Electronic Journal of Mathematical Analysis and Applications
Vol. 13(2) July 2025, No.3.
ISSN: 2090-729X (online)
ISSN: 3009-6731(print)
http://ejmaa.journals.ekb.eg/

EXISTENCE AND STABILITY OF THE SOLUTION OF AN IMPLICIT SET-VALUED FUNCTIONAL DIFFERENTIAL EQUATION WITH PARAMETERS

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Abstract. A set-valued function, also called a correspondence or set-valued relation, is a mathematical function that maps elements from one set, the domain of the function, to subsets of another set. Set-valued functions are used in a variety of mathematical fields, including optimization, control theory and game theory.

The set-valued functional differential equation has been widely applied in mathematics, physics, optimizations, optimal control, as well as economics and finance.[2, 3, 5, 6].
Here, we study the initial-value problem of a set-valued implicate functional differential equation with parameters.
The existence of solution and its continuous dependence on parameters will be proved.

1. Introduction

Let $F : I = [0,T] \times R \times R \to 2^R$ be a nonempty, compact and convex set-valued function on $I \times R$, [4] and S_F be the set of Lipschitz selection of the set-valued function F [7]. Let $\gamma \in (0,1)$ and μ be two parameters.
Here, we study the existence of solutions of the parametric set-valued implicit functional differential equation

$$\frac{dx(t)}{dt} \in F(t, \frac{dx(\gamma t)}{dt}, \mu\ ),\ t \in (0,T] \tag{1}$$

with the initial value

$$x(0) = x_0. \tag{2}$$

2020 *Mathematics Subject Classification.* 34A12, 34A30, 34D20.
Key words and phrases. Differential equations, stability, existence, uniqueness.
Submitted Jan. 20, 2025. Revised March 3, 2025.

The existence of solutions $x \in C[0,T]$, $C[0,T]$ is the Banach space of continuous functions defined on $[0,T]$, will be proved. The continuous dependence of the solutions on the initial value x_0 and the parameters γ. μ will be proved also.

2. Existence of solutions

Consider the problem (1)-(2) under the following assumptions

(1) $F : I \times R \to P(R)$ is nonlinear, nonempty, compact and convex set-valued function for all $(t, x, \mu) \in I \times R \times R$.

(2) F is Lipschitzian with a constant $L > 0$, i.e.

$$H(F(t,x,\mu), F(s,y,\mu)) \leq L(|t-s| + |x-y|).$$

Remark 1 [8] [9] From the assumptions (1)-(2) we can deduce that there exists $f \in S_F$, such that

$$\frac{dx(t)}{dt} = f(t, \frac{dx(\gamma t)}{dt}, \mu~),~ t \in (0,T] \tag{3}$$

where

(i) $f : I \times R \to R$ is a continuous for every $(t,x) \in I \times R$.

(ii) f is Lipschitzian with a constant $L > 0$, i.e.

$$|f(t,x(t),\mu) - f(s,y(s),\mu)| \leq L(|t-s| + |x(t) - y(s)|).$$

So, every solution of the problem (3)-(2) is a solution of the problem (1)-(2).

Now, let $\frac{dx}{dt} = y(t)$, $t \in [0,T]$, then we obtain

$$x(t) ~=~ x_0 ~+~ \int_0^t ~y(s)ds, \tag{4}$$

and

$$x(\gamma t) \quad = \quad x_0 ~+~ \int_0^{\gamma t} ~y(s)ds,$$

then

$$\frac{dx(\gamma t)}{dt} = \gamma y(\gamma t)$$

and from (3) we obtain the functional equation

$$y(t) = f(t, \gamma y(\gamma t), \mu),~ t \in (0,T] \tag{5}$$

Now, we have the following existence theorem

Theorem 2.1. *Let the assumptions (1)-(2) be satisfied. If $\gamma L < 1$, then the functional equation (5) has a unique solution $x \in C[0,T]$.*

