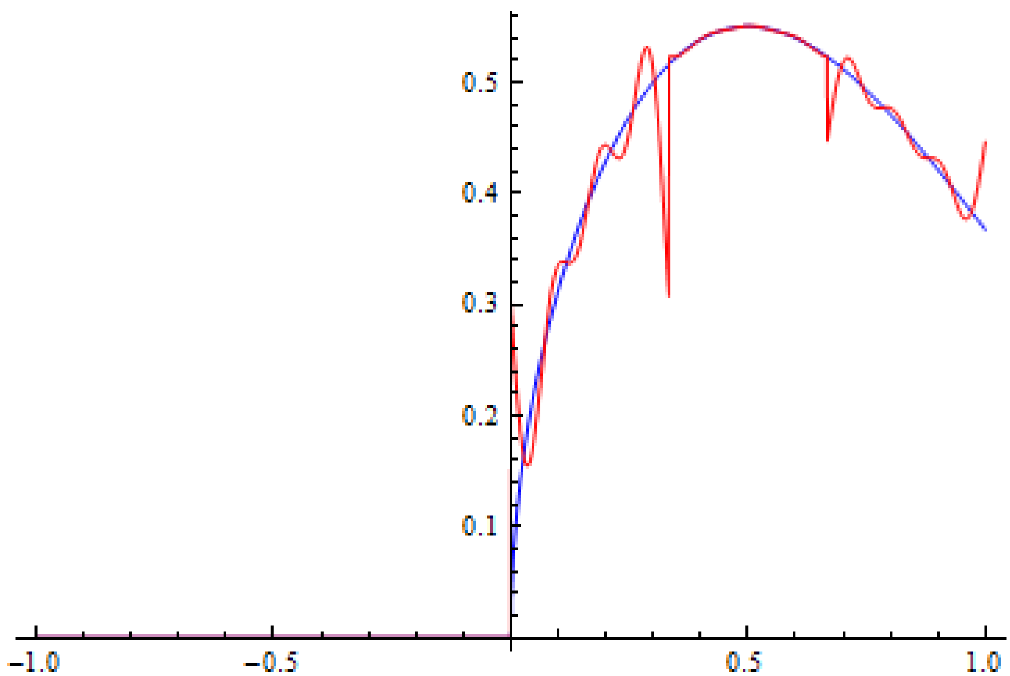

FIGURE 2. Graph of $S_{3^1,7}(t)$ and function $f(t)$

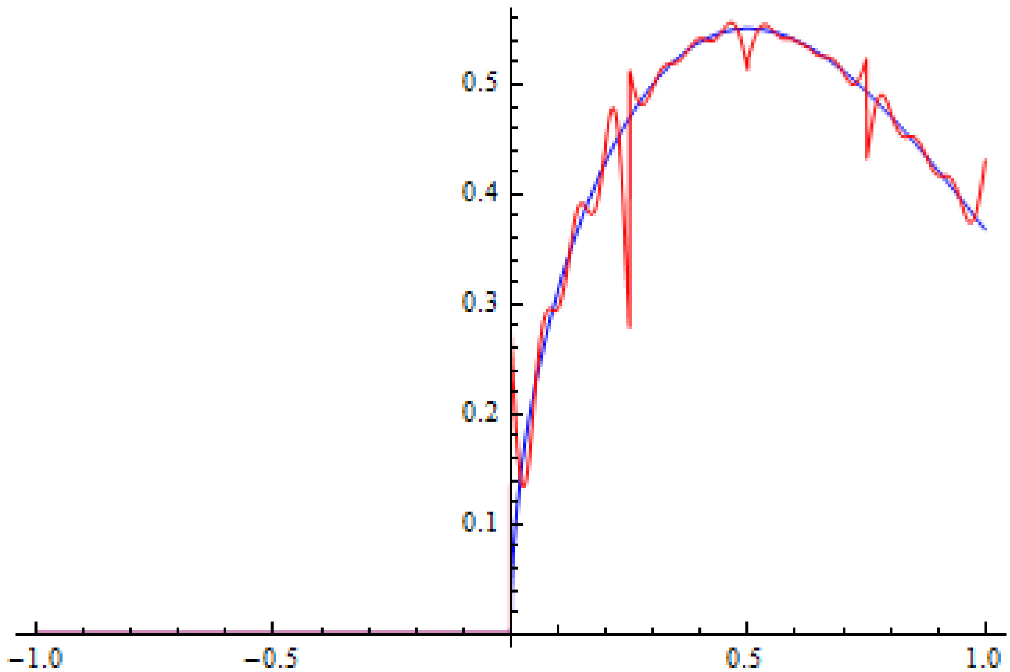

FIGURE 3. Graph of $S_{4^1,7}(t)$ and function $f(t)$

5. Evaluation of the operational matrix of integration for sine-cosine wavelets

5.1. Sine-Cosine wavelet Operational matrix of integration for $\mu = 2$. Here, we find the operational matrix of integration $P^{(2)}$ with $k = 1$ and $L = 3$ $\mu = 2$. The 14 basis functions are given by

$$\begin{cases} \phi_{0,0} = \sqrt{2} \\ \phi_{0,1} = 2\,cos(4\pi t) \\ \phi_{0,2} = 2\,cos(8\pi t) \\ \phi_{0,3} = 2\,cos(12\pi t) \\ \phi_{0,4} = 2\,sin(4\pi t) \\ \phi_{0,5} = 2\,sin(8\pi t) \\ \phi_{0,6} = 2\,sin(12\pi t) \end{cases} \quad 0 \le t < \frac{1}{2} \;(11) \quad \begin{cases} \phi_{1,0} = \sqrt{2} \\ \phi_{1,1} = 2\,cos(2\pi(2t-1)) \\ \phi_{1,2} = 2\,cos(4\pi(2t-1)) \\ \phi_{1,3} = 2\,cos(6\pi(2t-1)) \\ \phi_{1,4} = 2\,sin(2\pi(2t-1)) \\ \phi_{1,5} = 2\,sin(4\pi(2t-1)) \\ \phi_{1,6} = 2\,sin(6\pi(2t-1)) \end{cases} \quad \frac{1}{2} \le t < 1. \;(12)$$

