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ON FIXED CIRCLES IN C^*-ALGEBRA VALUED S-METRIC SPACES AND APPLICATION TO EXPONENTIAL LINEAR UNIT FUNCTION

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Abstract. In this article, we introduce the concept of a fixed circle in a C^*-algebra valued S-metric space and explore some interesting existence and uniqueness theorems for self-mappings that have fixed circles in various directions. With a geometric viewpoint, we delve into the properties of these self-mappings and provide a deeper understanding of their mathematical foundations and applications. Additionally, we present several illustrative examples to verify the accuracy of our findings and to concretely demonstrate the applicability of the concept. These examples serve to validate the theoretical results related to fixed circles and their extendability. We also investigate the interplay between the algebraic structure of the C^*-algebra and the geometric properties of the fixed circles, highlighting how these interactions contribute to the richness of the theory. Finally, we apply the obtained fixed-circle results to activation functions used in neural networks, providing a meaningful example of how these mathematical structures can be utilized in practice. In doing so, we offer a broader perspective on the potential applications and practical implications of these theoretical insights, particularly in fields such as machine learning and nonlinear analysis, where the understanding of such mappings can lead to advancements in both theory and application.

1. Introduction and preliminaries

The Banach contraction principle [4] has studied by many mathematicians in a long period of time not only in many branches of mathematics but also in mathematical physics and engineering sciences with wide range of applications to many exciting problems in various directions. It is a power tool used for the existence and uniqueness of solutions of many nonlinear problems appearing in engineering

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sciences and physics. In this regard, two main purposes of researchers are to generalize the Banach contraction principle in many ways and to obtain new results in different metric spaces (for example, [11] and the references therein).

Sedghi *et al.* [30] have presented the concept of an S-metric space as a generalized form of metric spaces and stated some fixed-point results in S-metric spaces that are corresponding in S-metric spaces of the Banach contraction principle valid for metric spaces.

In 2014, Ma *et al.* [17] have introduced the notion of C^*-algebra valued metric space and furnished some types of celebrated Banach's fixed-point theorem for self-mappings satisfying the expansive or contractive conditions on such spaces. So far, there have been many great attempts by mathematicians to extend and generalize C^*-algebra valued metric spaces [14, 12, 31, 1, 2, 19, 22, 34, 3]. Kalaivani and Kalpana [14] introduced the concept of a C^*-algebra valued S-metric space as a new addition to the existing literature and studied fixed-point problem on such spaces. Other works are noted in [12, 31, 16].

The fixed-circle problem in metric spaces initiated by Özgür and Taş [25] has been developed very fast in recent times due to their use in different fields of mathematical sciences such as neural networks. Some articles have appeared for the improvement and generalization of contractive conditions with various aspects in both metric spaces and S-metric spaces. Main articles on fixed circles of self-mappings on S-metric spaces are [26], [23] and [9] in which authors have obtained some noteworthy fixed-circle results using different techniques with some examples to substantiate the importance and effectiveness of their findings. For further studies, we recommend [32, 20, 27, 33].

In [10], we have created an introduction and standard reference for specialized articles in the future works by giving geometric properties of fixed circles of a self-mapping and obtaining some solutions to fixed-circle problem in the setting of an C^*-algebra valued metric space.

In the present study, we consider the fixed-circle problem on C^*-algebra valued S-metric spaces and examine its solutions for self-mappings on such spaces using some auxiliary functions and defining some contractive conditions. Also, we construct some nontrivial illustrative examples to support our assumptions and to prove the usability of our findings and application to neural networks. The theory of C^*-algebra is a popular topic in operator theory and functional analysis, so the results of this article gain importance in new exciting applications to theoretical physics and noncommutative geometry.

Now, we introduce a basic review of C^*-algebras and C^*-algebra valued metric spaces.

We begin with the definition of a C^*-algebra and some related results used in this discussion.

Definition 1.1. [21] *A mapping $x \to x^*$ of a complex algebra $\mathbb{A}$ into $\mathbb{A}$ is called an involution on $\mathbb{A}$ if the following properties hold for all $x, y \in \mathbb{A}$ and $\lambda \in \mathbb{C}$:*

(i) $(x^*)^* = x$,

(ii) $(xy)^* = y^*x^*$,

(iii) $(\lambda x + y)^* = \overline{\lambda}x^* + y^*$.

A complex Banach algebra A with an involution such that for every x in A

$$\|x^*x\| = \|x\|^2$$

is called a C^*-algebra.

In the continuation of the article, $\mathbb{A}$ will denote a unital C^*-algebra with a unit I.

We note that $\|x^*\| = \|x\|$ for all $x \in \mathbb{A}$ and if $x \in \mathbb{A}$ is invertible, then x^* is invertible and $(x^*)^{-1} = (x^{-1})^*$.

The spectrum of x in $\mathbb{A}$ is defined by $\sigma(x) = \{\lambda \in \mathbb{R} : x - \lambda I \text{ is non-invertible}\}$.

Let $\mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$. An element $x \in \mathbb{A}$ is said to be positive if $x \in \mathbb{A}_h$ and $\sigma(x) \subset \mathbb{R}^+$. If $x \in \mathbb{A}$ is positive, we write it as $\theta \preceq x$, where θ is the zero element in $\mathbb{A}$. We denote the set of all positive elements of $\mathbb{A}$ by $\mathbb{A}_+$. Besides, $\preceq$ becomes a partial ordering on the set by defining $x \preceq y$ to mean $y - x \in \mathbb{A}_+$.

Regarding these, we also have the following properties that we will use in the next sections:

(i) If $x, y, z \in \mathbb{A}_h$, then $x \preceq y$ implies $x + z \preceq y + z$.

(ii) If $x, y \in \mathbb{A}_h$ and $z \in \mathbb{A}$, then $x \preceq y$ implies $z^*xz \preceq z^*yz$.

(iii) If $\theta \preceq x \preceq y$, then $\|x\| \leq \|y\|$

(vi) If $x, y \in \mathbb{A}_+$ and $\alpha, \beta \in \mathbb{R}^+ \cup \{0\}$, then $\alpha x + \beta y \in \mathbb{A}_+$.

(v) $\mathbb{A}_+ = \{x^*x : x \in \mathbb{A}\}$ [21].

We denote the set $\{a \in \mathbb{A} : ab = ba \text{ for all } b \in \mathbb{A}\}$ by $\mathbb{A}'$.

Now, let us remember the definitions of a notion of a C^*-algebra valued metric space and a notion of a C^*-algebra valued b-metric space.

Definition 1.2. [17] *Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies*:

(i) *$\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta \Leftrightarrow x = y$;*

(ii) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*

(iii) *$d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

Then, d is called a C^*-algebra valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra valued metric space.

Definition 1.3. [18] *Let X be a nonempty set, and $A \in \mathbb{A}'$ such that $A \succeq I$. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies*:

(i) *$\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta \Leftrightarrow x = y$;*

(ii) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*

(iii) *$d(x, y) \preceq A[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.*

Then, d is called a C^*-algebra valued b-metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra valued b-metric space.

Using the concept of a positive element in a C^*-algebra, Kalaivani and Kalpana [14], Ege and Alaca [12] and Shatarah and Özer [31] presented the notion of a C^*-algebra valued S-metric space in different articles in the following way:

Definition 1.4. [14, 12, 31] *Let X be a nonempty set. Suppose the mapping $\mathcal{S} : X \times X \times X \to \mathbb{A}$ satisfies*:

(i) *$\theta \preceq \mathcal{S}(x, y, z)$ for all $x, y, z \in X$;*

(ii) *$\mathcal{S}(x, y, z) = \theta$ if and only if $x = y = z$;*

(iii) *$\mathcal{S}(x, y, z) \preceq \mathcal{S}(x, x, a) + \mathcal{S}(y, y, a) + \mathcal{S}(z, z, a)$ for all $x, y, z, a \in X$.*

Then, $\mathcal{S}$ is called a C^*-algebra valued S-metric on X and $(X, \mathbb{A}, \mathcal{S})$ is called a C^*-algebra valued S-metric space.

Lemma 1.1. [14, 12, 31] *Let $(X, \mathbb{A}, \mathcal{S})$ be a complete C^*-algebra valued S-metric space. Then, $\mathcal{S}(x, x, y) = \mathcal{S}(y, y, x)$.*

Definition 1.5. [31] *Let $(X, \mathbb{A}, \mathcal{S})$ be a C^*-algebra valued S-metric space, $x \in X$ and $\rho > 0$. Then, the open ball $B_{\mathcal{S}_{\mathbb{A}}}(x, \rho)$ and the closed ball $B_{S_{\mathbb{A}}}[x, \rho]$ with center x and radius ρ are defined as follows*:

$$B_{\mathcal{S}_{\mathbb{A}}}(x, \rho) = \{y \in X : \|\mathcal{S}(y, y, x)\| < \rho\}$$

and

$$B_{\mathcal{S}_{\mathbb{A}}}[x, \rho] = \{y \in X : \|\mathcal{S}(y, y, x)\| \leq \rho\}.$$

The topology induced by the C^*-*algebra valued S-metric* space is the topology generated by the base of all open balls in X [31].

