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PROPERTIES OF A CERTAIN CLASS RELATED TO Q-DIFFERENCE OPERATOR

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Abstract. In this paper, using an analytic univalent function of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, that is defined in the open unit disk $U = \{z : |z| < 1\}$, Aouf et al.[3] operator $D^n_{q,\lambda}$ and Jackson q- derivative, we define a class of functions denoted by $R_q(n, \xi, \alpha, \beta, \lambda)$, which for different values of its parameters many special new classes can be obtained from it. For this class of functions, we obtained some of its properties such as coefficient estimates, neighborhood of function and the modified Hadamard product of two functions in it and also for function which its coefficients are the sum of the squares of the coefficients of two functions. Key results demonstrate the influence of the Aouf et al.[3] operator on the univalence, growth theorems and distortion properties for this class of functions. Also we obtained analogue results of these results for each of the subclasses obtained from this class.

1. Introduction

Denote the class of analytic univalent function

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

by A, $z \in U = \{z : |z| < 1\}$ and $\tau \subset A$ such that

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geq 0)\,. \tag{2}$$

Using the Jackson q- derivative [5], Aouf et al. [3] defined

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the operator

$$D_{q,\lambda}^{n}f(z)=z+\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^{n}a_kz^k \quad \left(n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\right), \tag{3}$$

where $[k]_q=\dfrac{1-q^k}{1-q}$, which for $q\to 1^-$ reduces to Al-Oboudi operator [1] and for $q\to 1^-$ and $\lambda=1$, we get Sălăgean operator [11] , also for $\lambda=1$, we get the q-Sălăgean operator [2].

For $f\ (z)\in\tau$, let $R_q(n,\xi,\alpha,\beta,\lambda)$ be the class of functions satisfying

$$\left|\frac{\frac{z\partial_q(D_{q,\lambda}^{n}f(z))}{D_{q,\lambda}^{n}f(z)}-1}{2\xi(\frac{z\partial_q(D_{q,\lambda}^{n}f(z))}{D_{q,\lambda}^{n}f(z)}-\alpha)-(\frac{z\partial_q(D_{q,\lambda}^{n}f(z))}{D_{q,\lambda}^{n}f(z)}-1)}\right|<\beta, \tag{4}$$

$$0\le\alpha<\frac{1}{2\xi},\ 0<\beta\le 1,\ \frac{1}{2}\le\xi\le 1,\ n\in\mathbb{N}_0 \text{ and } z\in U.$$

For different values of $n,\beta,\xi,\alpha,\lambda$ and $q\longrightarrow 1^-$, we have:

(i) $R(0,1,0,1,\lambda)=S^*$ is precisely the class of starlike functions in U;
(ii) $R(0,1,\alpha,1,\lambda)$ is the class of starlike functions of order α $(0\le\alpha<1)$;
(iii) $R(0,\frac{\sigma+1}{2},0,\beta,0)$ be the class studied by Lakshminarsimhan [7];
(iv) $R(0,\xi,\alpha,\beta,0)$be the class studied by Kulkarni [6].

Also we have:

(i) $R_q(n,1,\alpha,1,\lambda)=R_q^n(\alpha,\lambda)=\left\{f\in\tau: Re\left\{\dfrac{z\partial_q(D_{q,\lambda}^{n}f(z))}{D_{q,\lambda}^{n}f(z)}\right\}>\alpha\right\}$;

(ii) $R_q(n,1,\alpha,\beta,\lambda)=R_q^n(\alpha,\beta,\lambda)=\left\{f\in\tau:\left|\dfrac{\frac{z\partial_q(D_{q,\lambda}^{n}f(z))}{D_{q,\lambda}^{n}f(z)}-1}{\frac{z\partial_q(D_{q,\lambda}^{n}f(z))}{D_{q,\lambda}^{n}f(z)}+1-2\alpha}\right|<\beta\right\}$;

(iii) The class $R_q(0,\xi,\alpha,\beta,\lambda)=R^*(\xi,\alpha,\beta)=$

$$\left\{f\in\tau:\left|\frac{\frac{z\partial_qf(z)}{f(z)}-1}{2\xi(\frac{z\partial_qf(z)}{f(z)}-\alpha)-(\frac{z\partial_qf(z)}{f(z)}-1)}\right|<\beta\right\}.$$

