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Research article

Statistical Analysis of Alpha-Power Exponential Distribution Using Unified Hybrid Censored Data and Its Applications

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Abstract: A new extended exponential distribution called the alpha-power exponential (APE) distribution is widely used for modeling various data compared to three well-known models, namely Weibull, gamma, and generalized-exponential distributions. This paper provides several approaches for estimating the APE parameters, reliability, and hazard rate functions using both classical and Bayesian methods when dealing with unified hybrid censored data. In the classical setup, we considered two methods called the maximum likelihood and maximum product of spacings. Moreover, using the delta method with observed Fisher information, asymptotic confidence intervals for the APE parameters or any function related to them are estimated. In the Bayes framework, using the squared-error loss with independent gamma density priors, we suggest two frequentist functions to develop the Bayes and highest posterior density credible interval estimators for all parameters. The Bayes estimation is performed by the Markov chain Monte Carlo sampling technique. Through extensive Monte Carlo simulations, based on various options of the effective sample sizes and threshold levels, the effectiveness of the proposed methodologies is compared. To show the applicability of the suggested estimators, two real-life datasets from the engineering domain are analyzed; the first consists of the yarn failure cycle and the other represents the failure times of the electronic components. The numerical analysis demonstrated that the APE model is suitable for analyzing the specified censored data. Furthermore, it is noted that the Bayesian method via the product of the spacing function yields more accurate results when a dataset under consideration is gathered from the proposed censoring.

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1. Introduction

Hybrid censoring (or the mixture of Type-I and Type-II plans) is used when we stop the experiment at a specific time point, say $T^{\bullet} = \min(X_{k:n}, T)$, where $k(1 \leqslant k \leqslant n)$ denotes the number of failures and $T \in (0, \infty)$ is the threshold time. Although this oversight saves the total time spent on testing by no more than T, its main drawback is that very few failures may occur up to the prefixed time T. Therefore, Childs et al. [15] called this plan a Type-I hybrid censoring (HC-TI) and suggested a Type-II hybrid censoring (HC-TII) where the experiment ended at $T^{\bullet} = \max(X_{k:n}, T)$.

Because HC-TII promises a certain number of failures, it could take a very long time to collect k failures. Because of this, Chandrasekar et al. [10] improved both HC-TI and HC-TII by introducing two new censoring strategies called generalized Type-I and Type-II hybrid censored plans. Under generalized Type-I hybrid censoring (GHC-TI), n units are put on a test at time zero, the number of failures k and r such that $k, r \in 1, 2, ..., n$ and $T \in (0, \infty)$ are prespecified. If the lifetime of $X_{k:n}$ is less than T, end the test at $\min X_{r:n}, T$, otherwise at $X_{k:n}$. The modification in HC-TI is achieved through GHC-TI, which allows the experiment to exceed the time T. According to GHC-TI, a minimal number k of failures is assured.

In contrast, in the generalized Type-II hybrid censoring (GHC-TII), we put n units on a test, fix $k \in 1, 2, ..., n$, and $T_1, T_2 \in (0, \infty)$ such that $T_1 < T_2$; if $X_{k:n} < T_1$, stop the experiment at T_1; if $T_1 < X_{k:n} < T_2$, stop at $X_{k:n}$; otherwise, and stop the experiment at T_2. Even though GHC-TII changes HC-TII to make sure the experiment is done by time T_2, there are some drawbacks. In GHC-TI, if we only have one preassigned time T, we cannot ensure r failures will occur. Additionally, in GHC-TII, it is possible not to see any failure or only a few failures before a set time called T_2. This means that it has the same issue as HC-TI. Although GHC-TII modifies HC-TII by guaranteeing that the experiment will be completed by time T_2, there are some drawbacks with GHCs.

In GHC-TI, because of only one preassigned time T, we cannot guarantee r failures. Also, in GHC-TII, there is a possibility that no failure (or very few failures) will be observed until the prefixed time T_2, and thus it has the same problem as in HC-TI. To overcome these drawbacks, Balakrishnan et al. [8] proposed a unified hybrid censoring (UHC), which is a mixture of GHC-TI and GHC-TII plans. They described this censoring as: suppose n identical units are put in a lifetime experiment at start time zero, fix integers $k, r \in 1, ..., n$ such that $k < r$, and time points $T_1, T_2 \in (0, \infty)$ such that $T_1 < T_2$. If $X_{k:n} < T_1$, end the experiment at $\min\{T_2, \max(X_{r:n}, T_1)\}$, if $T_1 < X_{k:n} < T_2$, end the experiment at $\min(X_{r:n}, T_2)$, otherwise end the experiment at $X_{k:n}$. Under UHC, one guarantees to terminate the life test at the most time T_2 with at least k failures. The number of failures before time points T_1 and T_2 is denoted by d_1 and d_2, respectively.

Subsequently, based on UHC plan, the experimenter can collect one of the following six observation forms:

$$
\begin{array}{rll}
\text{Case-I:} & \{0 < X_{k:n} < X_{r:n} < T_1\}, & \text{stop the test at } \; T_1; \\
\text{Case-II:} & \{0 < X_{k:n} < T_1 < X_{r:n} < T_2\}, & \text{stop the test at } \; X_{r:n}; \\
\text{Case-III:} & \{0 < X_{k:n} < T_1 < T_2 < X_{r:n}\}, & \text{stop the test at } \; T_2; \\
\text{Case-IV:} & \{0 < T_1 < X_{k:n} < X_{r:n} < T_2\}, & \text{stop the test at } \; X_{r:n}; \\
\text{Case-V:} & \{0 < T_1 < X_{k:n} < T_2 < X_{r:n}\}, & \text{stop the test at } \; T_2; \\
\text{Case-VI:} & \{0 < T_1 < T_2 < X_{k:n} < X_{r:n}\}, & \text{stop the test at } \; X_{k:n}.
\end{array}
$$

However, suppose that $\mathbf{x}$ is a vector of censored data that contains n identical independent units taken from a continuous population with probability density function (PDF) $f(\cdot)$ and cumulative distribution function (CDF) $F(\cdot)$, then the likelihood function, where x_i is used instead of $x_{i:n}$, is defined as

$$L(\xi|\mathbf{x}) = Q\prod_{i=1}^{\omega} f(x_i;\xi)\left[1 - F(\upsilon;\xi)\right]^{n-\omega}, \tag{1.1}$$

where

$$(\omega,\upsilon) = \begin{cases} (d, T_1) & \text{for Case-I;} \\ (r, x_r) & \text{for Case-II and Case-IV;} \\ (d, T_2) & \text{for Case-III and Case-V;} \\ (k, x_k) & \text{for Case VI,} \end{cases}$$

where ξ is a parameter vector, Q is a constant, and ω denotes the number of collected failures up to υ.