By integrating equation (11) from 0 to t and using in equation 1 we get,

$$\begin{cases} \int_0^t \phi_{0,0}(t)dt = \left[\frac{1}{4}, 0, 0, 0, \frac{-1}{2\sqrt{2}\pi}, \frac{-1}{4\sqrt{2}\pi}, \frac{-1}{6\sqrt{2}\pi}, \frac{1}{2}, 0, 0, 0, 0, 0, 0\right]^T \phi_{14}^{(2)}(t), \\ \int_0^t \phi_{0,1}(t)dt = \left[0, 0, 0, 0, \frac{1}{4\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{14}^{(2)}(t), \\ \int_0^t \phi_{0,2}(t)dt = \left[0, 0, 0, 0, 0, \frac{1}{8\pi}, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{14}^{(2)}(t), \\ \int_0^t \phi_{0,3}(t)dt = \left[0, 0, 0, 0, 0, 0, \frac{1}{12\pi}, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{14}^{(2)}(t), \\ \int_0^t \phi_{0,4}(t)dt = \left[\frac{1}{2\sqrt{2}\pi}, -\frac{1}{4\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{14}^{(2)}(t), \\ \int_0^t \phi_{0,5}(t)dt = \left[\frac{1}{4\sqrt{2}\pi}, 0, -\frac{1}{8\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{14}^{(2)}(t), \\ \int_0^t \phi_{0,6}(t)dt = \left[\frac{1}{6\sqrt{2}\pi}, 0, 0, -\frac{1}{12\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{14}^{(2)}(t), \end{cases}$$

and similarly for (12),

$$\begin{cases} \int_0^t \phi_{1,0}(t)dt = \left[0, 0, 0, 0, 0, 0, 0, \frac{1}{4}, 0, 0, 0, \frac{-1}{2\sqrt{2}\pi}, \frac{-1}{4\sqrt{2}\pi}, \frac{-1}{6\sqrt{2}\pi}, \right]^T \phi_{14}^{(2)}(t), \\ \int_0^t \phi_{1,1}(t)dt = \left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{4\pi}, 0, 0, 0\right]^T \phi_{14}^{(2)}(t), \\ \int_0^t \phi_{1,2}(t)dt = \left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{8\pi}, 0, 0\right]^T \phi_{14}^{(2)}(t), \\ \int_0^t \phi_{1,3}(t)dt = \left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{12\pi}, 0\right]^T \phi_{14}^{(2)}(t), \\ \int_0^t \phi_{1,4}(t)dt = \left[0, 0, 0, 0, 0, 0, 0, \frac{1}{2\sqrt{2}\pi}, -\frac{1}{4\pi}, 0, 0, 0, 0, 0\right]^T \phi_{14}^{(2)}(t), \\ \int_0^t \phi_{1,5}(t)dt = \left[0, 0, 0, 0, 0, 0, 0, \frac{1}{4\sqrt{2}\pi}, 0, -\frac{1}{8\pi}, 0, 0, 0, 0\right]^T \phi_{14}^{(2)}(t), \\ \int_0^t \phi_{1,6}(t)dt = \left[0, 0, 0, 0, 0, 0, 0, \frac{1}{6\sqrt{2}\pi}, 0, 0, -\frac{1}{12\pi}, 0, 0, 0\right]^T \phi_{14}^{(2)}(t), \end{cases}$$

Thus, we can write

$$\int_0^t \phi_{14}(x)dx = P_{14\times 14}^{(2)} \phi_{14}^{(2)}(t).$$

where

$$\phi_{14}^{(2)}(t) = \left[\phi_{0,0}, \psi_{0,1}, \phi_{0,2}, \phi_{0,3}, \phi_{0,4}, \phi_{0,5}, \phi_{0,6}, \phi_{1,0}, \phi_{1,1}, \phi_{1,2}, \phi_{1,3}, \phi_{1,4}, \phi_{1,5}, \phi_{1,6}\right]^T.$$

and $P_{14\times 14}^{(2)}$ be an operational matrix of integration which is given as

$$P_{14\times 14}^{(2)} = \begin{bmatrix} A & B \\ O & A \end{bmatrix}.$$

Here

$$A_{7\times 7} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & \frac{-1}{2\sqrt{2}\pi} & \frac{-1}{4\sqrt{2}\pi} & \frac{-1}{6\sqrt{2}\pi} \\ 0 & 0 & 0 & 0 & \frac{1}{4\pi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12\pi} \\ \frac{1}{2\sqrt{2}\pi} & -\frac{1}{4\pi} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4\sqrt{2}\pi} & 0 & -\frac{1}{8\pi} & 0 & 0 & 0 & 0 \\ \frac{1}{6\sqrt{2}\pi} & 0 & 0 & -\frac{1}{12\pi} & 0 & 0 & 0 \end{bmatrix},$$

$$B_{7\times 7} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, O_{7\times 7} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

5.2. **Sine-Cosine wavelet Operational matrix of integration for $\mu = 3$.** In this section, we find the operational matrix of integration $P^{(3)}$ with $k = 1$ and $L = 3$ $\mu = 3$. The 21 basis function are given by,

$$\begin{cases} \phi_{0,0} = \sqrt{3} \\ \phi_{0,1} = \sqrt{6}\, cos(6\pi t) \\ \phi_{0,2} = \sqrt{6}\, cos(12\pi t) \\ \phi_{0,3} = \sqrt{6}\, cos(18\pi t) \\ \phi_{0,4} = \sqrt{6}\, sin(6\pi t) \\ \phi_{0,5} = \sqrt{6}\, sin(12\pi t) \\ \phi_{0,6} = \sqrt{6}\, sin(18\pi t) \end{cases} \quad 0 \le t < \frac{1}{3} \quad (13)$$

$$\begin{cases} \phi_{1,0} = \sqrt{3} \\ \phi_{1,1} = \sqrt{6}\, cos(2\pi(3t-1)) \\ \phi_{1,2} = \sqrt{6}\, cos(4\pi(3t-1)) \\ \phi_{1,3} = \sqrt{6}\, cos(6\pi(3t-1)) \\ \phi_{1,4} = \sqrt{6}\, sin(2\pi(3t-1)) \\ \phi_{1,5} = \sqrt{6}\, sin(4\pi(3t-1)) \\ \phi_{1,6} = \sqrt{6}\, sin(6\pi(3t-1)) \end{cases} \quad \frac{1}{3} \le t < \frac{2}{3} \quad (14)$$

$$\begin{cases} \phi_{1,0} = \sqrt{3} \\ \phi_{1,1} = \sqrt{6}\, cos(2\pi(3t-1)) \\ \phi_{1,2} = \sqrt{6}\, cos(4\pi(3t-1)) \\ \phi_{1,3} = \sqrt{6}\, cos(6\pi(3t-1)) \\ \phi_{1,4} = \sqrt{6}\, sin(2\pi(3t-1)) \\ \phi_{1,5} = \sqrt{6}\, sin(4\pi(3t-1)) \\ \phi_{1,6} = \sqrt{6}\, sin(6\pi(3t-1)) \end{cases} \quad \frac{2}{3} \le t < 1. \quad (15)$$