proof. Let A be defined by

$$Ay(t) = f(t, \gamma y(\gamma t), \mu).$$

Let $y \in C[0,T]$, and let $t_1, t_2 \in [0,T]$ such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
|Ay(t_2) - Ay(t_1)| &= |f(t_2, \gamma y((\gamma t_2), \mu) - f(t_1, \gamma y(\gamma t_1), \mu)| \\
&\leq |f(t_2, \gamma y((\gamma t_2), \mu) - f(t_1, \gamma y((\gamma t_2), \mu) \\
&\quad + f(t_1, \gamma y((\gamma t_2), \mu) - f(t_1, \gamma y((\gamma t_1), \mu)| \\
&\leq |f(t_2, \gamma y(\gamma t_2), \mu) - f(t_1, \gamma y(\gamma t_2), \mu)| \\
&\quad + |f(t_1, \gamma y(\gamma t_2), \mu) - f(t_1, \gamma y(\gamma t_1), \mu)| \\
&\leq |f(t_2, \gamma y(\gamma t_2))) - f(t_1, \gamma y(\gamma t_2), \mu)| + L|\gamma y(\gamma t_2)) - \gamma y(\gamma t_1))| \\
&\leq |f(t_2, \gamma y(\gamma t_2))) - f(t_1, \gamma y(\gamma t_2), \mu)| + \gamma L|y(\gamma t_2)) - y(t_1))| \\
&\leq L|t_2 - t_1| + \gamma L|y(\gamma t_2)) - y(\gamma t_1))|
\end{aligned}$$

This prove that $A : C[0, T] \to C[0, T]$
Now we prove that A is contraction.
Let $y, \bar{y} \in C[0, T]$, then

$$\begin{aligned}
|Ay(t) - A\bar{y}(t)| &= |f(t, \gamma y(\gamma t), \mu) - f(t, \gamma \bar{y}(\gamma t), \mu)| \\
&\leq \gamma L||y(\gamma t) - \gamma \bar{y}(\gamma t)| \leq \gamma L\|y - \bar{y}\|,
\end{aligned}$$

then

$$\| Ay - A\bar{y} \| \leq \gamma L \| y - \bar{y} \|$$

since $\gamma L < 1$, then A is contraction, then by using Banach fixed point Theorem [1, 10] there exists a unique solution $y \in C[0, T]$ of the functional equation (5).

Corollary 2.0. *Let the assumptions of Theorem (2.1) be satisfied, then* $\forall f \in S_F$ *the problem (1)-(2) has a unique solution* $x \in C[0, T]$.

3. Continuous dependence of the solution

Definition 1. The unique solution of the problem (4) depends continuously on initial data x_0, if $\epsilon > 0$,$\exists \delta > 0$ such that

$$|x_0 - x_0^*| \leq \delta \ \Rightarrow \|x - x^*\| \leq \epsilon$$

where x^* is the unique solution of

$$x^*(t) = x_0^* + \int_0^t y(s) ds$$

Theorem 3.2. *Let the assumption of Theorem (2.1) be satisfied, then the unique solution of (4) depends continuously on* x_0

Proof. Let $x(t)$ and $x^*(t)$ be the solutions of problem (4), then

$$\begin{aligned}
&\|x - x^*\| = |x_0 + \int_0^t y(s) ds - x_0^* - \int_0^t y(s) ds| \\
&\leq |x_0 - x_0^*| \\
&\leq \delta \leq \epsilon
\end{aligned}$$

then

$$\|x - x^*\| \leq \epsilon$$

This proves the continuous dependence of the solution on initial data x_0, of the problem (4).

