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A NOTE ON THE DEFINITION OF PSEUDO-REVERSIBLE RING

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Abstract. This paper examines the concept of pseudo-reversible rings, introduced by Huang et al., as a generalization of reversible rings. We establish that pseudo-reversible rings are precisely the union of reversible rings and rings with trivial idempotents. Furthermore, we demonstrate that the results presented in the literature on pseudo-reversible rings are not novel but are direct derivations from established results on reversible rings and rings with trivial idempotents.

1. Introduction

By "ring," we always mean an associative ring with unity. The ring of n-by$n$ upper triangular (resp. full) matrices over R is denoted by $\mathbb{T}_n(T)$ (resp. $\mathbb{M}_n(R)$) and the set of all idempotents of R is denoted by $\mathcal{I}(R)$. It is worth noting that the trivial idempotents encompass zero and one. A ring R is called *abelian* if all its idempotents are central. Cohn [1] called a ring R *reversible* if $ba = 0$ whenever $ab = 0$ for every $a, b \in R$. The next theorem gives equivalent conditions for reversible property of rings with respect to its set of idempotents. This next theorem incorporates certain results form [5, Lemma 1.1. (1)] and [4, Proposition 1.4].

Theorem 1.1. *For a ring R, the following statements are equivalent:*

(i) *R is a reversible ring;*
(ii) *$ab \in \mathcal{I}(R)$, for some $a, b \in R$, implies $ba \in \mathcal{I}(R)$;*
(iii) *$ab \in \mathcal{I}(R)$, for some $a, b \in R$, implies $ba \in abRab$;*
(iv) *$ab \in \mathcal{I}(R)$, for some $a, b \in R$, implies $ba = ab$.*

Proof. (i)⇔(ii)⇔(iv) is direct form [5, Lemma 1.1. (1)] and [4, Proposition 1.4]. (ii)⇒(iii): Let $ab \in \mathcal{I}(R)$, for some $a, b \in R$. Then $ab(1 - ab) = 0$ and $b(1 - ab)a = 0$ since R is reversible. Therefore, $ba = b(ab)a = ab^2a \in abR$ since every reversible ring is

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abelian. Similarly, $ba \in Rab$ and $ba \in abR \cap Rab = abRab$. (iii)⇒(i) is clear because if $ab = 0 \in \mathcal{I}(R)$, for some $a, b \in R$ gives $ba \in abRab = 0$ and R is reversible. □

However, none of the aforementioned conditions in the previous theorem can be reduced to become "$0 \neq ab \in \mathcal{I}(R)$" instead of "$ab \in \mathcal{I}(R)$," as demonstrated in the following examples.

Example 1. *According to* [5, Theorem 1.8]*, the ring $R = \mathbb{M}_2(\mathbb{Z}_2)$ satisfies that ba is an idempotent whenever ab is a nonzero idempotent for all $a, b \in R$. While R is not reversible since the elements $\alpha = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ satisfy $\alpha\beta = 0$ and $\beta\alpha \neq 0$.*

Example 2. *Let R be any ring such that $\mathcal{I}(R) = \{0, 1\}$. Then the ring*

$$S = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \mid a, b, c, d, \in R \right\}$$

has no nontrivial idempotents; that is, 1 is the only nonzero idempotent of R. Therefore, if ab is a nonzero idempotent, then $ba \in R = abRab$. However, R is not reversible since the elements $\alpha = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ satisfy $\alpha\beta = 0$ while $\beta\alpha \neq 0$.

The following proposition demonstrates that conditions (iii) and (iv) in Theorem 1.1 remain equivalent even when the condition is reduced to nonzero idempotents.