Kalaivani and Kalpana [14] established the following main theorems which implies the existence and uniqueness of fixed point on complete C^*-algebra valued S-metric spaces.

Theorem 1.1. [14] *Let $(X, \mathbb{A}, \mathcal{S})$ be a complete C^*-algebra valued S-metric space. Suppose that the mapping $T : X \to X$ satisfies*

$$\mathcal{S}(Tx, Tx, Ty) \preceq a^* \mathcal{S}(x, x, y)\, a,$$

where $a \in \mathbb{A}'_+$ with $\|a\| < 1$ for all $x, y \in X$. Then, there exists a unique fixed point in X.

Theorem 1.2. [14] *Let $(X, \mathbb{A}, \mathcal{S})$ be a complete C^*-algebra valued S-metric space. Suppose that the mapping $T : X \to X$ satisfies*

$$\mathcal{S}(Tx, Tx, Ty) \preceq a\left(\mathcal{S}(Tx, Tx, x) + \mathcal{S}(Ty, Ty, y)\right),$$

where $a \in \mathbb{A}'_+$ and $\|a\| < \frac{1}{2}$ for all $x, y \in X$. Then, there exists a unique fixed point in X.

Theorem 1.3. [14] *Let $(X, \mathbb{A}, \mathcal{S})$ be a complete C^*-algebra valued S-metric space. Suppose that the mapping $T : X \to X$ satisfies*

$$\mathcal{S}(Tx, Tx, Ty) \preceq a\left(\mathcal{S}(Tx, Tx, y) + \mathcal{S}(Ty, Ty, x)\right),$$

where $a \in \mathbb{A}'_+$ and $\|a\| < \frac{1}{2}$ for all $x, y \in X$. Then, there exists a unique fixed point in X.

Ege and Alaca [12] introduced the extension of Banach's fixed-point theorem [4] for self-mappings defined on C^*-algebra valued S-metric spaces, which guarantees the existence and uniqueness of fixed point as follows:

Definition 1.6. [12] *Let $(X, \mathbb{A}, \mathcal{S})$ be a complete C^*-algebra valued S-metric space. A map $T : X \to X$ is said to be C^*-algebra valued contractive mapping on X, if there exists $A \in \mathbb{A}$ with $\|A\| < 1$ such that*

$$\mathcal{S}(Tx, Tx, Ty) \preceq A^* \mathcal{S}(x, x, y)\, A$$

for all $x, y \in X$.

Theorem 1.4. [12] *Let $(X, \mathbb{A}, \mathcal{S})$ be a complete C^*-algebra valued S-metric space and $T : X \to X$ be a C^*-algebra valued contractive mapping. Then T has a unique fixed point $x_0 \in X$.*

2. Main results

Our purpose in this section is to define the notion of a fixed circle on a C^*-algebra valued S-metric space and establish some fixed-circle theorems for self-mappings on C^*-algebra valued S-metric spaces.

In the following proposition, we show that the relationship between a C^*-algebra valued b-metric and a C^*-algebra valued S-metric.

Proposition 1. *Let $(X, \mathbb{A}, \mathcal{S})$ be a C^*-algebra valued S-metric space and the mapping $d : X \times X \to \mathbb{A}$ be defined as*

$$d_{\mathcal{S}}(x, y) = \mathcal{S}(x, x, y)$$

for all $x, y \in X$. Then, $(X, \mathbb{A}, d_{\mathcal{S}})$ is a C^*-algebra valued b-metric space.

Proof. Using Definition 1.4 and Lemma 1.1, we can easily see that the conditions (i) and (ii) in Definition 1.3 are satisfied. Now, using the condition (iii) of Definition 1.4 and Lemma 1.1, we get

$$\begin{aligned} d_S(x, y) &= \mathcal{S}(x, x, y) \preceq \mathcal{S}(x, x, a) + \mathcal{S}(x, x, a) + \mathcal{S}(y, y, a) \\ &= 2\mathcal{S}(x, x, a) + \mathcal{S}(y, y, a) \\ &= 2d_{\mathcal{S}}(x, a) + d_{\mathcal{S}}(y, a) \end{aligned} \tag{1}$$

and

$$\begin{aligned} d_{\mathcal{S}}(x, y) &= \mathcal{S}(x, x, y) = \mathcal{S}(y, y, x) \\ &\preceq \mathcal{S}(y, y, a) + \mathcal{S}(y, y, a) + \mathcal{S}(x, x, a) \\ &= 2\mathcal{S}(y, y, a) + \mathcal{S}(x, x, a) \\ &= 2d_{\mathcal{S}}(y, a) + d_{\mathcal{S}}(x, a). \end{aligned} \tag{2}$$

From the inequalities (1) and (2), we obtain

$$2d_{\mathcal{S}}(x, y) \preceq 3\left[d_{\mathcal{S}}(x, a) + d_{\mathcal{S}}(a, y)\right]$$

and so

$$d_{\mathcal{S}}(x, y) \preceq \frac{3}{2}\left[d_{\mathcal{S}}(x, a) + d_{\mathcal{S}}(a, y)\right].$$

Consequently, $d_{\mathcal{S}}$ is a C^*-algebra valued b-metric with $A = \frac{3}{2}$ and $(X, \mathbb{A}, d_{\mathcal{S}})$ is a C^*-algebra valued b-metric space. □

In the subsequent proposition, we observe that the relationship between a C^*-algebra valued metric and a C^*-algebra valued S-metric.

Proposition 2. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space and the mapping $\mathcal{S}_d : X \times X \times X \to \mathbb{A}$ be defined as*

$$\mathcal{S}_d(x, y, z) = d(x, z) + d(y, z)$$

for all $x, y, z \in X$. Then, $(X, \mathbb{A}, \mathcal{S}_d)$ is a C^*-algebra valued S-metric space.

Proof. It can be easily proved that the conditions (i) and (ii) of Definition 1.4 are satisfied. Now, we show that the condition (iii) is satisfied. Using the triangle inequality and symmetry property of a C^*-algebra valued metric d, we get

$$\begin{aligned} \mathcal{S}_d(x, y, z) &= d(x, z) + d(y, z) \\ &\preceq d(x, a) + d(a, z) + d(y, a) + d(a, z) \\ &\preceq 2d(x, a) + 2d(y, a) + 2d(z, a) \\ &= \mathcal{S}_d(x, x, a) + \mathcal{S}_d(y, y, a) + \mathcal{S}_d(z, z, a). \end{aligned}$$

Consequently, $\mathcal{S}_d$ is a C^*-algebra valued S-metric space. $\square$

We call the mapping $\mathcal{S}_d$ as the C^*-algebra valued S-metric generated by C^*-algebra valued metric d. But, there exists a C^*-algebra valued S-metric which is not generated by d as mentioned in the following example.

Example 1. *Let $X = [0,1]$ and $\mathbb{A} = M_2(\mathbb{R})$ with $\|A\| = \max\{a_1, a_2, a_3, a_4\}$ where a_i $(i \in \{1,2,3,4\})$ are the entries of A. Then, $(X, \mathbb{A}, \mathcal{S})$ is a C^*-algebra valued S-metric space, where*

$$\mathcal{S}(x,y,z) = \begin{bmatrix} |x-z| + |x+z-2y| & 0 \\ 0 & |x-z| + |x+z-2y| \end{bmatrix},$$

and partial ordering on $\mathbb{A}$ is given by

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \succeq \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \Leftrightarrow a_i \geq b_i \text{ for } i \in \{1,2,3,4\}.$$

But C^*-algebra valued S-metric $\mathcal{S}$ is not generated by a C^*-algebra valued metric d, that is,

$$\mathcal{S} \neq \mathcal{S}_d.$$

Conversely, we suppose that there is a C^*-algebra valued metric d such that

$$\mathcal{S}_d(x,y,z) = d(x,z) + d(y,z)$$

for all $x, y, z \in [0,1]$. Then, we get

$$\mathcal{S}_d(x,x,z) = 2d(x,z) = \begin{bmatrix} 2|x-z| & 0 \\ 0 & 2|x-z| \end{bmatrix}$$

and so

$$d(x,z) = \begin{bmatrix} |x-z| & 0 \\ 0 & |x-z| \end{bmatrix}$$

for all $x, y, z \in [0,1]$. Also, we similarly get $d(y,z) = \begin{bmatrix} |y-z| & 0 \\ 0 & |y-z| \end{bmatrix}$ for all $x, y, z \in [0,1]$. So, we write

$$\mathcal{S}_d(x,y,z) = \begin{bmatrix} |x-z| + |y-z| & 0 \\ 0 & |x-z| + |y-z| \end{bmatrix} = d(x,z) + d(y,z),$$

a contradiction. Hence, $\mathcal{S} \neq \mathcal{S}_d$.

Remark 1. *Since there is at least one C^*-algebra valued S-metric that cannot be generated with any C^*-algebra valued metric and because of above relationships, it is of great importance to study in C^*-algebra valued S-metric spaces.*

Now, we define the concept of a circle on a C^*-algebra valued metric space with some nontrivial examples.