By integrating equation (13) from 0 to t and using in equation 1 we get,

$$\begin{cases} \int_0^t \phi_{0,0}(t)dt = \left[\frac{1}{6}, 0, 0, 0, \frac{-1}{3\sqrt{2}\pi}, \frac{-1}{6\sqrt{2}\pi}, \frac{-1}{9\sqrt{2}\pi}, \frac{1}{3}, 0, 0, 0, 0, 0, 0, \frac{1}{3}, 0, 0, 0, 0, 0, 0\right]^T \phi_{21}^{(3)}(t), \\ \int_0^t \phi_{0,1}(t)dt = \left[0, 0, 0, 0, \frac{1}{6\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{21}^{(3)}(t), \\ \int_0^t \phi_{0,2}(t)dt = \left[0, 0, 0, 0, 0, \frac{1}{12\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{21}^{(3)}(t), \\ \int_0^t \phi_{0,3}(t)dt = \left[0, 0, 0, 0, 0, 0, \frac{1}{18\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{21}^{(3)}(t), \\ \int_0^t \phi_{0,4}(t)dt = \left[\frac{1}{3\sqrt{2}\pi}, -\frac{1}{6\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{21}^{(3)}(t), \\ \int_0^t \phi_{0,5}(t)dt = \left[\frac{1}{6\sqrt{2}\pi}, 0, -\frac{1}{12\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{21}^{(3)}(t), \\ \int_0^t \phi_{0,6}(t)dt = \left[\frac{1}{9\sqrt{2}\pi}, 0, 0, -\frac{1}{18\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{21}^{(3)}(t), \end{cases}$$

And similarly for (14),

$$\begin{cases}\int_0^t \phi_{1,0}(t)dt = \left[0,0,0,0,0,0,0,\frac{1}{6},0,0,0,\frac{-1}{3\sqrt{2}\pi},\frac{-1}{6\sqrt{2}\pi},\frac{-1}{9\sqrt{2}\pi},\frac{1}{3},0,0,0,0,0,0\right]^T \phi_{21}^{(3)}(t),\\ \int_0^t \phi_{1,1}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,\frac{1}{6\pi},0,0,0,0,0,0,0,0,0\right]^T \phi_{21}^{(3)}(t),\\ \int_0^t \phi_{1,2}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{12\pi},0,0,0,0,0,0,0,0\right]^T \phi_{21}^{(3)}(t),\\ \int_0^t \phi_{1,3}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{18\pi},0,0,0,0,0,0,0\right]^T \phi_{21}^{(3)}(t),\\ \int_0^t \phi_{1,4}(t)dt = \left[0,0,0,0,0,0,0,\frac{1}{3\sqrt{3}\pi},-\frac{1}{6\pi},0,0,0,0,0,0,0,0,0,0,0,0\right]^T \phi_{21}^{(3)}(t),\\ \int_0^t \phi_{1,5}(t)dt = \left[0,0,0,0,0,0,0,\frac{1}{6\sqrt{2}\pi},0,-\frac{1}{12\pi},0,0,0,0,0,0,0,0,0,0,0\right]^T \phi_{21}^{(3)}(t),\\ \int_0^t \phi_{1,6}(t)dt = \left[0,0,0,0,0,0,0,\frac{1}{9\sqrt{2}\pi},0,0,-\frac{1}{18\pi},0,0,0,0,0,0,0,0,0,0\right]^T \phi_{21}^{(3)}(t),\end{cases}$$

Again for (15),

$$\begin{cases}\int_0^t \phi_{1,0}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{6},0,0,0,\frac{-1}{3\sqrt{2}\pi},\frac{-1}{6\sqrt{2}\pi},\frac{-1}{9\sqrt{2}\pi}\right]^T \phi_{21}^{(3)}(t),\\ \int_0^t \phi_{1,1}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{6\pi},0,0\right]^T \phi_{21}^{(3)}(t),\\ \int_0^t \phi_{1,2}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{12\pi},0\right]^T \phi_{21}^{(3)}(t),\\ \int_0^t \phi_{1,3}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{18\pi}\right]^T \phi_{21}^{(3)}(t),\\ \int_0^t \phi_{1,4}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{3\sqrt{3}\pi},-\frac{1}{6\pi},0,0,0,0,0\right]^T \phi_{21}^{(3)}(t),\\ \int_0^t \phi_{1,5}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{6\sqrt{2}\pi},0,-\frac{1}{12\pi},0,0,0,0\right]^T \phi_{21}^{(3)}(t),\\ \int_0^t \phi_{1,6}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{9\sqrt{2}\pi},0,0,-\frac{1}{18\pi},0,0,0\right]^T \phi_{21}^{(3)}(t),\end{cases}$$

Thus, we can write

$$\int_0^t \phi_{21}(x)dx = P_{21\times 21}^{(3)} \phi_{21}^{(3)}(t).$$

where

$$\phi_{21}^{(3)}(t) = \left[\phi_{0,0},\phi_{0,1},\phi_{0,2},...,\phi_{0,6},\phi_{1,0},\phi_{1,1},\phi_{1,2},...,\phi_{1,6},\phi_{2,0},\phi_{2,1},\phi_{2,2},...,\phi_{2,6}\right]^T.$$

and $P_{21\times 21}^{(3)}$ be an operational matrix of integration which is given as

$$P_{21\times 21}^{(3)} = \begin{bmatrix} C & D & D\\ O & C & D\\ O & O & C\end{bmatrix}.$$

Here

$$C_{7\times 7} = \begin{bmatrix} \frac{1}{6} & 0 & 0 & 0 & \frac{-1}{3\sqrt{2}\pi} & \frac{-1}{6\sqrt{2}\pi} & \frac{-1}{9\sqrt{2}\pi}\\ 0 & 0 & 0 & 0 & \frac{1}{6\pi} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{12\pi} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{18\pi}\\ \frac{1}{3\sqrt{2}\pi} & -\frac{1}{6\pi} & 0 & 0 & 0 & 0 & 0\\ \frac{1}{6\sqrt{2}\pi} & 0 & -\frac{1}{12\pi} & 0 & 0 & 0 & 0\\ \frac{1}{9\sqrt{2}\pi} & 0 & 0 & -\frac{1}{18\pi} & 0 & 0 & 0 \end{bmatrix},$$

$$D_{7\times 7}=\begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, O_{7\times 7}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