Recently, using the proposed censored sample, different lifetime models are investigated via frequentist and Bayes methods of parameter estimations; see, for example, Panahi and Sayyareh [40] for Burr Type-XII; Ghazal and Hasaballah [25] for exponentiated Rayleigh; Panahi [39] for Burr Type-III; Ateya [7] for inverse Weibull; Ghazal and Shihab [24] for exponentiated Pareto; EL-Sagheer et al. [21] for Burr-XII; Rabie and Li [42] for Burr-X; Mahmouda [33] for exponentiated-gamma; Jeon and Kang [29] for Rayleigh; Dutta and Kayal [18] and Dutta et al. [19] for Weibull and family of inverted exponentiated distributions based on partially observed competing risks, respectively; Dutta et al. [20] for Marshall–Olkin bivariate Kumaraswamy distribution based on partially observed dependent competing risks; among others.

A unique method of enhancing distributions by introducing an additional parameter to well-known distributions has been introduced by Mahdavi and Kundu [32]. They also created a new distribution called the alpha-power exponential (APE) distribution by using the exponential baseline distribution. Furthermore, they analyzed the statistics in the APE distribution and utilized the maximum likelihood approach to estimate the distribution's characteristics in the presence of complete information. A random variable X is said to have APE distribution, denoted by $X \sim \text{APE}(\xi)$, where $\xi = (\alpha,\lambda)^\top$, if its PDF and CDF are given by

$$f(x;\xi) = \frac{\lambda\log(\alpha)e^{-\lambda x}\alpha^{1-e^{-\lambda x}}}{\alpha - 1}, \; x > 0, \; \alpha,\lambda > 0, \; \alpha \neq 1, \tag{1.2}$$

and

$$F(x;\xi) = \frac{\alpha^{1-e^{-\lambda x}} - 1}{\alpha - 1}, \tag{1.3}$$

respectively, where α and λ are the shape and scale parameters. Moreover, the reliability function (RF) and the hazard rate function (HRF) of the APE distribution (at a specified time $t > 0$) are given by

$$R(t;\xi) = \frac{\alpha}{\alpha - 1}(1 - \alpha^{-e^{-\lambda t}}), \tag{1.4}$$

and

$$h(t;\xi) = \frac{\lambda\log(\alpha)e^{-\lambda t}\alpha^{e^{-\lambda t}}}{1 - \alpha^{-e^{-\lambda t}}}, \tag{1.5}$$

respectively. It is better to mention here that (i) the HRF (1.5) can exhibit increasing, decreasing, or constant shapes, making it more adaptable for reliability analysis and survival studies, and (ii) the extra shape parameter α (alpha-power parameter) allows for better fitting to empirical data, outperforming traditional models like the exponential and Weibull distributions.

Several works in the literature have considered various inferences about the unknown parameters and reliability characteristics of the APE model; for example, Nassar et al. [35] provided different estimators of APE parameters using a complete sample; Salah [44] developed different estimations of the APE parameters under progressive Type-II censoring; Salah et al. [45] obtained both the point and interval estimators of the APE parameters using HC-TII; Abo-Kasem et al. [1] studied the estimating problems of the unknown APE parameters under Type-II progressive hybrid censoring; Haj Ahmad et al. [28] studied statistical analysis of alpha power inverse Weibull distribution under hybrid censored scheme; Alotaibi et al. [3] studied the maximum likelihood and Bayes estimators of the unknown parameters, survival and hazard rate functions under an adaptive progressive Type-II hybrid censoring.Although several extensive works on estimating parameter(s) of different lifetime models using maximum likelihood and/or Bayesian methods based on UHC are available, we have not come across any study that considers the maximum product of spacing (MPS) function, which is the first competitive approach to the maximum likelihood (ML) estimation, from the unified hybrid censoring. To address this issue by demonstrating the APE distribution, the main objectives of this study are:

- The first objective develops both point and interval estimations of the unknown model parameters and the reliability characteristics using ML and MPS methods. Also, based on the observed Fisher information obtained from these classical methods, the approximate confidence intervals (ACIs) for the unknown parameters or any function of them are obtained.

- The second objective, using independent conjugate gamma priors, develops the Bayes estimators of the APE parameters and associated reliability characteristics based on ML and MPS functions against the squared-error loss (SEL). The explicit expressions of Bayesian estimators cannot be obtained due to the complex mathematical nature of maximum likelihood estimators (MLEs) and maximum product of spacing estimators (MPSEs) for unknown parameters. Therefore, to develop the Bayes point estimates and to construct the highest posterior density (HPD) credible intervals for all unknown parameters, Markov chain Monte Carlo (MCMC) techniques are utilized.

- The third objective aims to evaluate the behavior of the proposed methodologies through Monte Carlo simulations. The simulated estimates, thus computed, are compared using four criteria. The last objective, using different datasets from the engineering field, demonstrates the applicability of estimation procedures in real-life situations.

The rest of the paper is classified as follows: classical and Bayes inferences are obtained in Sections 2 and 3, respectively. In Section 4, simulation results are reported. Two real-life applications are examined in Section 5. Finally, concluding remarks are summarized in Section 6.

2. Classical Inference

This part develops, when the UHC dataset is available, the MLEs and MPSEs as well as their associated ACIs of the APE parameters α, λ, $R(t)$, and $h(t)$.