Definition 2.7. *Let $(X, \mathbb{A}, \mathcal{S})$ be a C^*-algebra valued S-metric space, $x_0 \in X$ and $r \in \mathbb{A}_+$. Then, the circle with the centered x_0 and the radius r is defined by*

$$C_{x_0,r}^{C^*,\mathcal{S}} = \{x \in X : \mathcal{S}(x,x,x_0) = r\}.$$

Example 2. *Let $X=\mathbb{R}$, $\mathbb{A}=\mathbb{R}^2$ and $\mathcal{S}(x,y,z)=(|x-z|+|y-z|,0)$. Then, $(X,\mathbb{A},\mathcal{S})$ is a C^*-algebra valued S-metric space* [12]. *Choose the center $x_0=-\frac{1}{2}$ and the radius $r=(1,0)$. Then, we get*

$$\begin{aligned} C^{C^*,\mathcal{S}}_{-\frac{1}{2},(1,0)} &= \left\{x\in\mathbb{R}:\mathcal{S}\left(x,x,-\frac{1}{2}\right)=(1,0)\right\} \\ &= \left\{x\in\mathbb{R}:\left(2\left|x+\frac{1}{2}\right|,0\right)=(1,0)\right\}=\{-1,0\}. \end{aligned}$$

Example 3. *Let $X=[0,1]$ and $\mathbb{A}=M_2(\mathbb{R})$ with $\|A\|=\max\{a_1,a_2,a_3,a_4\}$, where a_i's are the entries of A. Then, $(X,\mathbb{A},\mathcal{S})$ is a C^*-algebra valued S-metric space, where*

$$\mathcal{S}(x,y,z)=\begin{bmatrix} |x-z|+|y-z| & 0 \\ 0 & |x-z|+|y-z| \end{bmatrix},$$

and partial ordering on $\mathbb{A}$ is given by

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \succeq \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \Leftrightarrow a_i\geq b_i \text{ for } i=1,2,3,4$$

[12]. *Choose the center $x_0=1$ and the radius $r=\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$. Then, we obtain*

$$\begin{aligned} C^{C^*,\mathcal{S}}_{1,r} &= \left\{x\in[0,1]:\mathcal{S}(x,x,1)=\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}\right\} \\ &= \left\{x\in[0,1]:\begin{bmatrix} 2|x-1| & 0 \\ 0 & 2|x-1| \end{bmatrix}=\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}\right\} \\ &= \left\{x\in[0,1]:|x-1|=\frac{3}{4}\right\}=\left\{\frac{1}{4}\right\}. \end{aligned}$$

Example 4. *Let $M_2(\mathbb{C})$ denote the set of bounded linear operators on a Hilbert space $\mathbb{C}^2$. Define $S:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to M_2(\mathbb{C})$ by*

$$\mathcal{S}(x,y,z)=\begin{bmatrix} |x-z|+|y-z| & 0 \\ 0 & k|x-z|+|y-z| \end{bmatrix},$$

where $k>0$ is a constant. Then, $(\mathbb{R},M_2(\mathbb{C}),\mathcal{S})$ is a complete C^*-algebra valued S-metric space [14]. *Choose the center $x_0=0$ and the radius $r=\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$, we get for $k=3$*

$$\begin{aligned} C^{C^*,\mathcal{S}}_{0,r} &= \left\{x\in\mathbb{R}:\mathcal{S}(x,x,0)=\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\right\} \\ &= \left\{x\in\mathbb{R}:\begin{bmatrix} 2|x| & 0 \\ 0 & 4|x| \end{bmatrix}=\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\right\} \\ &= \{x\in\mathbb{R}:|x|=1\}=\{-1,1\}. \end{aligned}$$

Example 5. *Let E be a Lebesgue measurable set and $L(L^2(E))$ denote the set of bounded linear operators on Hilbert space $L^2(E)$. Define $S:L^\infty(E)\times L^\infty(E)\times L^\infty(E)\to L(L^2(E))$ by*

$$\mathcal{S}(f,g,p)=\pi_{|f-p|+|g-p|},$$

for all $f,g,p\in L^\infty(E)$, where $\pi_h:L^2(E)\to L^2(E)$ is the multiplication operator defined by $\pi_h(\phi)=h\cdot\phi$ for all $\phi\in L^2(E)$. Then, $\mathcal{S}$ is a C^*-algebra valued S-metric *and* $\left(L^{\infty}(E), L\left(L^{2}(E)\right), \mathcal{S}\right)$ *is a complete* C^{*}*-algebra valued* S*-metric space* [14]. *Let* $E=[0,1]$. *Choose the center* x_0 *as the function* $f \in L^{\infty}[0,1]$ *defined by*

$$f:[0,1] \rightarrow \mathbb{R},\ f(x)=\chi_{\left[\frac{1}{2}, 1\right]}(x)=\left\{\begin{array}{ccc}1 & , & x \in\left[\frac{1}{2}, 1\right] \\ 0 & , & x \notin\left[\frac{1}{2}, 1\right]\end{array}\right. ,$$

the radius r *as the multiplication operator* $\pi_h \in L\left(L^{2}[0,1]\right)$ *and the function* $h \in L^{\infty}[0,1]$ *be defined by*

$$h:[0,1] \rightarrow \mathbb{R},\ h(x)=\left\{\begin{array}{ccc}1 & , & x \in(\mathbb{R} \backslash \mathbb{Q}) \cap[0,1] \\ \infty & , & x \in \mathbb{Q} \cap[0,1]\end{array}\right.$$

in Example 5 in [10]. *Then, we get*

$$\begin{aligned}
C_{f, \pi_h}^{C^{*}, \mathcal{S}} &= \left\{g \in L^{\infty}[0,1]: \mathcal{S}(g, g, f)=\pi_h\right\}=\left\{g \in L^{\infty}[0,1]: 2 \pi_{|g-f|}=\pi_h\right\} \\
&= \left\{g \in L^{\infty}[0,1]: \pi_{2|g-f|}=\pi_h\right\}=\left\{g \in L^{\infty}[0,1]: 2|g-f|=h\right\} \\
&= \left\{g \in L^{\infty}[0,1]: 2|g(x)-f(x)|=h(x) \text{ for each } x \in[0,1]\right\} \\
&= \left\{g \in L^{\infty}[0,1]: 2|g(x)-0|=1 \text{ for each } x \in(\mathbb{R} \backslash \mathbb{Q}) \cap\left[0, \frac{1}{2}\right)\right\} \\
&\quad \cup\left\{g \in L^{\infty}[0,1]: 2|g(x)-0|=\infty \text{ for each } x \in \mathbb{Q} \cap\left[0, \frac{1}{2}\right)\right\} \\
&\quad \cup\left\{g \in L^{\infty}[0,1]: 2|g(x)-1|=1 \text{ for each } x \in(\mathbb{R} \backslash \mathbb{Q}) \cap\left[\frac{1}{2}, 1\right]\right\} \\
&\quad \cup\left\{g \in L^{\infty}[0,1]: 2|g(x)-1|=\infty \text{ for each } x \in \mathbb{Q} \cap\left[\frac{1}{2}, 1\right]\right\} \\
&= \left\{g \in L^{\infty}[0,1]: g(x)=-\frac{1}{2} \text{ or } g(x)=\frac{1}{2} \text{ for each } x \in(\mathbb{R} \backslash \mathbb{Q}) \cap\left[0, \frac{1}{2}\right)\right\} \\
&\quad \cup\left\{g \in L^{\infty}[0,1]: g(x)=-\infty \text{ or } g(x)=\infty \text{ for each } x \in \mathbb{Q} \cap\left[0, \frac{1}{2}\right)\right\} \\
&\quad \cup\left\{g \in L^{\infty}[0,1]: g(x)=\frac{1}{2} \text{ or } g(x)=\frac{3}{2} \text{ for each } x \in(\mathbb{R} \backslash \mathbb{Q}) \cap\left[\frac{1}{2}, 1\right]\right\} \\
&\quad \cup\left\{g \in L^{\infty}[0,1]: g(x)=-\infty \text{ or } g(x)=\infty \text{ for each } x \in \mathbb{Q} \cap\left[\frac{1}{2}, 1\right]\right\} .
\end{aligned}$$

Now, we give a new concept as the foundation of this paper.

Definition 2.8. *Let* $(X, \mathbb{A}, \mathcal{S})$ *be a* C^{*}*-algebra valued* S*-metric space,* $T: X \rightarrow X$ *be a self-mapping and* $C_{x_0, r}^{C^{*}, \mathcal{S}}$ *be a circle on* X. *Then, the circle* $C_{x_0, r}^{C^{*}, \mathcal{S}}$ *is called as the fixed circle of* T, *if* $Tx=x$ *for all* $x \in C_{x_0, r}^{C^{*}, \mathcal{S}}$.

2.1. **The existence of fixed circles.** In this part, we explore the existence conditions of fixed-circles for self-mappings defining some contractive conditions with the help of some auxiliary functions in the context of C^{*}-algebra valued S-metric spaces.