5.3. **Sine-Cosine wavelet Operational matrix of integration for $\mu = 4$.** Finally, we have computed the operational matrix of integration $P^{(3)}$ with $k = 1$ and $L = 3$ $\mu = 4$. The 28 basis function are given by

$$\begin{cases} \phi_{0,0} = 2 \\ \phi_{0,1} = 2\sqrt{2}\,cos(8\pi t) \\ \phi_{0,2} = 2\sqrt{2}\,cos(16\pi t) \\ \phi_{0,3} = 2\sqrt{2}\,cos(24\pi t) \\ \phi_{0,4} = 2\sqrt{2}\,sin(8\pi t) \\ \phi_{0,5} = 2\sqrt{2}\,sin(16\pi t) \\ \phi_{0,6} = 2\sqrt{2}\,sin(24\pi t) \end{cases} \quad 0 \le t < \frac{1}{4} \qquad (16)$$

$$\begin{cases} \phi_{1,0} = 2 \\ \phi_{1,1} = 2\sqrt{2}\,cos(2\pi(4t-1)) \\ \phi_{1,2} = 2\sqrt{2}\,cos(4\pi(4t-1)) \\ \phi_{1,3} = 2\sqrt{2}\,cos(6\pi(4t-1)) \\ \phi_{1,4} = 2\sqrt{2}\,sin(2\pi(4t-1)) \\ \phi_{1,5} = 2\sqrt{2}\,sin(4\pi(4t-1)) \\ \phi_{1,6} = 2\sqrt{2}\,sin(6\pi(4t-1)) \end{cases} \quad \frac{1}{4} \le t < \frac{2}{4} \qquad (17)$$

$$\begin{cases} \phi_{2,0} = 2 \\ \phi_{2,1} = 2\sqrt{2}\,cos(2\pi(4t-2)) \\ \phi_{2,2} = 2\sqrt{2}\,cos(4\pi(4t-2)) \\ \phi_{2,3} = 2\sqrt{2}\,cos(6\pi(4t-2)) \\ \phi_{2,4} = 2\sqrt{2}\,sin(2\pi(4t-2)) \\ \phi_{2,5} = 2\sqrt{2}\,sin(4\pi(4t-2)) \\ \phi_{2,6} = 2\sqrt{2}\,sin(6\pi(4t-2)) \end{cases} \quad \frac{2}{4} \le t < \frac{3}{4} \qquad (18)$$

$$\begin{cases} \phi_{3,0} = 2 \\ \phi_{3,1} = 2\sqrt{2}\,cos(2\pi(4t-3)) \\ \phi_{3,2} = 2\sqrt{2}\,cos(4\pi(4t-3)) \\ \phi_{3,3} = 2\sqrt{2}\,cos(6\pi(4t-3)) \\ \phi_{3,4} = 2\sqrt{2}\,sin(2\pi(4t-3)) \\ \phi_{3,5} = 2\sqrt{2}\,sin(4\pi(4t-3)) \\ \phi_{3,6} = 2\sqrt{2}\,sin(6\pi(4t-3)) \end{cases} \quad \frac{3}{4} \le t < 1 \qquad (19)$$

By integrating equation (16) from 0 to t and using in equation 1 we get,

$$\begin{cases} \int_0^t \phi_{0,0}(t)dt = \left[\frac{1}{8}, 0, 0, 0, \frac{-1}{4\sqrt{2}\pi}, \frac{-1}{8\sqrt{2}\pi}, \frac{-1}{12\sqrt{2}\pi}, \frac{1}{4}, 0, 0, 0, 0, 0, 0, \frac{1}{4}, 0, 0, 0, 0, 0, 0, \frac{1}{4}, 0, 0, 0, 0, 0, 0\right]^T \phi_{28}^{(4)}(t), \\ \int_0^t \phi_{0,1}(t)dt = \left[0, 0, 0, 0, \frac{1}{8\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{28}^{(4)}(t), \\ \int_0^t \phi_{0,2}(t)dt = \left[0, 0, 0, 0, 0, \frac{1}{16\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{28}^{(4)}(t), \\ \int_0^t \phi_{0,3}(t)dt = \left[0, 0, 0, 0, 0, 0, \frac{1}{24\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{28}^{(4)}(t), \\ \int_0^t \phi_{0,4}(t)dt = \left[\frac{1}{4\sqrt{2}\pi}, -\frac{1}{8\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{28}^{(4)}(t), \\ \int_0^t \phi_{0,5}(t)dt = \left[\frac{1}{8\sqrt{2}\pi}, 0, -\frac{1}{16\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{28}^{(4)}(t), \\ \int_0^t \phi_{0,6}(t)dt = \left[\frac{1}{12\sqrt{2}\pi}, 0, 0, -\frac{1}{24\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T \phi_{28}^{(4)}(t), \end{cases}$$

and similarly for (17),

$$\begin{cases}
\int_0^t \phi_{1,0}(t)dt = \left[0,0,0,0,0,0,0,\frac{1}{8},0,0,0,\frac{-1}{4\sqrt{2}\pi},\frac{-1}{8\sqrt{2}\pi},\frac{-1}{12\sqrt{2}\pi},\frac{1}{4},0,0,0,0,0,0,\frac{1}{4},0,0,0,0,0,0\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{1,1}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,\frac{1}{8\pi},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{1,2}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{16\pi},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{1,3}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{24\pi},0,0,0,0,0,0,0,0,0,0,0,0,0\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{1,4}(t)dt = \left[0,0,0,0,0,0,0,\frac{1}{4\sqrt{2}\pi},-\frac{1}{8\pi},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{1,5}(t)dt = \left[0,0,0,0,0,0,0,\frac{1}{8\sqrt{2}\pi},0,-\frac{1}{16\pi},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{1,6}(t)dt = \left[0,0,0,0,0,0,0,\frac{1}{12\sqrt{2}\pi},0,0,-\frac{1}{24\pi},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right]^T \phi_{28}^{(4)}(t),
\end{cases}$$

for (18),

$$\begin{cases}
\int_0^t \phi_{2,0}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{8},0,0,0,\frac{-1}{4\sqrt{2}\pi},\frac{-1}{8\sqrt{2}\pi},\frac{-1}{12\sqrt{2}\pi},\frac{1}{4},0,0,0,0,0,0\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{2,1}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{8\pi},0,0,0,0,0,0,0,0,0\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{2,2}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{16\pi},0,0,0,0,0,0,0,0\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{2,3}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{24\pi},0,0,0,0,0,0,0\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{2,4}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{4\sqrt{2}\pi},-\frac{1}{8\pi},0,0,0,0,0,0,0,0,0,0,0,0\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{2,5}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{8\sqrt{2}\pi},0,-\frac{1}{16\pi},0,0,0,0,0,0,0,0,0,0,0\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{2,6}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{12\sqrt{2}\pi},0,0,-\frac{1}{24\pi},0,0,0,0,0,0,0,0,0,0\right]^T \phi_{28}^{(4)}(t),
\end{cases}$$