2.1. Likelihood estimation

This subsection deals with obtaining the MLE and their ACI estimator from ML-based (ACI-ML) of α, λ, $R(t)$, and $h(t)$. Let x_i, $i = 1, 2, \dots, \omega$, $(1 \leqslant \omega \leqslant n)$ be a UHC sample obtained from APE(ξ). Substituting (1.2) and (1.3) into (1.1), taking $\alpha^\star = \log(\alpha)$, the likelihood function (1.1) becomes

$$L(\xi|\mathbf{x}) \propto \left(\frac{\alpha}{\alpha-1}\right)^n (\lambda\alpha^\star)^\omega \left(1-\alpha^{-e^{-\lambda\upsilon}}\right)^{n-\omega} \exp\left(-\lambda\sum_{i=1}^{\omega} x_i - \alpha^\star \sum_{i=1}^{\omega} e^{-\lambda x_i}\right) \tag{2.1}$$

and its log-likelihood function $l(\cdot) \propto \log L(\cdot)$ becomes

$$l(\mathbf{x}|\xi) \propto n\left(\frac{\alpha^\star}{(\alpha-1)}\right) + \omega\log(\lambda\alpha^\star) - \lambda\sum_{i=1}^{\omega} x_i - \alpha^\star\sum_{i=1}^{\omega} e^{-\lambda x_i} + (n-\omega)\log\left(1-\alpha^{-e^{-\lambda\upsilon}}\right). \tag{2.2}$$

From (2.2), the MLEs $\hat{\alpha}$ and $\hat{\lambda}$ can be obtained by equating the following first-partial derivatives of $l(\cdot)$ with respect to α and λ to zero as

$$\frac{1}{\alpha}\left[\frac{\omega}{\alpha^\star} - \frac{n}{(\alpha-1)} + \frac{(n-\omega)\,e^{-\lambda\upsilon}}{\left(\alpha^{e^{-\lambda\upsilon}}-1\right)} - \sum\nolimits_{i=1}^{\omega} e^{-\lambda x_i}\right]\Bigg|_{\xi=\hat{\xi}} = 0, \tag{2.3}$$

and

$$\frac{\omega}{\lambda} - \sum\nolimits_{i=1}^{\omega} x_i + \alpha^\star\sum\nolimits_{i=1}^{\omega} x_i e^{-\lambda x_i} - \frac{\alpha^\star\,(n-\omega)\,\upsilon e^{-\lambda\upsilon}}{\left(\alpha^{e^{-\lambda\upsilon}}-1\right)}\Bigg|_{\xi=\hat{\xi}} = 0, \tag{2.4}$$

respectively.

It is clear, from (2.3) and (2.4), that the MLEs of α and λ cannot be developed in closed forms. Thus, for given n, k, r, T_1, and T_2, we can use the Newton-Raphson (N-R) method with the 'maxLik' package proposed by Henningsen and Toomet [27] to calculate the derived estimators. Moreover, once the MLEs $\hat{\alpha}$ and $\hat{\lambda}$ are calculated, the MLEs $\hat{R}(t)$ and $\hat{h}(t)$ of $R(t)$ and $h(t)$, respectively, are acquired by replacing the APE parameters α and λ by their MLEs $\hat{\alpha}$ and $\hat{\lambda}$, from (1.4) and (1.5), as

$$\hat{R}(t;\hat{\xi}) = \frac{\hat{\alpha}}{\hat{\alpha}-1}(1-\hat{\alpha}^{-e^{-\hat{\lambda}t}}),\ t>0, \quad \text{and} \quad \hat{h}(t;\hat{\xi}) = \frac{\hat{\lambda}\log(\hat{\alpha})e^{-\hat{\lambda}t}\alpha^{e^{-\hat{\lambda}t}}}{1-\alpha^{-e^{-\hat{\lambda}t}}},\ t>0.$$

Next, the interval estimators of α and λ will be constructed based on their asymptotic distributions of $\hat{\alpha}$ and $\hat{\lambda}$. The asymptotic variance-covariance elements of the MLEs $\hat{\xi}$ for the parameters ξ are given by inverting the Fisher information matrix (say, $\mathbb{I}(\cdot)$), which is given by

$$\mathbb{I}(\xi) = -E\left[\frac{\partial^2 l}{\partial\xi_i\partial\xi_j}\right];\ \text{for } i,j = 1,2.$$

Due to the complex expression in (2.2), the exact mathematical solution of $\mathbb{I}_{ij}(\xi)$, $i,j = 1,2$, is tedious to obtain. Hence, by ignoring the expectation operator, the approximate variance-covariance matrix (say $\mathbb{I}_L^{-1}(\cdot)$) is created at $\xi = \hat{\xi}$ as

$$\mathbb{I}_L^{-1}(\hat{\xi}) = \begin{bmatrix} -l_{11} & -l_{12} \\ -l_{21} & -l_{22} \end{bmatrix}^{-1}_{\xi=\hat{\xi}} = \begin{bmatrix} \hat{\Upsilon}_{11} & \hat{\Upsilon}_{12} \\ \hat{\Upsilon}_{21} & \hat{\Upsilon}_{22} \end{bmatrix}, \tag{2.5}$$

where

$$l_{11}=\frac{n}{\alpha^2(\alpha-1)}-\frac{\omega\,(1+\alpha^\star)}{(\alpha\alpha^\star)^2}+\frac{1}{\alpha^2}\sum_{i=1}^{\omega}e^{-\lambda x_i}-\frac{(n-\omega)\,e^{-\lambda\upsilon}(\alpha^{e^{-\lambda\upsilon}}\left(e^{-\lambda\upsilon}+1\right)-1)}{\alpha^2\left(\alpha^{e^{-\lambda\upsilon}}-1\right)^2},$$

$$l_{12}=\frac{(n-\omega)\upsilon e^{-\lambda\upsilon}[\alpha^\star e^{-\lambda\upsilon}\alpha^{e^{-\lambda\upsilon}}-\alpha^{e^{-\lambda\upsilon}}+1]}{\alpha(\alpha^{e^{-\lambda\upsilon}}-1)^2}+\frac{1}{\alpha}\sum_{i=1}^{\omega}x_ie^{-\lambda x_i},$$

and

$$l_{22}=-\frac{\omega}{\lambda^2}-\alpha^\star\sum_{i=1}^{\omega}x_i^2e^{-\lambda x_i}-\frac{\alpha^\star\,(n-\omega)\,\upsilon^2e^{-\lambda\upsilon}[\alpha^\star e^{-\lambda\upsilon}\alpha^{e^{-\lambda\upsilon}}-\alpha^{e^{-\lambda\upsilon}}+1]}{\left(\alpha^{e^{-\lambda\upsilon}}-1\right)^2}.$$

Thus, $(1-\gamma)100\%$ ACI-ML estimators of α and λ are given, respectively, by

$$(\hat{\alpha}\pm z_{\gamma/2}\sqrt{\hat{\Upsilon}_{11}})\quad\text{and}\quad(\hat{\lambda}\pm z_{\gamma/2}\sqrt{\hat{\Upsilon}_{22}}),$$

where $z_{\gamma/2}$ is the right-tail $(\gamma/2\ )-th$ point of the standard normal distribution.