Theorem 2.5. *Let* $(X, \mathbb{A}, \mathcal{S})$ *be a* C^{*}*-algebra valued* S*-metric space,* $C_{x_0, r}^{C^{*}, \mathcal{S}}$ *be any circle on* X. *Define the mapping* $\varphi: X \rightarrow \mathbb{A}_{+}$ *as*

$$\varphi(x)=\mathcal{S}\left(x, x, x_0\right), \tag{3}$$

for all $x \in X$. If T is a self-mapping defined on X satisfying the conditions

$$\mathcal{S}(x,x,Tx) \preceq \varphi(x) + \varphi(Tx) - 2r \tag{4}$$

and

$$\mathcal{S}(x,x,Tx) + \mathcal{S}(Tx,Tx,x_0) \preceq r, \tag{5}$$

for all $x \in C^{C^*,\mathcal{S}}_{x_0,r}$, then the circle $C^{C^*,\mathcal{S}}_{x_0,r}$ is a fixed circle of T.

Proof. Let x be any point in the circle $C^{C^*,\mathcal{S}}_{x_0,r}$. Then, using the (3), (4), (iii) given in Definition 1.4, Lemma 1.1, (5) and the definition of the relation $\preceq$, we get

$$\begin{aligned}
\mathcal{S}(x,x,Tx) &\preceq \varphi(x) + \varphi(Tx) - 2r \\
&= \mathcal{S}(x,x,x_0) + \mathcal{S}(Tx,Tx,x_0) - 2r \\
&\preceq \mathcal{S}(x,x,Tx) + \mathcal{S}(x,x,Tx) + \mathcal{S}(x_0,x_0,Tx) + \mathcal{S}(Tx,Tx,x_0) - 2r \\
&= 2\mathcal{S}(x,x,Tx) + 2\mathcal{S}(Tx,Tx,x_0) - 2r \\
&\preceq 2r - 2r = \theta
\end{aligned}$$

and so $\mathcal{S}(x,x,Tx) = \theta$ which means that $Tx = x$. As a result, we obtain that $C^{C^*,\mathcal{S}}_{x_0,r}$ is a fixed circle of T. □

Remark 2. *1. The inequality (4) means that Tx is not in the interior of the circle $C^{C^*,\mathcal{S}}_{x_0,r}$ for each $x \in C^{C^*,\mathcal{S}}_{x_0,r}$. In the same way, the inequality (5) says that Tx is not in the exterior of the circle $C^{C^*,\mathcal{S}}_{x_0,r}$ for each $x \in C^{C^*,\mathcal{S}}_{x_0,r}$. It follows that $T\left(C^{C^*,\mathcal{S}}_{x_0,r}\right) \subset C^{C^*,\mathcal{S}}_{x_0,r}$ regarding the conditions (4) and (5).*

2. Notice that Theorem 2.5 in C^*-algebra valued S-metric spaces is analogous to Theorem 3.1 given in metric spaces in [26] *and Theorem 2.5 generalizes Theorem 3.1.*

3. By Proposition 1, the obtained result in Theorem 2.5 is also valid for C^*-algebra valued b-metric $d_{\mathcal{S}}$ generated by any C^*-algebra valued S-metric $\mathcal{S}$.

Example 6. *Consider the C^*-algebra valued S-metric space $\left(\mathbb{R},\mathbb{R}^2,\mathcal{S}\right)$ and the circle $C^{C^*,\mathcal{S}}_{-\frac{1}{2},(1,0)}$ given in Example 2. Let us define the self-mapping $T:\mathbb{R}\to\mathbb{R}$ as*

$$Tx = \begin{cases} x & , \quad x \in C^{C^*,\mathcal{S}}_{-\frac{1}{2},(1,0)} \\ 5 & , \quad x \notin C^{C^*,\mathcal{S}}_{-\frac{1}{2},(1,0)} \end{cases}.$$

Then, by doing the necessary calculations one can see that T satisfies the conditions (4) and (5). That is to say that the circle $C^{C^*,\mathcal{S}}_{-\frac{1}{2},(1,0)}$ is a fixed circle of T.

Theorem 2.6. *Let $(X,\mathbb{A},\mathcal{S})$ be a C^*-algebra valued S-metric space, the mapping φ be as in (3) and $C^{C^*,\mathcal{S}}_{x_0,r}$ be any circle on X. If T is a self-mapping defined on X providing the conditions*

$$\mathcal{S}(x,x,Tx) \preceq \varphi(x) + \varphi(Tx) - 2r \tag{6}$$

and

$$\mathcal{S}(Tx,Tx,x_0) \preceq r \tag{7}$$

for all $x \in C^{C^*,\mathcal{S}}_{x_0,r}$, then the circle $C^{C^*,\mathcal{S}}_{x_0,r}$ is a fixed circle of T.

Proof. Let x be any point in the circle $C_{x_0,r}^{C^*,\mathcal{S}}$. If we use (3), (6), (7) and the definition of the relation $\preceq$, then we get

$$\begin{aligned}\mathcal{S}(x,x,Tx) &\preceq \varphi(x)+\varphi(Tx)-2r \\ &= \mathcal{S}(x,x,x_0)+\mathcal{S}(Tx,Tx,x_0)-2r \\ &\preceq 2r-2r=\theta\end{aligned}$$

and so $\mathcal{S}(x,x,Tx)=\theta$ which implies that $Tx=x$. Thus, we derive the desired result. □

Remark 3. *1. The inequality* (*6*) *says that* Tx *is not in the interior of the circle* $C_{x_0,r}^{C^*,\mathcal{S}}$ *for each* $x\in C_{x_0,r}^{C^*,\mathcal{S}}$. *Similarly, the inequality* (*7*) *guarantees that* Tx *is not in the exterior of the circle* $C_{x_0,r}^{C^*,\mathcal{S}}$ *for each* $x\in C_{x_0,r}^{C^*,\mathcal{S}}$. *These two results show that* $T\left(C_{x_0,r}^{C^*,\mathcal{S}}\right)\subset C_{x_0,r}^{C^*,\mathcal{S}}$ *under the conditions* (*6*) *and* (*7*).

2. Note that Theorem 2.6 is a new version of Theorem 2.2 given in metric spaces in [25].

3. Theorem 2.6 obtained in C^**-algebra valued* S*-metric spaces corresponds to Theorem 3.11 given in* S*-metric spaces in* [23].

4. Proposition 1 says that Theorem 2.6 is also provided for C^**-algebra valued* b*-metric* $d_{\mathcal{S}}$ *generated by any* C^**-algebra valued* S*-metric* $\mathcal{S}$.

Example 7. *Consider the* C^**-algebra valued* S*-metric space* $([0,1],M_2(\mathbb{R}),\mathcal{S})$ *and the circle* $C_{1,r}^{C^*,\mathcal{S}}$ *given in Example 3. Let us define the self-mapping* $T:[0,1]\to[0,1]$ *as*

$$Tx=\begin{cases} x & , \quad x\in C_{1,r}^{C^*,\mathcal{S}} \\ \frac{\sqrt{2}}{2} & , \quad x\notin C_{1,r}^{C^*,\mathcal{S}} \end{cases}.$$

It is apparent that T *satisfies the conditions* (*6*) *and* (*7*), *and we derive that the circle* $C_{1,r}^{C^*,\mathcal{S}}$ *is a fixed circle of* T *applying Theorem 2.6 with mapping* T.

Theorem 2.7. *Let* $(X,\mathbb{A},\mathcal{S})$ *be a* C^**-algebra valued* S*-metric space, the mapping* φ *be as in* (*3*) *and* $C_{x_0,r}^{C^*,\mathcal{S}}$ *be any circle on* X. *If* T *is a self-mapping defined on* X *satisfying the conditions*

$$\mathcal{S}(x,x,Tx)\preceq\varphi(x)-\varphi(Tx) \tag{8}$$

and

$$\mathcal{S}(Tx,Tx,x_0)\succeq r \tag{9}$$

for all $x\in C_{x_0,r}^{C^*,\mathcal{S}}$, *then the circle* $C_{x_0,r}^{C^*,\mathcal{S}}$ *is a fixed circle of* T.

Proof. Let x be arbitrary point in $C_{x_0,r}^{C^*,\mathcal{S}}$. Then, using (3), (8), (9) and the definition of the relation $\preceq$, we get

$$\begin{aligned}\mathcal{S}(x,x,Tx) &\preceq \varphi(x)-\varphi(Tx) \\ &= \mathcal{S}(x,x,x_0)-\mathcal{S}(Tx,Tx,x_0) \\ &\preceq r-\mathcal{S}(Tx,Tx,x_0) \\ &\preceq r-r=\theta\end{aligned}$$

and so $\mathcal{S}(x,x,Tx)=\theta$. Then, it should be $Tx=x$. Therefore, we deduce that $C_{x_0,r}^{C^*,\mathcal{S}}$ is a fixed circle of T. □

Remark 4. *1. The condition (8) shows that Tx is not in the exterior of the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ for each $x \in C_{x_0,r}^{C^*,\mathcal{S}}$. In the same manner, the condition (9) means that Tx is not in the interior of the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ for each $x \in C_{x_0,r}^{C^*,\mathcal{S}}$. Accordingly, these results show that $T\left(C_{x_0,r}^{C^*,\mathcal{S}}\right) \subset C_{x_0,r}^{C^*,\mathcal{S}}$ due to the conditions (8) and (9).*

2. Wee observe that Theorem 2.7 is a generalization of Theorem 2.1 given in metric spaces in [25].

3. We become aware of the fact that Theorem 2.7 is a new version of Theorem 3.2 given in S-metric spaces in [23].

4. Theorem 2.7 is also true for C^*-algebra valued b-metric $d_{\mathcal{S}}$ generated by any C^*-algebra valued S-metric $\mathcal{S}$ from Proposition 1.