Definition 2. The unique solution of the problem (5) depends continuously on γ, if $\epsilon > 0$,$\exists \delta > 0$ such that

$$|\gamma - \gamma^*| \leq \delta \ \Rightarrow \|y - y^*\| \leq \epsilon$$

where y^* is the unique solution of

$$y^*(t) \quad = \quad f(t, \gamma^* y^*(\gamma^* t), \mu) \tag{6}$$

Theorem 3.3. *Let the assumption of Theorem 1 be satisfied, then the unique solution of (5) depends continuously on* γ

Proof. Let $y(t)$ and $y^*(t)$ be the solutions of problem (5), then

$$\begin{aligned}
&\|y - y^*\| = |f(t, \gamma y(\gamma t), \mu) - f(t, \gamma^* y^*(\gamma^* t), \mu)| \\
&\leq |f(t, \gamma y(\gamma t), \mu) - f(t, \gamma^* y(\gamma t), \mu) \\
&+ f(t, \gamma^* y(\gamma t), \mu) - f(t, \gamma^* y^*(\gamma t), \mu)| \\
&\leq |f(t, \gamma y(\gamma t), \mu) - f(t, \gamma^* y(\gamma t), \mu)| \\
&+ |f(t, \gamma^* y(\gamma t), \mu) - f(t, \gamma^* y^*(\gamma t), \mu)| \\
&\leq L(|\gamma y(\gamma t) - \gamma^* y(\gamma t)| + |\gamma^* y(\gamma t) - \gamma^* y^*(\gamma t)|) \\
&\leq L(\|y\||\gamma - \gamma^*| + \gamma^* |y(\gamma t) - y(\gamma^* t)|) \\
&\leq L\|y\|\delta + L\delta_1 \leq \epsilon
\end{aligned}$$

then

$$\|y - y^*\| \leq \epsilon$$

This proves the continuous dependence of the solution on γ, of the problem (5).

Corollary 3.0. *Let the assumptions of Theorem (2.1) be satisfied, then from definition 2, we obtain*

$$x^*(t) = x_0 + \int_0^t y^*(s) ds, \tag{7}$$

and the equation (4), we get

$$\begin{aligned}
&\|x - x^*\| = |x_0 + \int_0^t y(s) ds - x_0 + \int_0^t y^*(s) ds| \\
&\leq \int_0^t |y(s) - y^*(s)| ds \\
&\leq \|y - y^*\| \\
&\leq \epsilon,
\end{aligned}$$

then

$$\|x - x^*\| \leq \epsilon$$

Definition 3. The unique solution of the problem (5) depends continuously on μ, if $\epsilon > 0$,$\exists \delta > 0$ such that

$$|\mu - \mu^*| \leq \delta \ \Rightarrow \|y - y^*\| \leq \epsilon$$

where y^* is the unique solution of

$$y^*(t) \quad = \quad f(t, \gamma y^*(\gamma t), \mu^*) \tag{8}$$

Theorem 3.4. *Let the assumption of Theorem 1 be satisfied, then the unique solution of (5) depends continuously on μ*

Proof. Let $y(t)$ and $y^*(t)$ be the solutions of problem (5), then

$$\begin{aligned}
&\|y - y^*\| = |f(t, \gamma y(\gamma t), \mu) - f(t, \gamma y^*(\gamma t), \mu^*)| \\
&\leq |f(t, \gamma y(\gamma t), \mu) - f(t, \gamma y^*(\gamma t), \mu) \\
&+f(t, \gamma y^*(\gamma t), \mu) - f(t, \gamma y^*(\gamma t), \mu^*)| \\
&\leq |f(t, \gamma y(\gamma t), \mu) - f(t, \gamma y^*(\gamma t), \mu)| \\
&+|f(t, \gamma y^*(\gamma t), \mu) - f(t, \gamma y^*(\gamma t), \mu^*)| \\
&\leq L(\gamma|y(\gamma t - y^*(\gamma t)| + |\mu - \mu^*|) \\
&\leq \gamma L|y(\gamma t) - y^*(\gamma t)| + L\delta \\
&\leq \gamma L\|y - y^*\| + L\delta \leq \epsilon,
\end{aligned}$$

then

$$\|y - y^*\| \leq L\delta(1 - \gamma L)^{-1} \leq \epsilon$$

then

$$\|y - y^*\| \leq \epsilon$$

This proves the continuous dependence of the solution on μ, of the problem (5).