2. Characterizations for the class $R_q(n,\xi,\alpha,\beta,\lambda)$

Theorem 2.1. The function $f\in R_q(n,\xi,\alpha,\beta,\lambda)$ if and only if

$$\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^{n}\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]a_k\le 2\beta\xi(1-\alpha). \tag{5}$$

Proof. Assume that (5) is true. We have

$$\left|z\partial_q(D^n_{q,\lambda}f(z)-D^n_{q,\lambda}f(z)\right|-\beta\left|2\xi(z\partial_q\left[D^n_{q,\lambda}f(z))-\alpha D^n_{q,\lambda}f(z)\right]\right.$$

$$\left.-\left[z\partial_q(D^n_{q,\lambda}f(z))-D^n_{q,\lambda}f(z)\right]\right|=\left|\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^n([k]_q-1)a_kz^k\right|$$

$$-\beta\left|2\xi(1-\alpha)-2\xi\sum_{k=2}^{\infty}([k]_q-\alpha)\left[1+\left([k]_q-1\right)\lambda\right]^n a_kz^k\right.$$

$$\left.+\sum_{k=2}^{\infty}([k]_q-1)\left[1+\left([k]_q-1\right)\lambda\right]^n a_kz^k\right|$$

$$\leq\left[\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^n\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]a_k-2\beta\xi(1-\alpha)\right]\leq 0.$$

Hence, by the maximum modulus theorem, we have $f\in R_q(n,\xi,\alpha,\beta,\lambda)$.
Conversely, let $f\in R_q(n,\xi,\alpha,\beta,\lambda)$.Then

$$\left|\frac{\frac{z\partial_q(D^n_{q,\lambda}f(z))}{D^n_{q,\lambda}f(z)}-1}{2\xi(\frac{z\partial_q(D^n_{q,\lambda}f(z))}{D^n_{q,\lambda}f(z)}-\alpha)-(\frac{z\partial_q(D^n_{q,\lambda}f(z))}{D^n_{q,\lambda}f(z)}-1)}\right|=$$

$$\left|\frac{\sum\limits_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^n([k]_q-1)a_kz^{k-1}}{2\xi(1-\alpha)-2\xi\sum\limits_{k=2}^{\infty}([k]_q-\alpha)\left[1+\left([k]_q-1\right)\lambda\right]^n a_kz^{k-1}+\sum\limits_{k=2}^{\infty}([k]_q-1)\left[1+\left([k]_q-1\right)\lambda\right]^n a_kz^{k-1}}\right|<\beta.$$

Since $Re(z)<|z|$ for all z, we obtain

$$Re\left\{\frac{\sum\limits_{k=2}^{\infty}\left[(1+([k]_q-1)\lambda\right]^n([k]_q-1)a_kz^{k-1}}{2\xi(1-\alpha)-2\xi\sum\limits_{k=2}^{\infty}([k]_q-\alpha)\left[1+\left([k]_q-1\right)\lambda\right]^n a_kz^{k-1}+\sum\limits_{k=2}^{\infty}([k]_q-1)\left[1+\left([k]_q-1\right)\lambda\right]^n a_kz^{k-1}}\right\}<\beta.$$

Let $z\to 1^-$ through real values, so we have (5).

Corollary 2.1. For $f\in R_q(n,\xi,\alpha,\beta,\lambda)$, we have

$$a_k\leq\frac{2\beta\xi(1-\alpha)}{\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right][1+([k]_q-1)\lambda]^n}\qquad(k\geq 2)\,. \tag{6}$$

The result is sharp for

$$f(z)=z-\frac{2\beta\xi(1-\alpha)}{\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right][1+([k]_q-1)\lambda]^n}z^k\qquad(k\geq 2)\,. \tag{7}$$

In the following theorem we prove the distortion results.