Moreover, to create the associated ACI-ML estimators of $R(t)$ and $h(t)$, we first need to approximate their variances. Thus, the delta method, which is a general approach for constructing the ACI of a function containing MLEs, is considered see Greene [26]. According to this method, the estimated variances of $\hat{R}(t)$ and $\hat{h}(t)$, are given, respectively, by

$$\hat{\Upsilon}_{\hat{R}}=\hat{\aleph}_1^\top\mathbb{I}_L^{-1}(\hat{\xi})\hat{\aleph}_1\quad\text{and}\quad\hat{\Upsilon}_{\hat{h}}=\hat{\aleph}_2^\top\mathbb{I}_L^{-1}(\hat{\xi})\hat{\aleph}_2,$$

where $\hat{\aleph}_1^\top=\begin{bmatrix}\frac{\partial R}{\partial\alpha} & \frac{\partial R}{\partial\lambda}\end{bmatrix}_{\xi=\hat{\xi}}$ and $\hat{\aleph}_2^\top=\begin{bmatrix}\frac{\partial h}{\partial\alpha} & \frac{\partial h}{\partial\lambda}\end{bmatrix}_{\xi=\hat{\xi}}$.

Thus, the $(1-\gamma)100\%$ ACI-ML estimators for $R(t)$ and $h(t)$ are given by

$$\left(\hat{R}(t)\pm z_{\gamma/2}\sqrt{\hat{\Upsilon}_{\hat{R}}}\right)\quad\text{and}\quad\left(\hat{h}(t)\pm z_{\gamma/2}\sqrt{\hat{\Upsilon}_{\hat{h}}}\right),$$

respectively.

2.2. *Product of spacings estimation*

The MPS method offers an alternative approach to estimation instead of the conventional ML method. It was first introduced separately by Cheng and Amin [12] and Ranneby [43]. Just like how the MLEs have certain properties, it was stated in Cheng and Iles [13] and Cheng and Traylor [14] that the MPSEs also have consistent properties. Also, Anatolyev and Kosenok [6] stated that the MPSE demonstrates efficient small sample behavior compared to the MLE. The MPSE also possesses an invariance property similar to the MLE; see Coolen and Newby [16].

Using the MPS, several authors have developed the problem of estimating parameter(s) in the presence of different censored samples; for example, Ng et al. [38] for progressive Type-II censoring; Basu et al. [9] for Type-I progressive hybrid censoring with binomial removals; El-Sherpieny et al. [22] for Type-II progressive hybrid censoring; Almetwally et al. [2] for adaptive type-II progressive censoring; Nassar et al. [37] for Type-II censored accelerated life testing; Alshenawy et al. [5] for progressive hybrid censored stress–strength data; Alrumayh et al. [4] for adaptive Type-II progressive censoring; and Nassar et al. [36] for adaptive Type-I progressively censoring, among others. To the best of our knowledge, we have not recorded any work on the MPS in the presence of UHC data. Thus, we can formulate the product of the spacing function for UHC data as

$$S(\xi) = (1 - F(\upsilon;\xi))^{n-\omega} \prod_{i=1}^{\omega+1} (F(x_i;\xi) - F(x_{i-1};\xi)), \tag{2.6}$$

where $F(x_0;\xi) \equiv 0$ and $F(x_{\omega+1};\xi) \equiv 1$. From (1.3) and (2.6), the MPS function (say $S(\cdot)$) of the APE distribution based on UHC can be written as

$$S(\xi) = \left(\frac{\alpha}{\alpha-1}\right)^{n+1} \left(1 - \alpha^{-e^{-\lambda\upsilon}}\right)^{n-\omega} \prod_{i=1}^{\omega+1} \left(\alpha^{-e^{-\lambda x_i}} - \alpha^{-e^{-\lambda x_{i-1}}}\right). \tag{2.7}$$

The corresponding log-MPS function (say $s(\cdot) \propto \log S(\cdot)$) of (2.7) is

$$s(\xi) = (n+1)\log\left(\frac{\alpha}{\alpha-1}\right) + \sum_{i=1}^{\omega+1} \log\left(\psi_i(\xi) - \psi_{i-1}(\xi)\right) + (n-\omega)\log\left(1 - \psi_\upsilon(\xi)\right), \tag{2.8}$$

where $\psi_i(\xi) = \alpha^{-e^{-\lambda x_i}}, \psi_{i-1}(\xi) = \alpha^{-e^{-\lambda x_{i-1}}}$ and $\psi_\upsilon(\xi) = \alpha^{-e^{-\lambda\upsilon}}$.

Differentiating (2.8) with respect to α and λ, we get

$$\frac{\partial s(\xi)}{\partial\alpha} = -\frac{n+1}{\alpha(\alpha-1)} - \sum\nolimits_{i=1}^{\omega+1} \frac{\psi_i'(\alpha) - \psi_{i-1}'(\alpha)}{(\psi_i(\xi) - \psi_{i-1}(\xi))} + \frac{(n-\omega)\psi_\upsilon'(\alpha)}{(1-\psi_\upsilon(\xi))} \tag{2.9}$$

and

$$\frac{\partial s(\xi)}{\partial\lambda} = \sum\nolimits_{i=1}^{\omega+1} \frac{\psi_i'(\lambda) - \psi_{i-1}'(\lambda)}{(\psi_i(\xi) - \psi_{i-1}(\xi))} - \frac{(n-\omega)\psi_\upsilon'(\lambda)}{(1-\psi_\upsilon(\xi))}, \tag{2.10}$$

where
$\psi_i'(\alpha) = -e^{-\lambda x_i}\alpha^{-(e^{-\lambda x_i}+1)}, \psi_i'(\lambda) = x_i e^{-\lambda x_i}\log(\alpha)\alpha^{-e^{-\lambda x_i}}$, $\psi_\upsilon'(\alpha) = -e^{-\lambda\upsilon}\alpha^{-(e^{-\lambda\upsilon}+1)}$ and $\psi_\upsilon'(\lambda) = \upsilon e^{-\lambda\upsilon}\log(\alpha)\alpha^{-e^{-\lambda\upsilon}}$.