Corollary 3.0. *Let the assumptions of Theorem (2.1) be satisfied, then from definition 3, we obtain*

$$x^*(t) = x_0 + \int_0^t y^*(s)ds, \tag{9}$$

and the equation (4), we get

$$\begin{aligned}
&\|x - x^*\| = |x_0 + \int_0^t y(s)ds - x_0 + \int_0^t y^*(s)ds| \\
&\leq \int_0^t |y(s) - y^*(s)|ds \\
&\leq \|y - y^*\| \\
&\leq \epsilon,
\end{aligned}$$

then

$$\|x - x^*\| \leq \epsilon$$

4. Example 1.

Consider the following equation

$$y(t) = t^4 e^- t + \gamma y(\gamma t) + \frac{1}{4}\mu,$$

then

$$|f(s, \gamma y(\gamma t), \mu)| = |t^4 e^- t + \gamma y(\gamma t) + \frac{1}{4}\mu|$$
$$\leq L(|t^4 e^{-t}| + |\gamma y(\gamma t)| + \frac{1}{4}\mu)$$

It is clear that the assumptions (1) and (2) of Theorem (2.1) are satisfied with $a(t) = t^4 e^- t \in C[0, T]$,
and let, $L = \frac{1}{4}$, therefor, by applying theorem (2.1),
the equation (5) has continuous solution.

5. Example 2.

Consider the following equation, since the equation (3) transformed to function equation (5), then

$$\frac{dx(t)}{dt} = \sqrt{t+2} + \frac{1}{2}\frac{d(x(\gamma t))}{dt} + \frac{1}{2}\mu,$$

then

$$|f(t, \frac{dx(\gamma t)}{dt}, \mu)| = |f(\sqrt{t+2} + \frac{dx(\gamma t))}{dt} + \frac{1}{2}\mu)|$$
$$\leq \frac{1}{2}(|\frac{dx(\gamma t))}{dt}| + \mu)$$

It is clear that the assumptions (1) and (2) of Theorem (2.1) are satisfied with $a(t) = \sqrt{t+2} \in C(0, T]$,
and let, $\gamma = \frac{1}{2}, L = \frac{1}{2}$, therefor, by applying theorem (2.1),
the equation (3) has continuous solution.

References

[1] R. Agarwal, M. Meehan and D.O'regan, Fixed Point Theory And Applications, Cambridge Tracts in Mathematics, (2001)
[2] J. P. Aubin and A. Cellina, Differential Inclusions, Springer-Verlag, (1984).
[3] R. Baier and M. Gerdts, Intensive Course on Set-valued numerical Analysis and Optimal Control Borovets and Sofia, Bulgaria,9(2005), 9-16.
[4] J. Barret Foulkes, Set-valued Functions and Convexity. May(1980).
[5] A. I. Bulgakov and V. V. Vasilyev, On the theory of functional-differential inclusion of neutral type, 9 (2002), 3352.
[6] M. Kisielewicz, Differential Inclusions and Optimal Control, Dordrecht, the Nether-Lands, (1991).
[7] I. Kupka, Continuous selection for Lipschitz multifunctions, Acta Math. Univ. Comenianae, LXXIV. 1 (2005), 133-141.

[8] G. Magerl, Metrizability of compact sets and continuous selections, Proc. Amer. Math. Soc. 72 (1978), 607-612.

[9] S. Park, Applications of Michael's selection theorems to fixed point theory, Topology and it's Applications, 155 (2008), 861-870 and Appl. 7 (2004), 419-428.

[10] Curtain R. F. and Pritchard A. J.*Functional Analysis in Modern Applied Mathematics*. cambride press: Cambride, MA, USA, 1977.

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