Example 8. *Consider the C^*-algebra valued S-metric space $\left(L^\infty(E), L\left(L^2(E)\right), \mathcal{S}\right)$ for $E = [0,1]$ and the circle $C_{f,\pi_h}^{C^*,\mathcal{S}}$ given in Example 5, and also, the function $g_0 \in L^\infty[0,1]$ defined by*

$$g_0 : [0,1] \to \mathbb{R},\ g_0(x) = \begin{cases} 1 & , \quad x \in (\mathbb{R}\backslash\mathbb{Q}) \cap [0,1] \\ \infty & , \quad x \in \mathbb{Q} \cap [0,1] \end{cases}$$

and the self-mapping $T : L^\infty[0,1] \to L^\infty[0,1]$ defined by

$$Tg = \begin{cases} g & , \quad g \in C_{f,\pi_h}^{C^*,\mathcal{S}} \\ g_0 & , \quad g \notin C_{f,\pi_h}^{C^*,\mathcal{S}} \end{cases}$$

in Example 10 in [10]. *Then, with a direct computation it can be seen that the self-mapping T satisfies the conditions (8) and (9). Observe that the circle $C_{f,\pi_h}^{C^*,\mathcal{S}}$ is a fixed circle of T.*

Theorem 2.8. *Let $(X, \mathbb{A}, \mathcal{S})$ be a C^*-algebra valued S-metric space, the mapping φ be as in (3) and $C_{x_0,r}^{C^*,\mathcal{S}}$ be any circle on X. If T is a self-mapping defined on X satisfying the conditions*

$$\mathcal{S}(x,x,Tx) \preceq \varphi(x) - \varphi(Tx) \tag{10}$$

and

$$A^*\mathcal{S}(x,x,Tx)A + \mathcal{S}(Tx,Tx,x_0) \succeq r \tag{11}$$

for all $x \in C_{x_0,r}^{C^*,\mathcal{S}}$ and some $A \in \mathbb{A}$ with $\|A\| < 1$, then the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ is a fixed circle of T.

Proof. Assuming $x \in C_{x_0,r}^{C^*,\mathcal{S}}$ such that $x \neq Tx$, we obtain

$$\begin{aligned} \theta &\preceq \mathcal{S}(x,x,Tx) \preceq \varphi(x) - \varphi(Tx) \\ &= \mathcal{S}(x,x,x_0) - \mathcal{S}(Tx,Tx,x_0) \\ &= r - \mathcal{S}(Tx,Tx,x_0) \\ &\preceq A^*\mathcal{S}(x,x,Tx)A + \mathcal{S}(Tx,Tx,x_0) - \mathcal{S}(Tx,Tx,x_0) \\ &= A^*\mathcal{S}(x,x,Tx)A. \end{aligned}$$

Using the (3), (10) and (11), we get

$$\begin{aligned} 0 &\leq \|\mathcal{S}(x,x,Tx)\| \leq \|A^*\mathcal{S}(x,x,Tx)A\| \\ &\leq \|A^*\| \|\mathcal{S}(x,x,Tx)\| \|A\| \\ &= \|A\|^2 \|\mathcal{S}(x,x,Tx)\| \\ &< \|\mathcal{S}(x,x,Tx)\|. \end{aligned}$$

This inequality is a contradiction with our assumption. Thus, we deduce that $x = Tx$ for all $x \in C_{x_0,r}^{C^*,\mathcal{S}}$ and it is apparent that $C_{x_0,r}^{C^*,\mathcal{S}}$ is a fixed circle of T. □

Remark 5. *1. The inequality (10) guarantees that Tx is not in the exterior of the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ for each $x \in C_{x_0,r}^{C^*,\mathcal{S}}$. Similarly, the inequality (11) means that Tx is not in the interior of the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ for each $x \in C_{x_0,r}^{C^*,\mathcal{S}}$. These results indicate that $T\left(C_{x_0,r}^{C^*,\mathcal{S}}\right) \subset C_{x_0,r}^{C^*,\mathcal{S}}$ taking the conditions (10) and (11) into account.*

2. We notice that Theorem 2.8 in C^*-algebra valued S-metric spaces is corresponding of Theorem 2.3 given in metric spaces in [25].

3. Theorem 2.8 is a generalization of Theorem 3.2 given in S-metric spaces in [26].

4. Note that Theorem 2.8 can be rewritten for C^*-algebra valued b-metric $d_{\mathcal{S}}$ generated by any C^*-algebra valued S-metric $\mathcal{S}$ by Proposition 1.

Example 9. *Consider the C^*-algebra valued S-metric space $(\mathbb{R}, M_2(\mathbb{C}), \mathcal{S})$ and the circle $C_{0,r}^{C^*,\mathcal{S}}$ given in Example 4. Define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as*

$$Tx = \begin{cases} x & , \quad x \in C_{0,r}^{C^*,\mathcal{S}} \\ 2 & , \quad x \notin C_{0,r}^{C^*,\mathcal{S}} \end{cases} .$$

It is not hard to prove that T satisfies the conditions (10) and (11) for

$$A = \begin{bmatrix} -\frac{1}{9} & 0 \\ 0 & \frac{3}{10} \end{bmatrix} \in M_2(\mathbb{C})$$

with $\|A\| = \frac{3}{10} < 1$, and we see that the circle $C_{0,r}^{C^*,\mathcal{S}}$ is a fixed circle of T.

Remark 6. *Notice that the number of elements of the circles given in Example 2, Example 3, Example 4 and Example 5 indicates the number of fixed points of the self-mappings T given in Example 6, Example 7, Example 8 and Example 9.*

Theorem 2.9. *Let $(X, \mathbb{A}, \mathcal{S})$ be a C^*-algebra valued S-metric space, the mapping φ be as in (3) and $C_{x_0,r}^{C^*,\mathcal{S}}$ be any circle on X. If T is a self-mapping defined on X satisfying the conditions*

$$\mathcal{S}(x, x, Tx) \preceq \varphi(x) - r \tag{12}$$

or

$$\mathcal{S}(x, x, Tx) \preceq \varphi(Tx) - r \tag{13}$$

and

$$\mathcal{S}(Tx, Tx, x_0) \preceq r + A^*\mathcal{S}(x, x, Tx) A, \tag{14}$$

for all $x \in C_{x_0,r}^{C^*,\mathcal{S}}$ and some $A \in \mathbb{A}$ with $\|A\| < 1$, then the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ is a fixed circle of T.

Proof. Let $x \in C_{x_0,r}^{C^*,\mathcal{S}}$ such that $x \neq Tx$. Then, if the condition (12) holds, we find

$$\mathcal{S}(x, x, Tx) \preceq \varphi(x) - r = \mathcal{S}(x, x, x_0) - r = r - r = \theta,$$

and so $\mathcal{S}(x, x, Tx) = \theta$, a contradiction. This implies that $Tx = x$.

On the other hand, if the condition (13) holds, we get by (14)

$$\begin{aligned} \mathcal{S}(x, x, Tx) &\preceq \varphi(Tx) - r = \mathcal{S}(Tx, Tx, x_0) - r \\ &\preceq r + A^*\mathcal{S}(x, x, Tx) A - r \\ &= A^*\mathcal{S}(x, x, Tx) A, \end{aligned}$$

and so

$$\begin{aligned} 0 &\leq \|\mathcal{S}(x,x,Tx)\| \leq \|A^*\mathcal{S}(x,x,Tx)A\| \\ &\leq \|A^*\| \|\mathcal{S}(x,x,Tx)\| \|A\| \\ &= \|A\|^2 \|\mathcal{S}(x,x,Tx)\| \\ &< \|\mathcal{S}(x,x,Tx)\|. \end{aligned}$$

This is a contradiction with our assumption. Thus, we deduce that $x = Tx$ for all $x \in C_{x_0,r}^{C^*,\mathcal{S}}$ and more precisely, $C_{x_0,r}^{C^*,\mathcal{S}}$ is a fixed circle of T. □

Remark 7. *1. The conditions (12) and (13) guarantees that Tx is not in the interior of the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ for each $x \in C_{x_0,r}^{C^*,\mathcal{S}}$. Similarly, taking the inequality (14) into account, Tx is not in the exterior of the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ for each $x \in C_{x_0,r}^{C^*,\mathcal{S}}$. These results indicate that $T\left(C_{x_0,r}^{C^*,\mathcal{S}}\right) \subset C_{x_0,r}^{C^*,\mathcal{S}}$ under the conditions (12) or (13) and (14).*

2. We emphasize that Theorem 2.9 obtained in C^*-algebra valued S-metric spaces corresponds to Theorem 4.2 given in S-metric spaces in [9].