Theorem 2.2. Let $f\in R_q(n,\xi,\alpha,\beta,\lambda)$, then for $|z|\leq r<1$ and $0\leq i\leq n$, we have

$$r-r^2\frac{2\beta\xi(1-\alpha)}{(1+q\lambda)^{n-i}\left[q(1-\beta)+2\beta\xi([2]_q-\alpha)\right]} \tag{8}$$

$$\leq \left|(D_{q,\lambda}^{i}f(z)\right| \leq r + r^2 \frac{2\beta\xi(1-\alpha)}{(1+q\lambda)^{n-i}\left[q(1-\beta)+2\beta\xi([2]_q-\alpha)\right]},$$

the above bounds are sharp.

Proof. Note that $f \in R_q(n,\xi,\alpha,\beta,\lambda)$ if and only if $D_{q,\lambda}^{i}f(z) \in R_q(n-i,\xi,\alpha,\beta,\lambda)$ and

$$D_{q,\lambda}^{i}f(z) = z - \sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^{i} a_k z^k.$$

By Theorem 2.1, we have

$$(1+q\lambda)^{n-i}\left[q(1-\beta)+2\beta\xi([2]_q-\alpha)\right]\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^{i} a_k$$

$$\leq \sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^{n}\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]a_k$$

$$\leq 2\beta\xi(1-\alpha),$$

then

$$\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^{i} a_k \leq \frac{2\beta\xi(1-\alpha)}{(1+\lambda q)^{n-i}\left[(1-\beta)q+2\beta\xi(1-\alpha)\right]}.$$

Hence

$$\left|D_{q,\lambda}^{i}f(z)\right| \leq |z| + |z|^2\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^{i} a_k$$

$$\leq r + r^2\frac{2\beta\xi(1-\alpha)}{\left[q(1-\beta)+2\beta\xi([2]_q-\alpha)\right](1+\lambda q)^{n-i}}$$

and

$$\left|D_{q,\lambda}^{i}f(z)\right| \geq r - r^2\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^{i} a_k \geq$$

$$r - r^2\frac{2\beta\xi(1-\alpha)}{(1+\lambda q)^{n-i}\left[q(1-\beta)+2\beta\xi([2]_q-\alpha)\right]},$$

thus (8) is true. The result is sharp for the function $f\ (z)$ defined by

$$f(z) = z - \frac{2\beta\xi(1-\alpha)}{\left[q(1-\beta)+2\beta\xi([2]_q-\alpha)\right]}z^2, z = \mp r.$$

For $i = 0$ in Theorem 2.2, we get:

Corollary 2.2. For$f \in R_q(n,\xi,\alpha,\beta,\lambda)$, we have

$$r - \frac{2\beta\xi(1-\alpha)}{\left[q(1-\beta)+2\beta\xi([2]_q-\alpha)\right](1+\lambda q)^{n}}r^2$$

$$\leq \quad |f(z)| \leq r + \frac{2\beta\xi(1-\alpha)}{\left[q(1-\beta)+2\beta\xi([2]_q-\alpha)\right](1+\lambda q)^{n}}r^2.$$

Theorem 2.3. The set $R_q(n,\xi,\alpha,\beta,\lambda)$ is convex set.
proof. Let $f_i(z)=z-\sum_{k=2}^{\infty}a_{k,i}z^k \quad (i=1,2)$ belong to $R_q(n,\xi,\alpha,\beta,\lambda)$ and let $g(z)=\varsigma_1 f_1(z)+\varsigma_2 f_2(z)$, with $\varsigma_1,\ \varsigma_2>0$ and $\varsigma_1+\varsigma_2=1$,
we can write

$$g(z)=z-\sum_{k=2}^{\infty}(\varsigma_1 a_{k,1}+\varsigma_2 a_{k,2})z^k.$$

It is sufficient to show that $g(z)\in R_q(n,\xi,\alpha,\beta,\lambda)$, we have

$$\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^n\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right](\varsigma_1 a_{k,1}+\varsigma_2 a_{k,2})$$

$$=\varsigma_1\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^n\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]a_{k,1}$$

$$+\varsigma_2\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^n\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]a_{k,2}$$

$$\leq\varsigma_1\left[2\beta\xi(1-\alpha)\right]+\varsigma_2\left[2\beta\xi(1-\alpha)\right]=(\varsigma_1+\varsigma_2)\left[2\beta\xi(1-\alpha)\right]=2\beta\xi(1-\alpha).$$

Thus $g(z)\in R_q(n,\xi,\alpha,\beta,\lambda)$.