and for (19),

$$\begin{cases}
\int_0^t \phi_{3,0}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{8},0,0,0,\frac{-1}{4\sqrt{2}\pi},\frac{-1}{8\sqrt{2}\pi},\frac{-1}{12\sqrt{2}\pi}\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{3,1}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{8\pi},0,0\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{3,2}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{16\pi},0\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{3,3}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{24\pi}\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{3,4}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{4\sqrt{2}\pi},-\frac{1}{8\pi},0,0,0,0,0\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{3,5}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{8\sqrt{2}\pi},0,-\frac{1}{16\pi},0,0,0,0\right]^T \phi_{28}^{(4)}(t),\\
\int_0^t \phi_{3,6}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{12\sqrt{2}\pi},0,0,-\frac{1}{24\pi},0,0,0\right]^T \phi_{28}^{(4)}(t).
\end{cases}$$

Hence, we can write

$$\int_0^t \phi_{28}(t)dt = P_{28\times 28}^{(4)}\phi_{28}^{(4)}(t).$$

where

$\phi_{28}^{(4)}(t) = [\phi_{0,0},\phi_{0,1},\phi_{0,2},...,\phi_{0,6},\phi_{1,0},\phi_{1,1},\phi_{1,2},...,\phi_{1,6},\phi_{2,0},\phi_{2,1},\phi_{2,2},...,\phi_{2,6},\phi_{3,0},\phi_{3,1},\phi_{3,2},...,\phi_{3,6}]^T$.

and, $P_{28\times 28}^{(4)}$ be an operational matrix of integration which is given as,

$$P_{28\times 28}^{(4)} = \begin{bmatrix} E & F & F & F \\ O & E & F & F \\ O & O & E & F \\ O & O & O & E \end{bmatrix}.$$

Here

$$E_{7\times 7}=\begin{bmatrix} \frac{1}{8} & 0 & 0 & 0 & \frac{-1}{4\sqrt{2}\pi} & \frac{-1}{8\sqrt{2}\pi} & \frac{-1}{12\sqrt{2}\pi} \\ 0 & 0 & 0 & 0 & \frac{1}{8\pi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{16\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{24\pi} \\ \frac{1}{4\sqrt{2}\pi} & -\frac{1}{8\pi} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{8\sqrt{2}\pi} & 0 & -\frac{1}{16\pi} & 0 & 0 & 0 & 0 \\ \frac{1}{12\sqrt{2}\pi} & 0 & 0 & -\frac{1}{24\pi} & 0 & 0 & 0 \end{bmatrix},$$

$$F_{7\times 7}=\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, O_{7\times 7}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

6. Numerical Examples

6.1. **Application in solving the differential equation. Example 1** Consider the differential equation,

$$y'(t)+y(t)=1,\ y(0)=0. \tag{20}$$

Exact solution of equation (20) is given by,

$$y(t)=(-e^{-t}+1).$$

Now, we will solve this equation by sine-cosine wavelet method for $k=1$, $L=3$, $\mu=2$,$\mu=3$ and $\mu=4$

Let

$$y(t)=C^{(\mu)^T}\phi^{(\mu)}(t),$$

on integrating both sides of the above equation on (0, t) and simplifying, we get

$$\int_0^t \phi^{(\mu)}(x)dx=P^{(\mu)}\phi^{(\mu)}(t).$$

Expanding $\beta^{(\mu)}(t)=1$, in the form basis functions we have, $\beta^{(\mu)}(t)=d^{(\mu)^T}\phi^{(\mu)}(t)$.

Where, $d^{(2)}=\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$,

$d^{(3)}=\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$,

$d^{(4)}=\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$.

Putting $y(t)=C^{(\mu)^T}\phi^{(\mu)}(t)$ in equation (20), and integrating between 0 to t, and simplifying, we get,

$$C^{(\mu)}=(P^{(\mu)^T}+I^{(\mu)})^{-1}P^{(\mu)^T}d^{(\mu)}.$$

This is a system of $14, 21, 28$ algebraic equations, which is solvable for $C^{(\mu)}$. After solving, we obtain the unknown vector $C^{(\mu)}$ as,

$$C^{(2)}=\begin{pmatrix}0.1490790\\-0.0049660\\-0.0012474\\-0.0005548\\-0.0624051\\-0.0313505\\-0.0209187\\0.3692893\\-0.0030229\\-0.0007562\\-0.0000956\\-0.0379878\\-0.0190064\\-0.0126726\end{pmatrix};\ C^{(3)}=\begin{pmatrix}0.0857002\\-0.00195141\\-0.000488881\\-0.000217365\\-0.0367832\\-0.0184304\\-0.0122917\\0.225257\\-0.00139749\\-0.00035011\\-0.000155665\\-0.0263421\\-0.0131988\\-0.00880267\\0.3252\\-0.00100081\\-0.00025073\\-0.000111479\\-0.0188648\\-0.0094523\\-0.00630399\end{pmatrix};\quad C^{(4)}=\begin{pmatrix}0.0572495\\-0.000989708\\-0.000247721\\-0.000110122\\-0.0248741\\-0.0124518\\-0.00830303\\0.155264\\-0.000770612\\-0.000192882\\-0.000085744\\-0.0193676\\-0.00969528\\-0.00646494\\0.23158\\-0.000600017\\-0.000150182\\-0.0000667624\\-0.0150801\\-0.00754899\\-0.00503377\\0.291001\\-0.000467188\\-0.000116936\\-0.0000519828\\-0.0117417\\-0.00587783\\-0.00391941\end{pmatrix}.$$

In table [1], the solution obtained by extended sine-cosine wavelet method and Euler method (with step size 0.1) is compared with exact solution.