3. Since any C^*-algebra valued S-metric $\mathcal{S}$ generates C^*-algebra valued b-metric $d_{\mathcal{S}}$ given in Proposition 1, Theorem 2.9 can be rearranged for $d_{\mathcal{S}}$.

Let $I_X : X \to X$ be the identity map defined as $I_X(x) = x$ for all $x \in X$. We note that the identity map satisfies the conditions in Theorem 2.5, Theorem 2.6, Theorem 2.7, Theorem 2.8 and Theorem 2.9 for any circle. Now we determine a condition which excludes I_X in Theorem 2.5, Theorem 2.6, Theorem 2.7, Theorem 2.8 and Theorem 2.9 modifying Caristi's fixed point theorem [5] and Caristi type contractive condition in C^*-algebra valued metric spaces [29] as follows:

Theorem 2.10. *Let $(X, \mathbb{A}, \mathcal{S})$ be a C^*-algebra valued S-metric space, the mapping φ be as in (3) and $C_{x_0,r}^{C^*,\mathcal{S}}$ be any circle on X. T is a self-mapping defined on X satisfying the condition*

$$A^*\mathcal{S}(x,x,Tx)A \preceq \varphi(x) - \varphi(Tx) \tag{15}$$

for all $x \in X$ where $A \in \mathbb{A}$ is an invertible element and $\left\|A^{-1}\right\| < \frac{1}{\sqrt{2}}$ if and only if T fixes the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ and $T = I_X$.

Proof. Suppose that T be a self-mapping defined on X satisfying the rule (15). Let x be any point in X. We assert that $x = Tx$. Suppose, on contrary that $x \neq Tx$. Then, using the (3), (15), (*iii*) given in Definition 1.4 and Lemma 1.1, we get

$$\begin{aligned} A^*\mathcal{S}(x,x,Tx)A &\preceq \varphi(x) - \varphi(Tx) \\ &= \mathcal{S}(x,x,x_0) - \mathcal{S}(Tx,Tx,x_0) \\ &\preceq \mathcal{S}(x,x,Tx) + \mathcal{S}(x,x,Tx) + \mathcal{S}(x_0,x_0,Tx) - \mathcal{S}(Tx,Tx,x_0) \\ &= 2\mathcal{S}(x,x,Tx) + \mathcal{S}(Tx,Tx,x_0) - \mathcal{S}(Tx,Tx,x_0) \\ &= \mathcal{S}S(x,x,Tx), \end{aligned}$$

and so

$$\mathcal{S}(x,x,Tx) \preceq (A^*)^{-1} 2\mathcal{S}(x,x,Tx)A^{-1} = \left(A^{-1}\right)^* 2\mathcal{S}(x,x,Tx)A^{-1}.$$

After some elementary calculations, it follows that

$$\begin{aligned}\|\mathcal{S}(x,x,Tx)\| &\leq \left\|\left(A^{-1}\right)^{*}2\mathcal{S}(x,x,Tx)A^{-1}\right\| \\ &\leq 2\left\|\left(A^{-1}\right)^{*}\right\|\|\mathcal{S}(x,x,Tx)\|\left\|A^{-1}\right\| \\ &= 2\left\|A^{-1}\right\|^{2}\|\mathcal{S}(x,x,Tx)\| \\ &< \|\mathcal{S}(x,x,Tx)\|.\end{aligned}$$

However it is not possible and $T = I_X$.

Conversely, suppose that T fixes the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ and $T = I_X$. Then, since $Tx = x$ for all $x \in X$, the condition (15) holds for any invertible element $A \in \mathbb{A}$ with $\left\|A^{-1}\right\| < \frac{1}{\sqrt{2}}$. This completes the proof. $\square$

Remark 8. *Theorem 2.10 says that if a self-mapping fixes a circle by satisfying (4) and (5) (or (6) and (7), or (8) and (9), or (10) and (11), or [(12) or (13)] and (14)), but does not satisfy the condition (15), then the self-mapping cannot be identity map.*

Now, we state a new existence theorem for fixed circles using another auxiliary function.

Theorem 2.11. *Let $(X,\mathbb{A},\mathcal{S})$ be a C^*-algebra valued S-metric space, $C_{x_0,r}^{C^*,\mathcal{S}}$ be any circle on X. Define the mapping $\varphi_r : \mathbb{A}_+ \to \mathbb{A}$ as*

$$\varphi_r(u) = \begin{cases} u - r & , \quad u \in \mathbb{A}_+ - \{\theta\} \\ \theta & , \quad u = \theta \end{cases},$$

for all $u \in \mathbb{A}_+$. If T is a self-mapping defined on X satisfying the conditions

$$\mathcal{S}(Tx,Tx,x_0) = r \tag{16}$$

for all $x \in C_{x_0,r}^{C^*,\mathcal{S}}$,

$$\mathcal{S}(Tx,Tx,Ty) - r \in \mathbb{A}_+ - \{\theta\} \tag{17}$$

for all $x, y \in C_{x_0,r}^{C^*,\mathcal{S}}$ and $x \neq y$, and

$$\mathcal{S}(Tx,Tx,Ty) \preceq \mathcal{S}(x,x,y) - \varphi_r(\mathcal{S}(x,x,Tx)) \tag{18}$$

for all $x, y \in C_{x_0,r}^{C^*,\mathcal{S}}$, then the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ is a fixed circle of T.

Proof. Let x be any point in the circle $C_{x_0,r}^{C^*,\mathcal{S}}$. Let $x \neq Tx$ and $y = Tx$. Then, using the conditions (16) and (17), we write

$$\mathcal{S}\left(Tx,Tx,T^2x\right) - r \in \mathbb{A}_+ - \{\theta\}, \tag{19}$$

for $y = Tx \in C_{x_0,r}^{C^*,\mathcal{S}}$. Also, on account of the condition (18) we have

$$\begin{aligned}\mathcal{S}\left(Tx,Tx,T^2x\right) &\preceq \mathcal{S}(x,x,Tx) - \varphi_r(\mathcal{S}(x,x,Tx)) \\ &= \mathcal{S}(x,x,Tx) - \mathcal{S}(x,x,Tx) + r = r.\end{aligned}$$

This implies that $r - \mathcal{S}\left(Tx,Tx,T^2x\right) \in \mathbb{A}_+$. It is a contradiction with (19). Hence $C_{x_0,r}^{C^*,\mathcal{S}}$ is a fixed circle of T. $\square$

Remark 9. *1. The condition (16) in Theorem 2.11 says that Tx is on the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ for every $x \in C_{x_0,r}^{C^*,\mathcal{S}}$.*

2. We see that Theorem 2.11 is a new version of Theorem 3 given in metric spaces in [24].

3. Notice that Theorem 2.11 is a generalization to C^*-algebra valued S-metric spaces of Theorem 4.1 given in S-metric spaces in [9].

4. From Proposition 1, we know that Theorem 2.11 is also valid for C^*-algebra valued b-metric $d_{\mathcal{S}}$ generated by any C^*-algebra valued S-metric $\mathcal{S}$.

Note that I_X satisfies the conditions in Theorem 2.11 for any circle. In the following theorem, we investigate a condition which excludes I_X in Theorem 2.11.

Theorem 2.12. *Let $(X, \mathbb{A}, \mathcal{S})$ be a C^*-algebra valued S-metric space, the mapping φ_r be as in (**??**) and $C_{x_0,r}^{C^*,\mathcal{S}}$ be any circle on X. T is a self-mapping defined on X satisfying the condition*

$$\varphi_r \left(\mathcal{S}\left(x, x, Tx\right)\right) + r - \mathcal{S}\left(x, x, Tx\right) \in \mathbb{A}_+ - \{\theta\} \tag{20}$$

for all $x \in X$ if and only if T fixes the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ and $T = I_X$.

Proof. Suppose that T be a self-mapping defined on X satisfying the rule (20). Let x be any point in X. Let $x \neq Tx$. Then, we get

$$\begin{aligned} \varphi_r \left(\mathcal{S}\left(x, x, Tx\right)\right) + r - \mathcal{S}\left(x, x, Tx\right) &= \mathcal{S}\left(x, x, Tx\right) - r + r - \mathcal{S}\left(x, x, Tx\right) \\ &= \mathcal{S}\left(x, x, Tx\right) - \mathcal{S}\left(x, x, Tx\right) \\ &= \theta, \end{aligned}$$

a contradiction. It follows that $x = Tx$ for all $x \in X$ and $T = I_X$.

Contrarily, assume that T fixes the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ and $T = I_X$. Then, since $Tx = x$ for all $x \in X$, the condition (20) holds for all $x \in X$. The proof is completed. □

2.2. **The uniqueness of fixed circles.** In this subsection, we prove some uniqueness theorems for fixed circles in the existence theorems given in Subsection 2.1. To do this, the following example emphasizes that fixed circles of a self-mapping may not be unique.