We obtained the extreme points in the following theorem.

Theorem 2.4. Let $f_1(z)=z$ and

$$f_k(z)=z-\frac{2\beta\xi(1-\alpha)}{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}z^k \qquad (k\geq 2),$$

then $f(z)\in R_q(n,\xi,\alpha,\beta,\lambda)$ if and only if it can be expressed in the form $f(z)=\sum_{k=1}^{\infty}\gamma_k f_k(z)$, where $\gamma_k\geq 0$ and $\sum_{k=1}^{\infty}\gamma_k=1$ or $\gamma_1+\sum_{k=2}^{\infty}\gamma_k=1$.

Proof. Let $f(z)=\sum_{k=1}^{\infty}\gamma_k f_k(z)$. Thus

$$f(z)=z-\sum_{k=2}^{\infty}\frac{2\beta\xi(1-\alpha)}{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}\gamma_k z^k,$$

thus

$$\sum_{k=2}^{\infty}\frac{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}{2\beta\xi(1-\alpha)}\times$$

$$\frac{2\beta\xi(1-\alpha)\gamma_k}{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}$$

$$=\sum_{k=2}^{\infty}\gamma_k=1-\gamma_1\leq 1.$$

In view of Theorem 2.1, $f(z)\in R_q(n,\xi,\alpha,\beta,\lambda)$. Conversely suppose $f(z)\in R_q(n,\xi,\alpha,\beta,\lambda)$, then

$$a_k\leq\frac{2\beta\xi(1-\alpha)}{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]} \qquad (k\geq 2).$$

Putting

$$\gamma_k = \frac{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}{2\beta\xi(1-\alpha)}\ a_k$$

and $\gamma_1 = 1-\sum_{k=2}^{\infty}\gamma_k$, then we have $f(z)=\gamma_1 f_1(z)+\sum_{k=2}^{\infty}\gamma_k f_k(z)$.

3. Neighborhood and Hadamard product

Definition 3.1. For $\gamma \geq 0$ Aouf et al. [2] defined the k,q,γ neighborhood of function $f(z)\in\tau$ by (see [8, 9]):

$$N_{k,q,\gamma}(f;g)=\left\{g\in\tau : g(z)=z-\sum_{k=2}^{\infty}b_k z^k \text{ and } \sum_{k=2}^{\infty}[k]_q\left|a_k-b_k\right|\leq\gamma_q\right\} \tag{9}$$

and for $e(z)=z$,

$$N_{k,q,\gamma}(e;g)=\left\{g\in\tau : g(z)=z-\sum_{k=2}^{\infty}b_k z^k \text{ and } \sum_{k=2}^{\infty}[k]_q\left|b_k\right|\leq\gamma_q\right\}. \tag{10}$$

Note that when $q\to 1^-$ in Definition 3.1, we have the definition for Goodman [4] and Ruscheweyh [10].

Now, we get some neighborhood results.

Theorem 3.1. Let

$$\gamma_q=\frac{2\,[2]_q\,\beta\xi(1-\alpha)}{[1+q\lambda]^n\left[q(1-\beta)+2\beta\xi([2]_q-\alpha)\right]},$$

then $R_q(n,\xi,\alpha,\beta,\lambda)\subset N_{k,\gamma_q}(e)$.

Proof. Let $f\in R_q(n,\xi,\alpha,\beta,\lambda)$, then we have

$$\frac{\left[(1-\beta)q+2\beta\xi([2]_q-\alpha)\right](1+q\lambda)^n}{[2]_q}\sum_{k=2}^{\infty}[k]_q\,a_k$$

$$\leq\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^n\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]a_k$$

$$\leq 2\beta\xi(1-\alpha),$$

therefore,

$$\sum_{k=2}^{\infty}[k]_q\,a_k\leq\frac{2\,[2]_q\,\beta\xi(1-\alpha)}{(1+q\lambda)^n\left[(1-\beta)q+2\beta\xi([2]_q-\alpha)\right]}, \tag{11}$$

then from (11), $f\in N_{k,\gamma_q}(e)$.