6.2. **Application in solving the integral equation.** Here we consider the linear integral equation

$$y(x) = m(x) + \int_0^1 q(x,t)y(t)dt, \tag{21}$$

where $m(x)$ and $q(x,t)$ are continuous on $[0,1]$ and the region $[0,1]\times[0,1]$ respectively. Also here we have to evaluate $y(x)$ which is an unknown function. Now here we approximating the functions $y(x)$, $m(x)$ and $q(x,t)$ which is given as follows

$$y(x) = \phi^T(x)Y, m(x) = \phi^T(x)M, q(x,t) = \phi^T(x)Q\phi^T(t).$$

where Y, M are $\mu^k(2L+1)\times 1$ vectors and Q is a $\mu^k(2L+1)\times\mu^k(2L+1)$ matrix. Putting these values in equation (21) and then simplifying we get

$$Y = (I-Q)^{-1}M \tag{22}$$

This is system of $\mu^k(2L+1)\times 1$ algebraic equations which can be solved for Y. Now this method is illustrated with help of following example:

t	Approximate Solution $y(t) = {C^{(\mu)}}^T \phi^{(\mu)}(t)$ for $k = 1$, $L = 3$, $\mu = 2$	Approximate Solution $y(t) = {C^{(\mu)}}^T \phi^{(\mu)}(t)$ for $k = 1$, $L = 3$, $\mu = 3$	Approximate Solution $y(t) = {C^{(\mu)}}^T \phi^{(\mu)}(t)$ for $k = 1$, $L = 3$, $\mu = 4$	Exact Solution $y(t) = (-e^{-t} + 1)$	By Eulers Method with step size $h = 0.1$
0.0	0.19729	0.14192	0.11068	0.00000	0.00000
0.1	0.07971	0.10899	0.08625	0.09516	0.10000
0.2	0.16423	0.19042	0.18826	0.18126	0.19000
0.3	0.27127	0.26889	0.25302	0.25918	0.27100
0.4	0.34164	0.32240	0.33555	0.32968	0.34390
0.5	0.51450	0.39310	0.46084	0.39346	0.40951
0.6	0.44205	0.45779	0.44603	0.45118	0.46855
0.7	0.49400	0.49730	0.50787	0.50341	0.52170
0.8	0.55922	0.54514	0.54713	0.55067	0.56953
0.9	0.60146	0.58555	0.59717	0.59343	0.61257

TABLE 1. Comparison between approximate solution, exact solution and by Euler's method for $k = 1$, $L = 3$, $\mu = 2, 3, 4$.

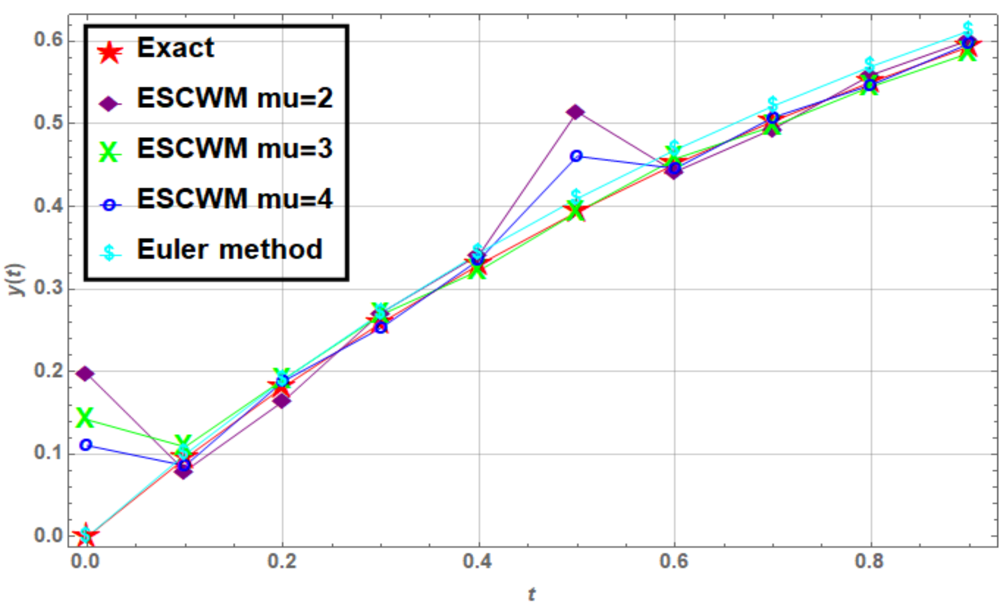

FIGURE 4. Graph of exact solution, approximate solution and Euler solution of Example (1)

t	Abs.Error for $k=1$, $L=3$, $\mu=2$	Abs.Error for $k=1$, $L=3$, $\mu=3$	Abs.Error for $k=1$, $L=3$, $\mu=4$
0.0	0.19729	0.14192	0.11068
0.1	0.01545	0.01383	0.00891
0.2	0.01703	0.00916	0.007
0.3	0.01209	0.00971	0.00616
0.4	0.01196	0.00728	0.00587
0.5	0.12104	0.00036	0.06738
0.6	0.00913	0.00661	0.00515
0.7	0.00941	0.00611	0.00446
0.8	0.00855	0.00553	0.00354
0.9	0.00803	0.00788	0.00374

TABLE 2. Error table

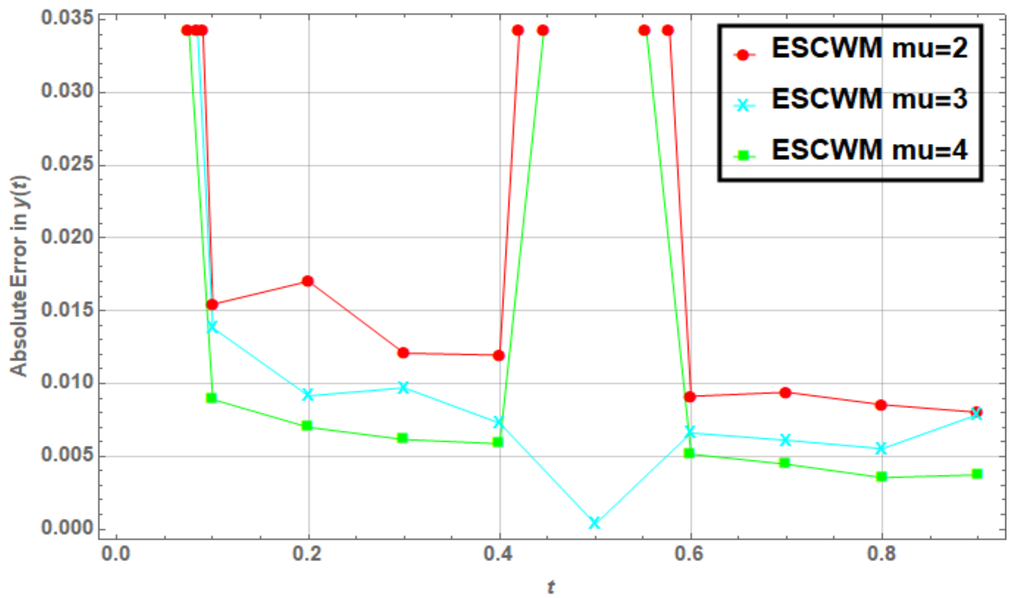

FIGURE 5. The comparison between the absolute errors and by sine-cosine wavelet solution of Example (1)