Example 10. *Let us consider the C^*-algebra valued S-metric space $([0,1], M_2(\mathbb{R}), \mathcal{S})$* [12]*. If we define the self-mapping $T : [0,1] \to T[0,1]$ as*

$$Tx = \begin{cases} x^2 + \frac{3}{16} & , \quad x \in \left\{\frac{1}{4}, \frac{3}{4}\right\} \\ 0 & , \quad x \in [0,1] \setminus \left\{\frac{1}{4}, \frac{3}{4}\right\} \end{cases}$$

for all $x \in X$. Then, T fixes the circles $C_{0,r}^{C^*,\mathcal{S}} = \left\{\frac{1}{4}\right\}$ and $C_{1,r}^{C^*,\mathcal{S}} = \left\{\frac{3}{4}\right\}$, where $r = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. In other words, the fixed circles of T is not unique.

Firstly, we discuss the uniqueness of fixed circles in Theorem 2.5 using Theorem 1.1 and Theorem 1.4 which are modified forms of Banach's fixed-point theorem [4] in the subsequent theorem.

Theorem 2.13. *Let $(X, \mathbb{A}, \mathcal{S})$ be a C^*-algebra valued S-metric space, $C_{x_0,r}^{C^*,\mathcal{S}}$ be any circle on X and T be a self-mapping satisfying the conditions (4) and (5) given in Theorem 2.5. If T satisfies*

$$\mathcal{S}\left(Tx, Tx, Ty\right) \preceq A^* \mathcal{S}\left(x, x, y\right) A, \tag{21}$$

for all $x \in C_{x_0,r}^{C^*,\mathcal{S}}$, $y \in X - C_{x_0,r}^{C^*,\mathcal{S}}$ and some $A \in \mathbb{A}'_+$ with $\|A\| < 1$, then the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ is unique fixed circle of T.

Proof. Assume that $C_{x_1,\delta}^{C^*,\mathcal{S}}$ is another fixed circle of T. Let $x \in C_{x_0,r}^{C^*,\mathcal{S}}$ and $y \in C_{x_1,\delta}^{C^*,\mathcal{S}}$. Then, considering the condition (21) we get

$$\mathcal{S}(x,x,y) = \mathcal{S}(Tx,Tx,Ty) \preceq A^*\mathcal{S}(x,x,y)A,$$

and so

$$\|\mathcal{S}(x,x,y)\| \leq \|A^*\mathcal{S}(x,x,y)A\| \leq \|A\|^2 \|\mathcal{S}(x,x,y)\| < \|\mathcal{S}(x,x,y)\|.$$

But this is not possible. Hence, the self-mapping T fixes only circle $C_{x_0,r}^{C^*,\mathcal{S}}$. □

Corollary 2.9. *Let $(X,\mathbb{A},\mathcal{S})$ be a C^*-algebra valued S-metric space, $C_{x_0,r}^{C^*,\mathcal{S}}$ be any circle on X and T be a self-mapping satisfying the conditions (4) and (5) given in Theorem 2.5. If T satisfies (21) for all $x \in C_{x_0,r}^{C^*,\mathcal{S}}$, $y \in X - C_{x_0,r}^{C^*,\mathcal{S}}$ and some $A \in \mathbb{A}$ with $\|A\| < 1$, then the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ is unique fixed circle of T.*

Proof. The proof is clear from Theorem 2.13. □

Now, we determine the uniqueness condition for the fixed circles in Theorem 2.6 utilizing the condition given in Theorem 1.2 which is an enlargement of Kannan's fixed-point condition [15].

Theorem 2.14. *Let $(X,\mathbb{A},\mathcal{S})$ be a C^*-algebra valued S-metric space, $C_{x_0,r}^{C^*,\mathcal{S}}$ be any circle on X and T be a self-mapping providing the conditions (6) and (7) given in Theorem 2.6. If T satisfies the contraction condition*

$$\mathcal{S}(Tx,Tx,Ty) \preceq A(\mathcal{S}(Tx,Tx,x) + \mathcal{S}(Ty,Ty,y)) \tag{22}$$

for all $x \in C_{x_0,r}^{C^*,\mathcal{S}}$, $y \in X - C_{x_0,r}^{C^*,\mathcal{S}}$ and some $A \in \mathbb{A}'_+$ with $\|A\| < \frac{1}{2}$, then the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ is unique fixed circle of T.

Proof. Suppose that $C_{x_1,\delta}^{C^*,\mathcal{S}}$ is another fixed circle of T. For arbitrary points $x \in C_{x_0,r}^{C^*,\mathcal{S}}$ and $y \in C_{x_1,\delta}^{C^*,\mathcal{S}}$, we get by (22)

$$\mathcal{S}(x,x,y) = \mathcal{S}(Tx,Tx,Ty) \preceq A(\mathcal{S}(Tx,Tx,x) + \mathcal{S}(Ty,Ty,y)),$$

so that

$$\begin{aligned} \|\mathcal{S}(x,x,y)\| &\leq \|A(\mathcal{S}(Tx,Tx,x) + \mathcal{S}(Ty,Ty,y))\| \\ &\leq \|A\| \|(\mathcal{S}(x,x,x) + \mathcal{S}(y,y,y))\| \\ &= 0, \end{aligned}$$

which is a contradiction which means that $x = y$. This shows that the self-mapping T fixes only circle $C_{x_0,r}^{C^*,\mathcal{S}}$. □

Subsequently, we find the uniqueness condition for the fixed circles in Theorem 2.7 by employing the condition given in Theorem 1.3 which is a new form of Chatterjea's contractive condition [6].

Theorem 2.15. *Let $(X,\mathbb{A},\mathcal{S})$ be a C^*-algebra valued S-metric space, $C_{x_0,r}^{C^*,\mathcal{S}}$ be any circle on X and T be a self-mapping satisfying the conditions (8) and (9) given in Theorem 2.7. If T satisfies the contraction condition*

$$\mathcal{S}(Tx,Tx,Ty) \preceq A(\mathcal{S}(Tx,Tx,y) + \mathcal{S}(Ty,Ty,x)) \tag{23}$$

for all $x \in C_{x_0,r}^{C^*,\mathcal{S}}$, $y \in X - C_{x_0,r}^{C^*,\mathcal{S}}$ and some $A \in \mathbb{A}'_+$ with $\|A\| < \frac{1}{2}$, then the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ is unique fixed circle of T.

Proof. Assume that $C_{x_1,\delta}^{C^*,\mathcal{S}}$ is another fixed circle of T. For any points $x \in C_{x_0,r}^{C^*,\mathcal{S}}$ and $y \in C_{x_1,\delta}^{C^*,\mathcal{S}}$, we have the following statement by (23):

$$\mathcal{S}(x,x,y) = \mathcal{S}(Tx,Tx,Ty) \preceq A\left(\mathcal{S}(Tx,Tx,y) + \mathcal{S}(Ty,Ty,x)\right).$$

So, we see that

$$\begin{array}{rcl} \|\mathcal{S}(x,x,y)\| & \leq & \|A\left(\mathcal{S}(Tx,Tx,y) + \mathcal{S}(Ty,Ty,x)\right)\| \\ & \leq & \|A\| \, \|\mathcal{S}(Tx,Tx,y) + \mathcal{S}(Ty,Ty,x)\| \\ & = & \|A\| \, \|2\mathcal{S}(x,x,y)\| \\ & < & \|\mathcal{S}(x,x,y)\|. \end{array}$$

This yields a contradiction which implies that $x = y$. So, T fixes only circle $C_{x_0,r}^{C^*,\mathcal{S}}$. □

Finally, we state our three uniqueness theorems for the fixed circles in Theorem 2.8, Theorem 2.9 and Theorem 2.11 by revising Ćirić's and Reich's fixed point theorems [7], [28].

Theorem 2.16. *Let $(X, \mathbb{A}, \mathcal{S})$ be a C^*-algebra valued S-metric space, $C_{x_0,r}^{C^*,\mathcal{S}}$ be any circle on X and T be a self-mapping satisfying (10) and (11) given in Theorem 2.8. If T satisfies the contraction condition that there exists*

$$U \in \left\{\mathcal{S}(x,x,y), \mathcal{S}(Tx,Tx,x), \mathcal{S}(Ty,Ty,y), \mathcal{S}(Ty,Ty,x), \mathcal{S}(Tx,Tx,y)\right\}$$

such that

$$\mathcal{S}(Tx,Tx,Ty) \preceq A^*UA, \tag{24}$$

for all $x \in C_{x_0,r}^{C^*,\mathcal{S}}$, $y \in X - C_{x_0,r}^{C^*,\mathcal{S}}$ and some $A \in \mathbb{A}$ with $\|A\| < 1$, then the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ is unique fixed circle of T.