Proposition 1. *For a ring R, the following conditions are equivalent:*

(i) *ab is a nonzero idempotent, for some $a, b \in R$, implies $ba \in abRab$.*
(ii) *ab is a nonzero idempotent, for some $a, b \in R$, implies $ba = ab$.*

Proof. (i)⇒(ii): Let ab is a nonzero idempotent for some $a, b \in R$. So, $ba \in abRab$ and $ba = ba^2b = ab^2b$. Moreover, $(ba)^4 = b(ab)^3a = (ba)^2$ and $(ba)^2$ is an idempotent that can not be zero since $ab \neq 0$. Therefore, $ab = (ab)^2 \in (ba)^2R(ba)^2$ from the assumption. Consequently, $ab = (ba)^2ab = ba(ba^2b) = (ba)^2$ and $ab^2 = (ba)^2b = b(ab)^2 = bab$. So, $ba = ab^2a = (bab)a = (ba)^2 = ab$.

(ii)⇒(i): Suppose ab is a nonzero idempotent for some $a, b \in R$. Then, $ba = ab = ababab = ab(ba)ab \in abRab$. □

The next example shows that although conditions (iii) and (iv) of Theorem 1.1 are equivalent, they do not equate to condition (ii) of the theorem when applied only to non-zero idempotents.

Example 3. *For the ring $R = \mathbb{T}_2(\mathbb{Z})$, the set of nontrivial idempotents if R is*

$$\left\{ \begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & m \\ 0 & 1 \end{bmatrix} \mid m \in \mathbb{Z} \right\}.$$

The elements a and b of R with ab being a nonzero idempotent satisfy ba is also an idempotent. However, the elements $\alpha = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ satisfy $\alpha\beta$ and $\beta\alpha$ are idempotents while $\alpha\beta \neq \beta\alpha$.

In[3], Huang et al. call the ring satisfying any of the conditions in Proposition 1 a *pseudo-reversible* ring. Clearly, every reversible ring is pseudo-reversible by Theorem 1.1. However, the converse is not necessarily true, as demonstrated in Example 2. In this paper, we provide a rigorous analysis showing the equivalence between pseudo-reversible rings and other well-known rings. Furthermore, we demonstrate that the results obtained for pseudo-reversible rings are not new but rather direct restatements of previously established results for well-known rings.

2. Main Results

In this section, we give our main result that the class of pseudo-reversible rings is exactly the union of the classes of reverable rings and the rings with trivial idempotents.

Theorem 2.2. *A ring R is a pseudo-reversible ring if and only if R is either reversible or has only trivial idempotents.*

Proof. The sufficiency: Let R be a pseudo-reversible ring. If R has only trivial idempotents, it is done. If not, then there is a nontrivial idempotent e of R. For arbitrary $r \in R$, the element $e + er(1-e)$ is a nonzero idempotent. So, $e + er(1-e) = e(1 + r(1-e)) = e(1+r(1-e))e = e$, by Proposition 1. Hence, $eR(1-e) = 0$. Similarly, $(1-e)Re = 0$ and e is central. Assume that eR is not reversible, then there exist $a, b \in eR$ such that $ab = 0$ while $ba \neq 0$. Define the elements $c = a + (1-e)$ and $d = b + (1-e)$ in R. We have $cd = 1 - e$ is a nonzero idempotent and therefore $cd = dc$ from the pseudo-reversibility of R. Therefore, $1 - e = cd = dc = ba + (1-e)$and $ba = 0$, a contradiction. Thus eR is reversible and $(1-e)R$ is so. Therefore, $R = eR \oplus (1-e)R$ is also reversible. The necessity: If R is reversible, then R is pseudo-reversible, form Proposition 1 and Theorem 1.1. If R has only trivial idempotent and ab is nonzero idempotent, then $ab = 1$. So, $ba \in R = abRab$ and R is pseudo-reversible form condition (i) of Proposition 1. □

In [5], a generalization of the reversible ring is presented; that is, the ring that satisfies condition (ii) in Theorem 1.1 for the nonzero idempotents. A ring R is called *quasi-reversible* if ab is a nonzero idempotent, which implies that ba is also an idempotent for all $a, b \in R$. From the definitions, every pseudo-reversible ring is quasi-reversible. The following theorem provides a sufficient and necessary condition to make a quasi-reversible ring pseudo-reversible.