EXAMPLE 2:

Here we consider the fredholm integral equation

$$y(x) = \frac{x}{2} + \int_0^1 \frac{e^{(x+t)}}{4} y(t)dt. \tag{23}$$

Th exact solution of equation(23) is given by $y(x) = \frac{x}{2} + \frac{e^x}{9-e^2}$. Comparison between exact solution and by sine cosine wavelet solution for $k=1$, $L=3$, $\mu=2,3,4$.

t	sine cosine wavelet solution for $k=1$, $L=3$, $\mu=2$	sine cosine wavelet solution for $k=1$, $L=3$, $\mu=3$	sine cosine wavelet solution for $k=1$, $L=3$, $\mu=4$	Exact Solution $y(t)=\frac{x}{2}+\frac{e^x}{9-e^2}$
0.0	0.924618	0.819051	0.767174	0.620754
0.1	0.697348	0.751727	0.721603	0.736039
0.2	0.816766	0.866532	0.864414	0.858191
0.3	0.991712	0.996963	0.971654	0.98793
0.4	1.12522	1.10032	1.13426	1.12606
0.5	1.69294	1.26406	1.47431	1.27345
0.6	1.37231	1.43785	1.41026	1.43109
0.7	1.53725	1.56439	1.60756	1.60005
0.8	1.78137	1.74588	1.75742	1.78151
0.9	1.96945	1.92912	1.98735	1.97681

TABLE 3. Comparison between extended sine cosine wavelet solution (ESCWM) and exact solution for $k=1$, $L=3$, $\mu=2,3,4$

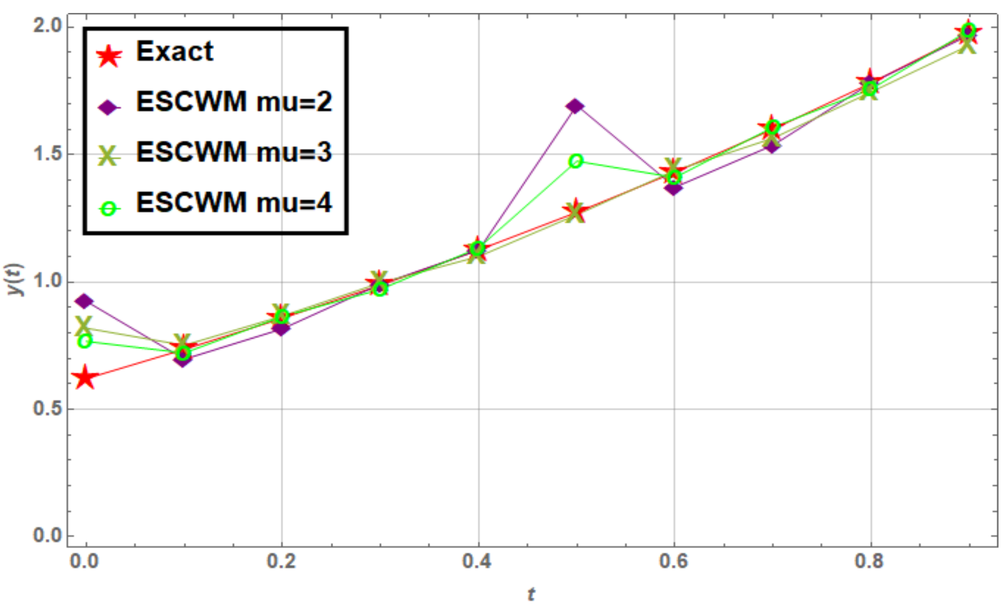

FIGURE 6. Graph of exact solution, approximate solution of Example (2)

t	Abs.Error for $k = 1$, $L = 3$, $\mu = 2$	Abs.Error for $k = 1$, $L = 3$, $\mu = 3$	Abs.Error for $k = 1$, $L = 3$, $\mu = 4$
0.0	0.303864	0.198297	0.14642
0.1	0.038691	0.015688	0.014436
0.2	0.041425	0.008341	0.006223
0.3	0.003782	0.009033	0.016276
0.4	0.00084	0.02574	0.0082
0.5	0.41949	0.00939	0.20086
0.6	0.05878	0.00676	0.02083
0.7	0.0628	0.03566	0.00751
0.8	0.00014	0.03563	0.02409
0.9	0.00736	0.04769	0.01054

TABLE 4. Error table

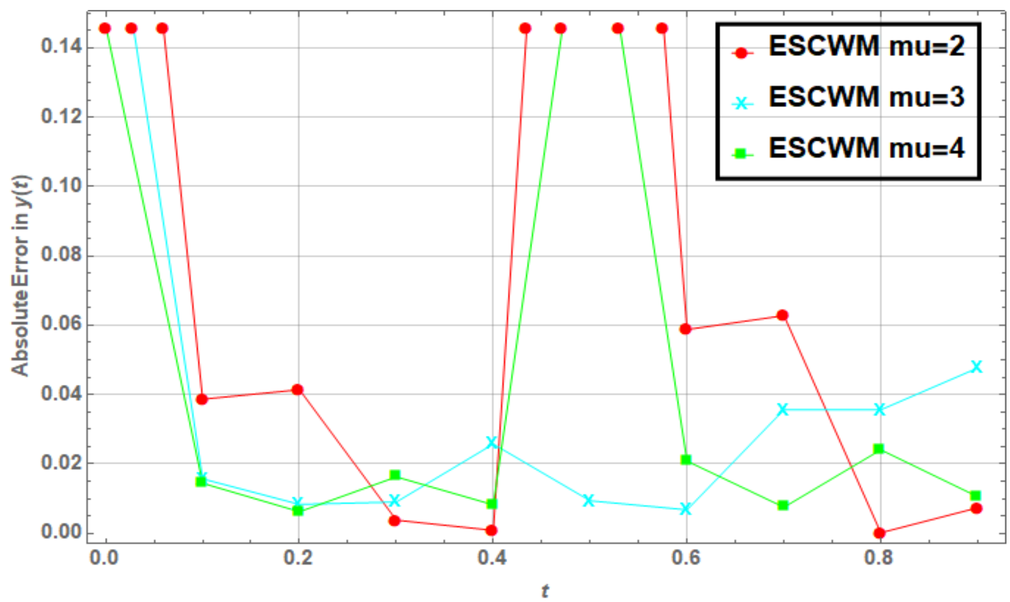

FIGURE 7. The comparison between the absolute errors and by sine-cosine wavelet solution of Example (2)