Proof. Assume that $C_{x_1,\delta}^{C^*,\mathcal{S}}$ is another fixed circle of T, and x and y be any points in $C_{x_0,r}^{C^*,\mathcal{S}}$ and $C_{x_1,\delta}^{C^*,\mathcal{S}}$, respectively. Then, we get

$$\mathcal{S}(x,x,y) = \mathcal{S}(Tx,Tx,Ty) \preceq A^*UA$$

from the condition (24). But we get a contradiction because of

$$\begin{array}{rcl} \|\mathcal{S}(x,x,y)\| & \leq & \|A^*UA\| \leq \|A\|^2 \|U\| < \|U\| \\ & \leq & \max\left\{ \begin{array}{c} \|\mathcal{S}(x,x,y)\|, \|\mathcal{S}(Tx,Tx,x)\|, \|\mathcal{S}(Ty,Ty,y)\|, \\ \|\mathcal{S}(Ty,Ty,x)\|, \|\mathcal{S}(Tx,Tx,y)\| \end{array} \right\} \\ & = & \max\left\{\|\mathcal{S}(x,x,y)\|, 0\right\} = \|\mathcal{S}(x,x,y)\|. \end{array}$$

Hence $x = y$ and T fixes only circle $C_{x_0,r}^{C^*,\mathcal{S}}$. □

Theorem 2.17. *Let $(X, \mathbb{A}, \mathcal{S})$ be a C^*-algebra valued S-metric space, $C_{x_0,r}^{C^*,\mathcal{S}}$ be any circle on X and the T be a self-mapping satisfying the conditions (12) or (13) and (14) given in Theorem 2.9. If T satisfies the contraction condition that there exists*

$$V \in \left\{\mathcal{S}(Tx,Tx,x), \mathcal{S}(Ty,Ty,y), \mathcal{S}(Ty,Ty,x), \mathcal{S}(Tx,Tx,y)\right\}$$

such that

$$\mathcal{S}(Tx,Tx,Ty) \preceq A^*\mathcal{S}(x,x,y)A + B^*VB, \tag{25}$$

for all $x \in C_{x_0,r}^{C^*,\mathcal{S}}$, $y \in X - C_{x_0,r}^{C^*,\mathcal{S}}$ and some $A, B \in \mathbb{A}$ with $\|A\| < \frac{1}{\sqrt{2}}$ and $\|B\| < \frac{1}{\sqrt{2}}$, then the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ is unique fixed circle of T.

Proof. Assume that $C_{x_1,\delta}^{C^*,\mathcal{S}}$ is another fixed circle of T. Suppose x and y be arbitrary points in $C_{x_0,r}^{C^*,\mathcal{S}}$ and $C_{x_1,\delta}^{C^*,\mathcal{S}}$, respectively. Then, we get by (25)

$$\mathcal{S}(x,x,y)=\mathcal{S}(Tx,Tx,Ty)\preceq A^*\mathcal{S}(x,x,y)A+B^*VB$$

so that

$$\begin{aligned}
\|\mathcal{S}(x,x,y)\| &\leq \|A^*\mathcal{S}(x,x,y)A+B^*VB\| \\
&\leq \|A\|^2\|\mathcal{S}(x,x,y)\|+\|B\|^2\|V\| \\
&\leq \|A\|^2\|\mathcal{S}(x,x,y)\| \\
&\quad +\|B\|^2\max\left\{\begin{array}{c}\|\mathcal{S}(Tx,Tx,x)\|,\|\mathcal{S}(Ty,Ty,y)\|, \\ \|\mathcal{S}(Ty,Ty,x)\|,\|\mathcal{S}(Tx,Tx,y)\|\end{array}\right\} \\
&= \|A\|^2\|\mathcal{S}(x,x,y)\|+\|B\|^2\max\{\|\mathcal{S}(x,x,y)\|,0\} \\
&= \left(\|A\|^2+\|B\|^2\right)\|\mathcal{S}(x,x,y)\|<\|\mathcal{S}(x,x,y)\|.
\end{aligned}$$

Therefore a contradiction is reached. As a result we get $x=y$ indicates the self-mapping T fixes only circle $C_{x_0,r}^{C^*,\mathcal{S}}$. □

Theorem 2.18. *Let $(X,\mathbb{A},\mathcal{S})$ be a C^*-algebra valued S-metric space, $C_{x_0,r}^{C^*,\mathcal{S}}$ be any circle on X and T be a self-mapping satisfying the conditions (16), (17) and (18) given in Theorem 2.11. If T satisfies the contraction condition such that*

$$\mathcal{S}(Tx,Tx,Ty)\preceq A^*\mathcal{S}(x,x,y)A+B^*\mathcal{S}(Tx,Tx,x)B+C^*\mathcal{S}(Ty,Ty,y))C \quad (26)$$

for all $x\in C_{x_0,r}^{C^*,\mathcal{S}}$, $y\in X-C_{x_0,r}^{C^*,\mathcal{S}}$ and some $A,B,C\in\mathbb{A}_+$ with $\|A\|<\frac{1}{\sqrt{3}}$, $\|B\|<\frac{1}{\sqrt{3}}$ and $\|C\|<\frac{1}{\sqrt{3}}$, then the circle $C_{x_0,r}^{C^*,\mathcal{S}}$ is unique fixed circle of T.

Proof. Assume that $C_{x_1,\delta}^{C^*,\mathcal{S}}$ is another fixed circle of T. Let x and y be arbitrary points in $C_{x_0,r}^{C^*,\mathcal{S}}$ and $C_{x_1,\delta}^{C^*,\mathcal{S}}$, respectively. Then, we get by (26)

$$\mathcal{S}(x,x,y)=\mathcal{S}(Tx,Tx,Ty)\preceq A^*\mathcal{S}(x,x,y)A+B^*\mathcal{S}(Tx,Tx,x)B+C^*\mathcal{S}(Ty,Ty,y)C$$

so that

$$\begin{aligned}
\|\mathcal{S}(x,x,y)\| &\leq \|A^*\mathcal{S}(x,x,y)A+B^*\mathcal{S}(Tx,Tx,x)B+C^*\mathcal{S}(Ty,Ty,y)C\| \\
&\leq \|A\|^2\|\mathcal{S}(x,x,y)\|+\|B\|^2\|\mathcal{S}(Tx,Tx,x)\|+\|C\|^2\|\mathcal{S}(Ty,Ty,y)\| \\
&= \|A\|^2\|\mathcal{S}(x,x,y)\|<\frac{1}{3}\|\mathcal{S}(x,x,y)\|<\|\mathcal{S}(x,x,y)\|,
\end{aligned}$$

a contradiction, and so $x=y$. Hence, the self-mapping T fixes only circle $C_{x_0,r}^{C^*,\mathcal{S}}$. □

Remark 10. *The uniqueness theorems in this subsection are equivalents of Theorem 3.1, Theorem 3.2 and Theorem 3.3 in* [25] *for metric spaces, and also, Theorem 3.4 in* [26]*, Theorem 3.10 and Theorem 3.16 in* [23] *and Theorem 4.3 in* [9] *for S-metric spaces.*

3. An application to exponential linear unit activation functions

Activation functions define the output of that node given an input or set of inputs in neural networks. In the literature, there are a lot of examples of activation functions. The most common activation functions are ridge functions, radial functions and fold functions. More details about neural networks and activation functions can be found in [13] and the references therein.

One of several activation functions is "Exponential Linear Unit Function" defined as

$$ELU(x) = \begin{cases} \alpha(e^x - 1) & , \quad x \leq 0 \\ x & , \quad x > 0 \end{cases},$$

with parameter α (see [8] for more details).

Let us consider C^*-algebra valued S-metric space $([0,1], M_2(\mathbb{R}), \mathcal{S})$ defined in Example 1. If we take $\alpha = 1$ and $X = [0,1]$, then we have

$$ELU(x) = \begin{cases} e^x - 1 & , \quad x = 0 \\ x & , \quad x \in (0,1] \end{cases},$$

for all $x \in [0,1]$. Then, ELU satisfies the conditions of the existence theorems and fixes the circles $C_{0,r}^{C^*,\mathcal{S}} = \left\{\frac{1}{16}\right\}$ and $C_{1,r}^{C^*,\mathcal{S}} = \left\{\frac{15}{16}\right\}$, where

$$r = \begin{bmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{8} \end{bmatrix}.$$

We note that the fixed circles of ELU is not unique. Also, the selection of the center or the radius is not unique. For example, the centers of $C_{0,r}^{C^*,\mathcal{S}}$ and $C_{1,r}^{C^*,\mathcal{S}}$ are different. If we get the radius

$$r = \begin{bmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{10} \end{bmatrix},$$

then ELU fixes the circles $C_{0,r}^{C^*,\mathcal{S}} = \left\{\frac{1}{20}\right\}$ and $C_{1,r}^{C^*,\mathcal{S}} = \left\{\frac{19}{20}\right\}$.

4. Conclusion and future works

In this article, we review the existence and uniqueness conditions for fixed circles of self-mappings satisfying different kinds of contractive conditions on C^*-algebra valued S-metric spaces with constructed various techniques by giving some numerical examples to support our newly fulfilled results. Also, we give an application to exponential linear unit activation functions used in the neural networks. Each of the uniqueness theorems given in subsection 2.2 can be also stated using the contraction conditions in other uniqueness theorems given in the same subsection instead of its own contraction conditions. So, we can obtain several more consequences. But since the approaches are the same as another, we avoid listing all possible corollaries.

Furthermore, there is no reference on the existence and uniqueness of fixed circles of self-mappings on such spaces. So, since we consider and develop the fixed-circle problem in C^*-algebra valued S-metric spaces, we believe that our results will motivated many authors to study continuing works and applications.

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