Theorem 3.2. Let $g \in R_q(n,\xi,\alpha,\beta,\lambda)$ and $\zeta = 1 - \frac{\gamma_q}{[2]_q} d\,(n,\xi,\alpha,\beta,\lambda)$. Then $N_{k,\gamma_q}(g) \subset R_q(n,\xi,\alpha,\beta,\lambda,\zeta)$, where

$$d\,(n,\xi,\alpha,\beta,\lambda) = \frac{(1+q\lambda)^n \left[(1-\beta)q + 2\beta\xi([2]_q - \alpha)\right]}{(1+q\lambda)^n \left[(1-\beta)q + 2\beta\xi([2]_q - \alpha)\right] - 2\beta\xi(1-\alpha)}.$$

Proof. Let $f \in N_{k,\gamma_q}(g)$. Then by (9), we have

$$\overset{\infty}{\underset{k=2}{}} [k]_q |a_k - b_k| \leq \gamma_q, \text{ then } \overset{\infty}{\underset{k=2}{}} |a_k - b_k| \leq \frac{\gamma_q}{[2]_q}.$$

Since $g \in R_q(n,\xi,\alpha,\beta,\lambda)$, we have

$$\overset{\infty}{\underset{k=2}{}} b_k \leq \frac{2\beta\xi(1-\alpha)}{(1+q\lambda)^n \left[(1-\beta)q + 2\beta\xi([2]_q - \alpha)\right]},$$

$$\left|\frac{f\ (z)}{g\,(z)} - 1\right| < \frac{\overset{\infty}{\underset{k=2}{}} |a_k - b_k|}{1 - \overset{\infty}{\underset{k=2}{}} b_k}$$

$$\leq \frac{\gamma_q}{[2]_q} \left(\frac{(1+q\lambda)^n \left[(1-\beta)q + 2\beta\xi([2]_q - \alpha)\right]}{(1+q\lambda)^n \left[(1-\beta)q + 2\beta\xi([2]_q - \alpha)\right] - 2\beta\xi(1-\alpha)}\right)$$

$$= \frac{\gamma_q}{[2]_q} d\,(n,\xi,\alpha,\beta,\lambda) = 1 - \zeta,$$

then $f \in R_q(n,\xi,\alpha,\beta,\lambda,\zeta)$.

In the next theorem, we get some Hadamard product results.

Theorem 3.3. Let $f(z)$ and $g(z) \in R_q(n,\xi,\alpha,\beta,\lambda)$ such that

$$f(z) = z - \overset{\infty}{\underset{k=2}{}} a_k z^k \text{ and } g\,(z) = z - \overset{\infty}{\underset{k=2}{}} b_k z^k,$$

then the Hadamard product $(f * g)\,(z)$ is defined by

$$(f * g)\,(z) = z - \overset{\infty}{\underset{k=2}{}} a_k b_k z^k$$

is in the subclass $R_q(n,\xi,\sigma,\beta,\lambda)$, where

$$\sigma = 1 - \frac{q\xi\beta(1-\alpha)^2}{(1+q\lambda)^n \left[q(1-\beta) + 2\beta\xi([2]_q - \alpha)\right]^2 - 4\beta^2\xi^2\,(1-\alpha)^2}.$$

Proof. Employing the method of Sehild and Silverman [12], we have

$$\overset{\infty}{\underset{k=2}{}} \frac{\left[1 + \left([k]_q - 1\right)\lambda\right]^n \left[\left([k]_q - 1\right)(1-\beta) + 2\beta\xi([k]_q - \alpha)\right]}{2\beta\xi(1-\alpha)} a_k \leq 1 \tag{12}$$

and

$$\overset{\infty}{\underset{k=2}{}} \frac{\left[1 + \left([k]_q - 1\right)\lambda\right]^n \left[\left([k]_q - 1\right)(1-\beta) + 2\beta\xi([k]_q - \alpha)\right]}{2\beta\xi(1-\alpha)} b_k \leq 1.$$