Similar to the case of the MLEs of α and λ, the closed-form solutions for the MPSEs of α and λ (denoted by $\widehat{\alpha}$ and $\widehat{\lambda}$) as in (14) and (15) are not available. Therefore, the N-R method can be used to calculate the MPSEs. Moreover, once MPSEs $\widehat{\alpha}$ and $\widehat{\lambda}$ are calculated, the MPSEs $\widehat{R}(t)$ and $\widehat{h}(t)$ of $R(t)$ and $h(t)$ are given respectively by

$$\widehat{R}(t;\widehat{\xi}) = \frac{\widehat{\alpha}}{\widehat{\alpha}-1}(1 - \widehat{\alpha}^{-e^{-\widehat{\lambda}t}}),\ t > 0, \quad \text{and} \quad \widehat{h}(t;\widehat{\xi}) = \frac{\widehat{\lambda}\log(\widehat{\alpha})e^{-\widehat{\lambda}t}\alpha^{e^{-\widehat{\lambda}t}}}{1 - \alpha^{-e^{-\widehat{\lambda}t}}},\ t > 0.$$

Using the offered MPSEs $\widehat{\alpha}$ and $\widehat{\lambda}$, the approximate variance-covariance matrix (say $\mathbb{I}_S^{-1}(\cdot)$) is obtained by replacing α and λ by their $\widehat{\alpha}$ and $\widehat{\lambda}$, respectively, as

$$\mathbb{I}_S^{-1}(\widehat{\xi}) = \begin{bmatrix} -s_{11} & -s_{12} \\ -s_{21} & -s_{22} \end{bmatrix}^{-1}_{\xi=\widehat{\xi}} = \begin{bmatrix} \widehat{\Upsilon}_{11} & \widehat{\Upsilon}_{12} \\ \widehat{\Upsilon}_{21} & \widehat{\Upsilon}_{22} \end{bmatrix}. \tag{2.11}$$

From (2.8), the second derivatives $s_{ij},\ i,j=1,2$ are obtained as

$$s_{11}=\frac{(n+1)(2\alpha-1)}{\alpha^2(\alpha-1)^2}-\frac{(n-\omega)\left((1-\upsilon(\xi))\psi''_{\upsilon}(\alpha)-(\psi'_{\upsilon}(\alpha))^2\right)}{(1-\upsilon(\xi))^2}$$
$$+\sum_{i=1}^{\omega+1}\frac{(\psi_i(\xi)-\psi_{i-1}(\xi))\left(\psi''_i(\alpha)-\psi''_{i-1}(\alpha)\right)-\left(\psi'_i(\alpha)-\psi'_{i-1}(\alpha)\right)^2}{(\psi_i(\xi)-\psi_{i-1}(\xi))^2},$$

$$s_{22}=\sum_{i=1}^{\omega+1}\frac{(\psi_i(\xi)-\psi_{i-1}(\xi))\left(\psi''_i(\lambda)-\psi''_{i-1}(\lambda)\right)-\left(\psi'_i(\lambda)-\psi'_{i-1}(\lambda)\right)^2}{(\psi_i(\xi)-\psi_{i-1}(\xi))^2}$$
$$-\frac{(n-\omega)\left((1-\upsilon(\xi))\psi''_{\upsilon}(\lambda)-(\psi'_{\upsilon}(\lambda))^2\right)}{(1-\upsilon(\xi))^2},$$

and

$$s_{12}=-\sum_{i=1}^{\omega+1}\frac{\left(\psi'_{i-1}(\alpha)-\psi'_i(\alpha)\right)\left(\psi'_{i-1}(\lambda)-\psi'_i(\lambda)\right)}{(\psi_i(\xi)-\psi_{i-1}(\xi))^2}+\frac{(n-\omega)\psi'_{\upsilon}(\alpha)\psi'_{\upsilon}(\lambda)}{(1-\psi_{\upsilon}(\xi))^2}.$$

where $\psi''_i(\alpha)=e^{-\lambda x_i}\left(e^{-\lambda x_i}+1\right)\alpha^{-(e^{-\lambda x_i}+2)}, \psi''_i(\lambda)=x_i^2\log(\alpha)e^{-\lambda x_i}\alpha^{-e^{-\lambda x_i}}\left(e^{-\lambda x_i}\log(\alpha)-1\right)$, $\psi''_{\upsilon}(\alpha)=e^{-\lambda\upsilon}\left(e^{-\lambda\upsilon}+1\right)\alpha^{-(e^{-\lambda\upsilon}+2)}$ and $\psi''_{\upsilon}(\lambda)=\upsilon^2\log(\alpha)e^{-\lambda\upsilon}\alpha^{-e^{-\lambda\upsilon}}\left(e^{-\lambda\upsilon}\log(\alpha)-1\right)$.

Thus, $(1-\gamma)100\%$ ACI-MPS estimators of α and λ are given, respectively, by

$$(\widehat{\alpha}\pm z_{\gamma/2}\sqrt{\widehat{\Upsilon}_{11}})\quad\text{and}\quad(\widehat{\lambda}\pm z_{\gamma/2}\sqrt{\widehat{\Upsilon}_{22}}),$$

where $\widehat{\Upsilon}_{\widehat{\alpha}}$ and $\widehat{\Upsilon}_{\widehat{\lambda}}$ obtained from (2.11).

Moreover, the delta method is reused to construct the ACI-MPS estimators of $R(t)$ and $h(t)$. First, we obtain the estimated variance of $\widehat{R}(t)$ and $\widehat{h}(t)$, respectively, as

$$\widehat{\Upsilon}_{\widehat{R}}=\widehat{\aleph}_1^{\top}\mathbb{I}_S^{-1}(\widehat{\xi})\widehat{\aleph}_1\quad\text{and}\quad\widehat{\Upsilon}_{\widehat{h}}=\widehat{\aleph}_2^{\top}\mathbb{I}_S^{-1}(\widehat{\xi})\widehat{\aleph}_2,$$

where $\widehat{\aleph}_1^{\top}=\left[\frac{\partial R}{\partial\alpha}\quad\frac{\partial R}{\partial\lambda}\right]_{\xi=\widehat{\xi}}$ and $\widehat{\aleph}_2^{\top}=\left[\frac{\partial h}{\partial\alpha}\quad\frac{\partial h}{\partial\lambda}\right]_{\xi=\widehat{\xi}}$.

Thus, the $(1-\gamma)100\%$ ACI-MPS estimators for $R(t)$ and $h(t)$ are given by

$$\left(\widehat{R}(t)\pm z_{\gamma/2}\sqrt{\widehat{\Upsilon}_{\widehat{R}}}\right)\quad\text{and}\quad\left(\widehat{h}(t)\pm z_{\gamma/2}\sqrt{\widehat{\Upsilon}_{\widehat{h}}}\right),$$

respectively.