Theorem 2.3. *A ring R is pseudo-reversible (reversible or having trivial idempotents) if and only if R is quasi-reversible and abelian.*

Proof. Let R be a pseudo-reversible ring. Then R is quasi-reversible by Proposition 1. For every nonzero idempotent $e \in R$ and arbitrary $r \in R$, the element $e + er(1-e)$ is a nonzero idempotent. So, $e + er(1-e) = e(1 + r(1-e)) = e(1 + r(1-e))e = e$, by Proposition 1. Hence, $eR(1-e) = 0$. Similarly, $(1-e)Re = 0$ and e is central. Thus, R is an abelian ring. Conversely, let R be a quasi-reversible and abelian ring. If ab is a nonzero idempotent, for some $a, b \in R$, then ba is a nonzero idempotent form quasi-reversibility of R. So, $ba = (ba)^3 = b(ab)(ab)a = (ab)(ba)(ab) \in aBRab$, from the abelianity of R. Thus, R is pseudo-reversible. □

Now, we show that the results in [3] about pseudo-reversible are trivial and obtained directly from well-known results for reversible rings or rings with trivial idempotents.

Theorem 2.4 ([3],Theorem 1.5). *Let $R = \prod_{i\in\Lambda} R_i$ be the direct product of rings R_i for $i \in \Lambda$ with $|\Lambda| \geq 2$. Then the following conditions are equivalent:*

(i) *R is pseudo-reversible;*
(ii) *R is quasi-reversible;*
(iii) *R_i is reversible for all $i \in I$;*
(iv) *R is reversible.*

Remark 1. *It is evident that the result of* [3, Theorem 1.5] *is equivalent to* [7, Lemma 1.9]*, with no substantive differences in conclusion, since in the case of $|\Lambda| \geq 2$, R has nontrivial idempotents and R is reversible.*

Theorem 2.5 ([3],Theorem 1.6). (i) *Let R be a semiperfect ring. If R is abelian, then R is either a local ring or a finite direct product of two or more local rings. Especially, Ris pseudo-reversible in the former case.*

(ii) *Let R be a semiperfect ring. If R is pseudo-reversible, then R is either a local ring or a finite direct product of two or more reversible local rings.Especially, R is reversible in the latter case.*

Remark 2. *In fact, the two statements of* [3, Theorem 1.6] *are identical, and they are exactly the result in* [2, Proposition 2.6]. *Also, the last additional part in each statement of this theorem is obvious with the fact that the pseudo-reversibility of a ring means it is reversible or local.*

Proposition 2 ([3],Proposition 1.11)**.** *Let R be a von Neumann regular ring. Then the following conditions are equivalent: (1) R is reduced; (2) R is reversible; (3) R is pseudo-reversible; (4) R is Abelian.*

Remark 3. *In the case of R has only trivial idempotents, R is a division ring, and all conditions hold. In case R is reversible, these equivalences and more are in* [4, Proposition 2.20].

In [6], a ring satisfying the quasi-reversible property is called *i-reversible*. Part (4) of [6, Proposition 2.1] shows that a quasi-reversible ring with a nontrivial central idempotent is reversible.

Theorem 2.6 ([3], Theorem 2.6)**.** *For a ring R, the following conditions are equivalent:*

(i) *$R \times_{dor} \mathbb{Z}$ is pseudo-reversible;*
(ii) *$R \times_{dor} \mathbb{Z}$ is quasi-reversible;*
(iii) *R is reversible;*
(iv) *$R \times_{dor} \mathbb{Z}$ is reversible.*

Remark 4. *In* [3, Theorem 2.6], *$R \times_{dor} \mathbb{Z}$ has nontrivial central idempotent $(-1, 1)$, and so the pseudo-reversible and quasi-reversible properties are already the reversible property. So, this result is a special case of Part (2) in* [7, Proposition 1.14].

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