7. CONCLUSIONS

(i) From theorems 3.1 and 3.2, it follows that

$$E^{(1)}_{\mu^k,2L+1} = ||f - S_{\mu^k,2L+1}||_2 = \begin{cases} O\left(\frac{1}{\mu^k}\right), & m = 0; \\ O\left(\frac{1}{\mu^k\sqrt{2L+1}}\right), & m > 0. \end{cases}$$

and

$$E^{(2)}_{\mu^k,2L+1} = ||f - S_{\mu^k,2L+1}||_2 = O\left(\frac{1}{\mu^{2k}}\frac{1}{(2L+1)^{\frac{3}{2}}}\right).$$

Since $E^{(1)}_{\mu^k,2L+1}$ and $E^{(2)}_{\mu^k,2L+1} \to 0$ as $k, L \to \infty$. Therefore, the approximations obtained in theorems 3.1 and 3.2 are best possible in wavelet analysis.

(ii) Approximations calculated in theorems 3.1 and theorem 3.2are verified by an example in section (4) which is significant part of the paper.

(iii) These approximations are used in solving the different types of differential and integral equations and solutions obtained by such method are compared to the exact solutions. We observe that the exact solution and approximate solution is very close to each other which is shown in the graphs.

References

[1] H. Azin, F. Mohammadi and J.T. Machado, A piecewise spectral-collocation method for solving fractional Riccati differential equation in large domains. Comput. Appl. Math. 38, 1-13, 2019.

[2] N. Azizi and R. Pourgholi, Applications of Sine–Cosine wavelets method for solving the generalized Hirota–Satsuma coupled KdV equation. Math. Sci. 17(4);503-516, 2023.

[3] N. Azizi and R. Pourgholi, Applications of Sine-Cosine wavelets method for solving Drinfel'd–Sokolov–Wilson system. Adv. Syst. Sci. Appl. 21(3);75-90, 2021.

[4] A.T. Dincel, A sine-cosine wavelet method for the approximation solutions of the fractional Bagley-Torvik equation. Sigma J. Eng. Nat. Sci. 40(1); 150-154, 2021.

[5] A.B. Gabis, Y. Meraihi, S. Mirjalili, and A. Ramdane-Cherif, A comprehensive survey of sine cosine algorithm: variants and applications. Artif. Intell. Rev. 54(7);5469-5540, 2021.

[6] Guf, Jin-Sheng, and Wei-Sun Jiang, The Haar wavelets operational matrix of integration. Int. J. Syst. Sci. 27(7); 623-628, 1996.

[7] Y. Guo, X. Zhou, J. Li, Ba, R., Xu, Z., Tu, S. and L. Chai, A Novel and Optimized Sine–Cosine Transform Wavelet Threshold Denoising Method Based on the sym4 Basis Function and Adaptive Threshold Related to Noise Intensity. Appl. Sci. 13(19); p.10789, 2023.

[8] J. Iqbal, R. Abass and P. Kumar, Approximate solution of fractional differential equations using Shannon wavelet operational matrix method. Int. J. Comput. Sci. Math. 13(3); 228-244, 2021.

[9] N. Irfan, and A. H. Siddiqi, Sine-Cosine wavelets approach in numerical evaluation of Hankel transform for seismology. Appl. Math. Model. 40(7-8); 4900-4907, 2016.

[10] F.C. Kung, and S. Y. Chen, Solution of integral equations using a set of block pulse functions. J. Frank. Inst. 306(4); 283-291, 1978.

[11] S. Lal, and Abhilasha, Approximation of functions in Hölder class by third kind Chebyshev wavelet and its application in solution of Fredholm integro-differential equations. Rend. Circ. Mat. Palermo (2) 1-20, 2023.

[12] B.M. Mohan, and K. B. Datta, Identification via Fourier series for a class of lumped and distributed parameter systems. IEEE Trans. Circuits Syst. 36(11); 1454-1458, 1989.

[13] U. Saeed, M. ur Rehman, K. Javid, Q. Din, and S. Haider, Fractional Gegenbauer wavelets operational matrix method for solving nonlinear fractional differential equations. Math. Sci. 15, 83-97, 2021.

[14] U. Saeed, Sine–cosine wavelets operational matrix method for fractional nonlinear differential equation. Int. J. Wavelets Multiresolution Inf. Process. 17(04); p.1950026, 2019.

[15] P. Rahimkhani,Y. Ordokhani and E. Babolian, Fractional-order Bernoulli functions and their applications in solving fractional Fredholem–Volterra integro-differential equations. Appl. Numer. Math. 122, 66-81, 2017.

[16] M. Razzaghi and S. Yousefi, The Legendre wavelets operational matrix of integration. Int. J. Syst. Sci. 32(4); 495-502, 2001.

[17] V. K. Sharma, V. Sharma, S. Lal and H. M. Srivastava, Approximation of a function having bounded derivatives upto the second order by sine-cosine wavelet expansion and its applications . Poincare j. anal. appl. 2023: n. pag.

[18] H.R. Yazdani, and M. Nadjafikhah, Solving differential equations by new wavelet transform method based on the quasi-wavelets and differential invariants. Punjab Univ. J. Math. 49(3)., 2020.

[19] B. Yilmaz, and Y. and Cetin, Numerical solutions of the Fredholm integral equations of the second type. New trend math. sci. 5(3);284-292, 2017.

[20] S. Yousefi, and A. Banifatemi, Numerical solution of Fredholm integral equations by using CAS wavelets. Appl Math Comput. 183(1); 458-463, 2006.

[21] L. Zhu and Y. Wang, Second Chebyshev wavelet operational matrix of integration and its application in the calculus of variations. Int. J. Comput. Math. 90(11); 2338-2352, 2013.

Vivek Kumar Sharma
Deen Dayal Upadhyaya Gorakhpur University, Gorakhpur-273009, India
Email address: vivek.jaimatadi.sharma7.@gmail.com

Virendra Sharma
Deen Dayal Upadhyaya Gorakhpur University, Gorakhpur-273009, India
Email address: 13v12s95@gmail.com

Sonoo Singh
Deen Dayal Upadhyaya Gorakhpur University, Gorakhpur-273009, India
Email address: sonoos96@gmail.com