3. Bayes Inference

Bayesian estimates of any unknown parametric function of α and λ under the SEL are developed in this part. We assumed that α and λ having independent gamma prior PDFs as $\alpha\sim Gamma(a_1,b_1)$ and $\lambda\sim Gamma(a_2,b_2)$. Thus, the joint prior PDF $\pi(\cdot)$, of α and λ, is

$$\pi(\xi)\propto\alpha^{a_1-1}\lambda^{a_2-1}e^{-(b_1\alpha+b_2\lambda)},\tag{3.1}$$

where $a_i, b_i > 0,\ i = 1, 2$ are assumed to be known. The gamma distribution is utilized as prior information on APE parameters due to its flexibility, especially in computational parts. It provides different knowledge about the unknown parameter based on its parameter values. In many cases, it gives a posterior distribution with less complex forms, and this property is desirable, especially when the number of parameters is large, as in our case; for additional detail see Dey et al. [17].

3.1. *Bayes estimators via likelihood-based*

Combining (2.1) with (3.1), the joint posterior PDF (say $\pi_L^*(\cdot)$) of α and λ becomes

$$\begin{aligned} \pi_L^*(\xi|\mathbf{x}) = &\frac{\alpha^{n+a_1-1}\lambda^{\omega+a_2-1}}{\eta_L(\alpha-1)^n}(\alpha^\star)^\omega(1-\alpha^{-e^{-\lambda\upsilon}})^{n-\omega} \\ &\times \exp\left(-\lambda\sum\nolimits_{i=1}^{\omega} x_i - \log(\alpha)\sum\nolimits_{i=1}^{\omega} e^{-\lambda x_i} - b_1\alpha - b_2\lambda\right), \end{aligned} \tag{3.2}$$

where $\eta_L = \int_0^\infty \int_0^\infty \pi(\xi) L(\xi|\mathbf{x}) d\alpha d\lambda$.

From (3.2), the Bayes estimator $\tilde{g}_L(\cdot)$ of any unknown function of α and λ, say $g(\alpha, \lambda)$, against the SEL is acquired by posterior expectation of $g(\alpha, \lambda)$ as follows

$$\tilde{g}_L(\xi) = E(g(\xi|\mathbf{x})) = \frac{1}{\eta_L}\int_0^\infty \int_0^\infty g(\xi)\pi_L(\xi|\mathbf{x})\, d\alpha d\lambda. \tag{3.3}$$

The Bayes estimators, based on (3.3), can be obtained by calculating the division of two integrals. Nevertheless, no mathematical equation has been discovered that can provide exact solutions for these integrals. The Metropolis-Hastings (M-H) algorithm (proposed by Metropolis et al. [34] and later discussed by Gelman et al. [23]) is a general technique of the MCMC approaches. It is used here to approximate the Bayes estimators by simulating random samples from a complex posterior distribution. From (3.2), the respective conditional PDFs for α and λ are

$$\pi_L^\alpha(\alpha|\lambda, \mathbf{x}) \propto \frac{\alpha^{n+a_1-1}}{(\alpha-1)^n}(1-\alpha^{-e^{-\lambda\upsilon}})^{n-\omega} \exp\left(-b_1\alpha + \omega\log(\alpha^\star) - \log(\alpha\sum_{i=1}^{\omega} e^{-\lambda x_i})\right), \tag{3.4}$$

and

$$\pi_L^\lambda(\lambda|\alpha, \mathbf{x}) \propto \lambda^{\omega+a_2-1}(1-\alpha^{-e^{-\lambda\upsilon}})^{n-\omega} \exp\left(-\lambda(b_2 + \sum_{i=1}^{\omega} x_i) - \alpha^\star \sum_{i=1}^{\omega} e^{-\lambda x_i}\right). \tag{3.5}$$

It is obvious, from (3.4) and (3.5), that the conditional posterior densities of α and λ cannot be simplified into familiar distributions through analytical methods. As a result, it is not feasible to directly sample using standard techniques. By generating one UHC random sample from APE(2, 1) (when $n = 100$, $(k, r) = (60, 80)$ and $(T_1, T_2) = (0.4, 0.8)$), Figure 1 shows that the fully conditional distributions $\pi_L^\alpha(\cdot)$ and $\pi_L^\lambda(\cdot)$ of α and λ, respectively, using this collected sample behave similarly to the normal density appropriately.

Therefore, using a normal proposal distribution, the M-H procedure steps are used in turn to simulate MCMC samples from (3.4) and (3.5) as follows:

Step-1 Start with initial guesses $(\alpha^{(0)}, \lambda^{(0)}) = (\hat{\alpha}, \hat{\lambda})$.

Step-2 Set $\varrho = 1$.

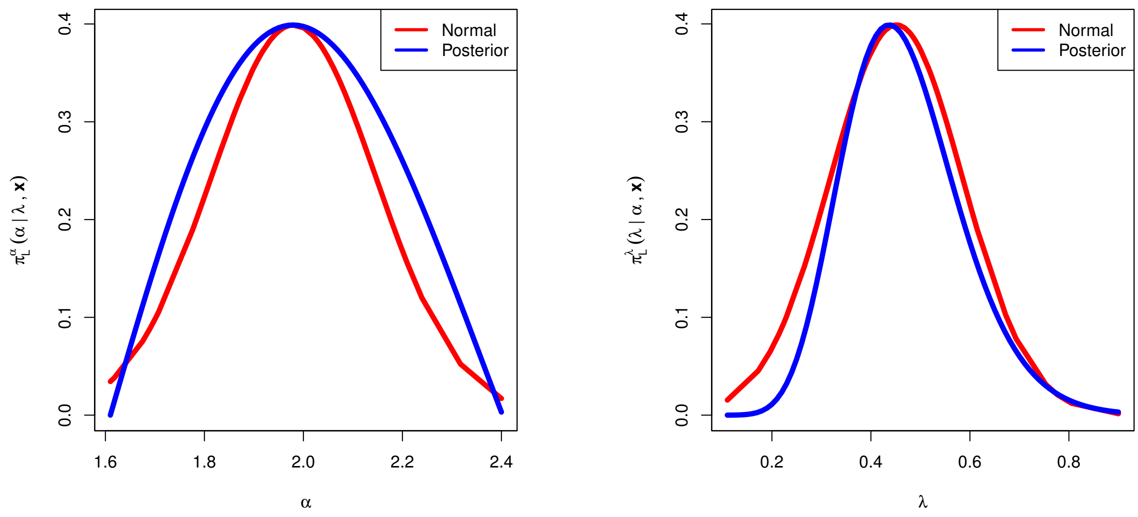


Figure 1. The normal and posterior density (likelihood-based) curves of α (left) and λ (right).