We have only to find the largest σ such that

$$\sum_{k=2}^{\infty}\frac{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\sigma)\right]}{2\beta\xi(1-\sigma)}a_kb_k\le 1. \tag{13}$$

Now, by Cauchy-Schwarz inequality, we obtain

$$\sum_{k=2}^{\infty}\frac{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}{2\beta\xi(1-\alpha)}\sqrt{a_kb_k}\le 1, \tag{14}$$

we need only to show that

$$\frac{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\sigma)\right]}{2\beta\xi\left(1-\sigma\right)}a_kb_k$$

$$\le\frac{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}{2\beta\xi\left(1-\alpha\right)}\sqrt{a_kb_k},$$

equivalently,

$$\sqrt{a_kb_k}\le\frac{2\beta\xi\left(1-\sigma\right)}{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\sigma)\right]}\times$$

$$\frac{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}{2\beta\xi\left(1-\alpha\right)},$$

then

$$\sqrt{a_kb_k}\le\frac{(1-\sigma)\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}{(1-\alpha)\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\sigma)\right]}\quad\left(k\ge 2\right).$$

But from (14), we have

$$\sqrt{a_kb_k}\le\frac{2\beta\xi(1-\alpha)}{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}.$$

Consequently, we need to prove that

$$\frac{2\beta\xi(1-\alpha)}{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}$$

$$\le\frac{(1-\sigma)\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}{(1-\alpha)\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\sigma)\right]}\quad\left(k\ge 2\right).$$

Or, equivalently, that

$$\sigma\le 1-$$

$$\frac{\left([k]_q-1\right)\xi\beta(1-\alpha)^2}{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]^2-4\beta^2\xi^2\left(1-\alpha\right)^2},$$

$$\Phi(k) = 1-$$

$$\frac{\left([k]_q - 1\right)\xi\beta(1-\alpha)^2}{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]^2 - 4\beta^2\xi^2(1-\alpha)^2}, \quad (15)$$

is an increasing function of k $(k \geq 2)$, letting $k = 2$ in (15), we obtain

$$\sigma \leq \Phi(2) = 1 - \frac{q\xi\beta(1-\alpha)^2}{(1+q\lambda)^n\left[q(1-\beta)+2\beta\xi([2]_q-\alpha)\right]^2 - 4\beta^2\xi^2(1-\alpha)^2},$$

Theorem 3.4. Let $f(z) \in R_q(n,\xi,\alpha,\beta,\lambda)$ and $c > -1$ be real number, then the function

$$G(z) = \frac{[c+1]_q}{z^c}\int_0^z s^{c-1} f(s)\, d_q s, c > -1,$$

also belongs to $R_q(n,\xi,\alpha,\beta,\lambda)$.

Proof. By virtue of $G(z)$ it follows from (2) that

$$G(z) = z - \sum_{k=2}^{\infty}\left(\frac{[c+1]_q}{[c+k]_q}\right)a_k z^k.$$

But

$$\sum_{k=2}^{\infty}\frac{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}{2\beta\xi(1-\alpha)}\left(\frac{[c+1]_q}{[c+k]_q}\right)a_k \leq 1,$$

since $\frac{[c+1]_q}{[c+k]_q} \leq 1$ and by Theorem 2.1, so the proof is completed.

Theorem 3.5. Let $f(z) \in R_q(n,\xi,\alpha,\beta,\lambda)$ and

$$F\mu(z) = (1-\mu)z + \mu\int_0^z \frac{f(s)}{s} d_q s \quad (\mu \geq 0,\ z \in U).$$

Then $F\mu(z) \in R_q(n,\xi,\alpha,\beta,\lambda)$ if $0 \leq \mu \leq [2]_q$.