Step-3 Use M-H sampler to generate $\alpha^{[\varrho]}$ and $\lambda^{[\varrho]}$ from (3.4) and (3.5) as:

(a) Generate α^* and λ^* from normal $N\left(\alpha^{[\varrho-1]}, \hat{\Upsilon}_{11}\right)$ and $N\left(\lambda^{[\varrho-1]}, \hat{\Upsilon}_{22}\right)$.

(b) Obtain $\mathcal{Q}_1 = \frac{\pi_L^{\alpha}(\alpha^*|\lambda^{[\varrho-1]},\mathbf{x})}{\pi_L^{\alpha}(\alpha^{[\varrho-1]}|\lambda^{[\varrho-1]},\mathbf{x})}$, $\mathcal{Q}_2 = \frac{\pi_L^{\lambda}(\lambda^*|\alpha^{[\varrho]},\mathbf{x})}{\pi_L^{\lambda}(\lambda^{[\varrho-1]}|\alpha^{[\varrho]},\mathbf{x})}$, $Ш_1 = \min\{1, \mathcal{Q}_1\}$, and $Ш_2 = \min\{1, \mathcal{Q}_2\}$.

(c) Generate $\sqcap_1$ and $\sqcap_2$ from Uniform (0,1) distribution.

(d) If $\sqcap_1 \leq Ш_1$, set $\alpha^{[\varrho]} = \alpha^*$, else set $\alpha^{[\varrho]} = \alpha^{[\varrho-1]}$.

(e) If $\sqcap_2 \leq Ш_2$, set $\lambda^{[\varrho]} = \lambda^*$, else set $\lambda^{[\varrho]} = \lambda^{[\varrho-1]}$.

Step-4 Obtain $R(t)$ and $h(t)$ at $t > 0$, respectively, as

$$R^{[\varrho]}(t) = \left(1 - \exp\left(-\lambda^{[\varrho]} t^{-1}\right)\right)^{\alpha^{[\varrho]}} \text{ and } h^{[\varrho]}(t) = \frac{\alpha^{[\varrho]}\lambda^{[\varrho]}}{t^2\left(\exp\left(\lambda^{[\varrho]} t^{-1}\right) - 1\right)}.$$

Step-5 Set $\varrho = \varrho + 1$.

Step-6 Repeat Steps 2-5 $\mathfrak{D}$ times to get $\varphi^{[\varrho]} = (\alpha^{[\varrho]}, \lambda^{[\varrho]}, R^{[\varrho]}(t), h^{[\varrho]}(t))$ for $\varrho = 1, ..., \mathfrak{D}$.

To eliminate the effect of the initial values and to guarantee the convergence of the MCMC sampler, the first $\mathfrak{D}_0$ simulated varieties are discarded as a burn-in period. Then, the selected samples $\varphi^{[\varrho]} = (\alpha^{[\varrho]}, \lambda^{[\varrho]}, R^{[\varrho]}(t), h^{[\varrho]}(t))$ will be used to develop the target Bayesian inferences. However, the Bayes estimate of any function of α and λ, say φ, based on the SEL is given by

$$\hat{\varphi}_{BS} = \frac{1}{\mathfrak{D}^\star} \sum_{i=\mathfrak{D}_0+1}^{\mathfrak{D}} \varphi^{[\varrho]}.$$

where $\mathfrak{D}^\star = \mathfrak{D} - \mathfrak{D}_0$.

To acquire the HPD credible interval of φ, firstly order $\varphi^{[\varrho]}$, $\varrho = \mathfrak{D}_0 + 1, ..., \mathfrak{D}$ as $(\varphi_{(\mathfrak{D}_0+1)}, \ldots, \varphi_{(\mathfrak{D})})$, hence the $100\,(1-\gamma)\,\%$ HPD credible interval of φ is given by

$$\left(\varphi_{(\varrho^*)}, \varphi_{(\varrho^*+(1-\gamma)(\mathfrak{D}^\star))}\right),$$

where ϱ^* is chosen such that

$$\varphi_{(\varrho^*+[(1-\gamma)(\mathfrak{D}^\star)])} - \varphi_{(J^*)} = \min_{1\leq\varrho\leq\gamma(\mathfrak{D}^\star)} (\varphi_{(\varrho+[(1-\gamma)(\mathfrak{D}^\star)])} - \varphi_{[\varrho]}),\ \varrho^* = \mathfrak{D}_0 + 1, ..., \mathfrak{D},$$

where $[y]$ denotes the largest integer lower than (or equal) to y, see Chen and Shao [11].

3.2. Bayes estimators via spacing-based

The concept of estimation using Bayesian MPS-based inference was first studied by Coolen and Newby [16]. They figured out a Bayesian estimate of the density of MPS that is similar to the usual estimate obtained by the likelihood function. Combining (2.7) with (3.1), the joint posterior PDF of α and λ based on MPS (say $\pi_S^*(\cdot)$) is

$$\pi_S\,(\xi) = \frac{\alpha^{n+a_1-1}e^{-b_1\alpha}}{\eta_S\,(\alpha-1)^n}\left(1-\alpha^{-e^{-\lambda\upsilon}}\right)^{n-\omega}\prod_{i=1}^{\omega+1}\left(\alpha^{-e^{-\lambda x_i}} - \alpha^{-e^{-\lambda x_{i-1}}}\right), \tag{3.6}$$

where $\eta_S = \int\limits_0^\infty\int\limits_0^\infty \pi\,(\xi)\,S\,(\xi)\,d\alpha d\lambda.$

From (3.6), the respective conditional PDFs of α and λ may be stated as

$$\pi_S^\alpha(\alpha\,|\lambda) \propto \frac{\alpha^{n+a_1-1}e^{-b_1\alpha}}{(\alpha-1)^n}\left(1-\alpha^{-e^{-\lambda\upsilon}}\right)^{n-\omega}\prod_{i=1}^{\omega+1}\left(\alpha^{-e^{-\lambda x_i}} - \alpha^{-e^{-\lambda x_{i-1}}}\right), \tag{3.7}$$

and

$$\pi_S^\lambda(\lambda\,|\alpha) \propto \lambda^{a_2-1}e^{-b_2\lambda}\left(1-\alpha^{-e^{-\lambda\upsilon}}\right)\prod_{i=1}^{\omega+1}\left(\alpha^{-e^{-\lambda x_i}} - \alpha^{-e^{-\lambda x_{i-1}}}\right). \tag{3.8}$$