Proof. Let f defined by (2), then

$$F\mu(z) = (1-\mu)z + \mu\int_0^z\left(\frac{s - \sum_{k=2}^{\infty} a_k s^k}{s}\right)d_q s = z - \sum_{k=2}^{\infty}\frac{\mu}{[k]_q}a_k z^k.$$

By Theorem 2.1 and since $(\frac{\mu}{[2]_q} \leq 1)$, we have

$$\sum_{k=2}^{\infty}\frac{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}{2\beta\xi(1-\alpha)}\left(\frac{\mu}{[k]_q}\right)a_k$$

$$\leq \sum_{k=2}^{\infty}\frac{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[\left([k]_q-1\right)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}{2\beta\xi(1-\alpha)}\left(\frac{\mu}{[2]_q}\right)a_k \leq 1,$$

then $F\mu(z) \in R_q(n,\xi,\alpha,\beta,\lambda)$.

Theorem 3.6. If $f_j(z) \in R_q(n, \xi, \alpha, \beta, \lambda) \quad (j = 1, 2)$. Then

$$h(z) = z - \sum_{k=2}^{\infty} \left(a_{k,1}^2 + a_{k,2}^2\right) z^k,$$

belongs to the class $R_q(n, \xi, \vartheta, \beta, \lambda)$, where

$$\vartheta = 1 - \frac{4\beta\xi q(1-\alpha)^2(1-\beta)}{[1+q\lambda]^n \left[q(1-\beta) + 2\beta\xi([2]_q - \alpha)\right]^2 - 8\beta^2\xi^2(1-\alpha)^2}.$$

Proof. By virtue of Theorem 2.1, we obtain

$$\sum_{k=2}^{\infty} \left[\frac{\left[1+\left([k]_q-1\right)\lambda\right]^n \left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}{2\beta\xi(1-\alpha)}\right]^2 a_{k,j}^2$$

$$\leq \left[\sum_{k=2}^{\infty} \frac{\left[1+\left([k]_q-1\right)\lambda\right]^n \left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}{2\beta\xi(1-\alpha)} a_{k,j}\right]^2$$

$$\sum_{k=2}^{\infty} \left[\frac{\left[1+\left([k]_q-1\right)\lambda\right]^n \left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]}{2\beta\xi(1-\alpha)}\right]^2 a_{k,j}^2$$

$$\leq 1 \quad (j = 1, 2).$$

It follows that

$$\sum_{k=2}^{\infty} \frac{1}{2} \frac{\left[1+\left([k]_q-1\right)\lambda\right]^{2n} \left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]^2}{4\beta^2\xi^2(1-\alpha)^2} \left(a_{k,1}^2 + a_{k,2}^2\right) \leq 1$$

Therefore, we need to find the largest ϑ such that

$$\frac{\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\vartheta)\right]}{1-\vartheta} \leq$$

$$\frac{1}{2} \frac{\left[1+\left([k]_q-1\right)\lambda\right]^n \left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]^2}{2\beta\xi(1-\alpha)^2} \quad (k \geq 2)$$

that is,

$$\vartheta \leq 1-$$

$$\frac{4\beta\xi(1-\alpha)^2([k]_q-1)(1-\beta)}{\left[1+\left([k]_q-1\right)\lambda\right]^n \left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]^2 - 8\beta^2\xi^2(1-\alpha)^2} \quad (k \geq 2).$$

Since

$$\varphi(k) = 1-$$

$$\frac{4\beta\xi(1-\alpha)^2([k]_q-1)(1-\beta)}{\left[1+\left([k]_q-1\right)\lambda\right]^n\left[([k]_q-1)(1-\beta)+2\beta\xi([k]_q-\alpha)\right]^2-8\beta^2\xi^2(1-\alpha)^2},$$

is an increasing function of k $(k\geq 2)$, setting $k=2$, we readily have

$$\vartheta\leq\varphi(2)=1-\frac{4\beta\xi q(1-\alpha)^2(1-\beta)}{[1+q\lambda]^n\left[q(1-\beta)+2\beta\xi([2]_q-\alpha)\right]^2-8\beta^2\xi^2(1-\alpha)^2}.$$

The functions $f_j(z)$ given by (15), give the sharpness.

Remark 1. For different values of n,ξ,α,β and λ in our results, we have results for the special classes defined in the introduction.

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