Hence, from (3.6), the Bayes estimator $\tilde{g}_S(\cdot)$ of any function of α and λ, say $g\,(\xi)$, using the SEL function is the posterior mean of $g\,(\xi)$ as

$$\tilde{g}_S(\xi) = E(g(\xi)) = \frac{1}{\eta_S}\int_0^\infty\int_0^\infty g(\xi)\pi_S(\xi)d\alpha d\lambda. \tag{3.9}$$

From (3.9), it can be noticed that estimator $\tilde{g}_S(\alpha,\lambda)$ is represented as a fraction of two integrals, but there is no known exact solution for these integrals. Similar to the posterior density from likelihood-based, we propose to use the MCMC approximation in the case of the posterior density from spacing-based. Using the same simulated UHC dataset discussed in Subsection 3.1, Figure 2 supports the same facts depicted in Figure 1 and confirms that the posterior PDFs $\pi_S^\alpha(\cdot)$ and $\pi_S^\lambda(\cdot)$ of α and λ, respectively, behave as normal density quite well. Thus, after using the M-H technique mentioned in Subsection 3.1, we simulate MCMC samples of α and λ from (3.7) and (3.8) in turn to calculate the Bayesian (or HPD credible interval) estimate of α, λ, $R(t)$, or $h(t)$.

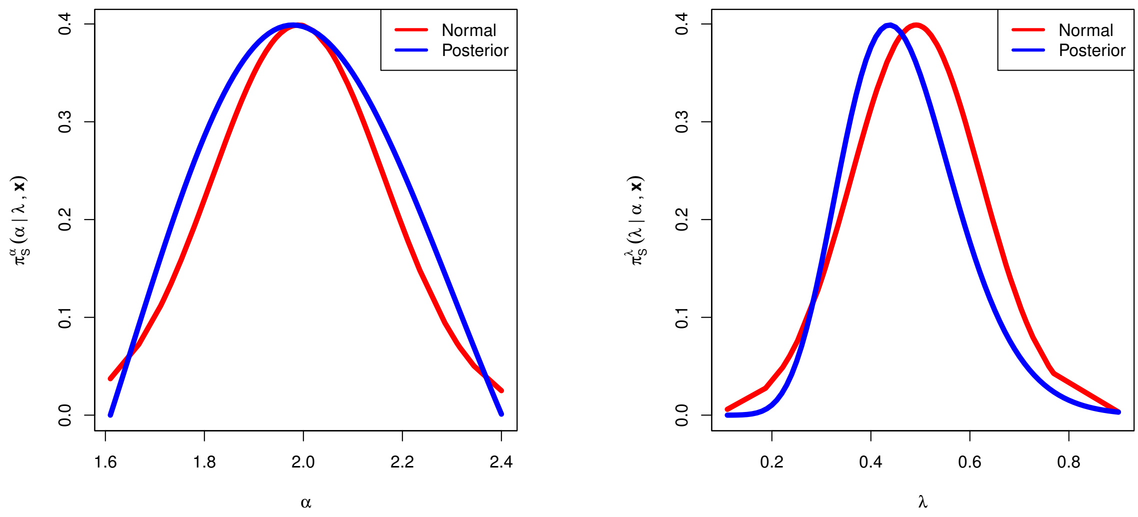


Figure 2. The normal and posterior density (spacings-based) curves of α (left) and λ (right).

4. Simulation Study

To examine the performance of the different proposed methodologies, including the maximum likelihood, product of spacings, and their extension to Bayesian inference, extensive Monte Carlo simulation studies are performed. When the true parameter values of α and λ are taken as 1.2 and 0.8, respectively, we simulate large 2,000 UHC samples using different options of n, k, r, and T_i, $i = 1, 2$. Also, for a given time $t = 0.1$, the true values of $R(t)$ and $h(t)$ are used as 0.9294 and 0.7345, respectively. Once 2,000 UHC samples are collected, the proposed classical point/interval estimators are calculated. To calculate the Bayes estimates of α, λ, $R(t)$ or $h(t)$, two different informative sets of the hyper-parameters of gamma prior distributions are used, namely Prior-1: $(a_1, a_2) = (6, 4)$ and $b_i = 5, i = 1, 2$; and Prior-2: $(a_1, a_2) = (12, 8)$ and $b_i = 10, i = 1, 2$. Following Kundu [30], the values of the prior parameters a_i and b_i for $i = 1, 2$, are chosen so that their average values match the true parameter values. If no prior information about α and λ is available, it is useful to use the classical approach rather than the other. Taking $\mathfrak{D} = 12,000$ and $\mathfrak{D}_0 = 2,000$, using the SEL function, the Bayes estimates are developed.

The average estimates (AEs) of α, λ, $R(t)$, or $h(t)$ (say φ) as well as their simulated root mean squared-errors (RMSE) and mean relative absolute biases (MRAB) are given, respectively, by

$$\bar{\hat{\varphi}}_\tau = \frac{1}{2,000} \sum_{j=1}^{2,000} \hat{\varphi}_\tau^{(j)}, \ \tau = 1, 2, 3, 4,$$

$$\text{RMSE}\,(\hat{\varphi}_\tau) = \sqrt{\frac{1}{2,000} \sum_{j=1}^{2,000} \left(\hat{\varphi}_\tau^{(j)} - \varphi_\tau\right)^2}, \ \tau = 1, 2, 3, 4,$$

and

$$\text{MRAB}\,(\hat{\varphi}_\tau) = \frac{1}{2,000} \sum_{j=1}^{2,000} \frac{1}{\varphi_\tau} \left|\hat{\varphi}_\tau^{(j)} - \varphi_\tau\right|, \ \tau = 1, 2, 3, 4,$$

where $\varphi_\tau^{(j)}$ is the classical (or Bayes) estimate of φ_τ at jth sample, $\varphi_1 = \alpha$, $\varphi_2 = \lambda$, $\varphi_3 = R(t)$, and $\varphi_4 = h(t)$. Moreover, comparison between the offered $100(1-\gamma)\%$ interval estimates of φ is made with respect to their average confidence lengths (ACLs) and coverage percentages (CPs) as follows:

$$\text{ACL}_{\varphi_\tau}^{95\%} = \frac{1}{2,000} \sum\nolimits_{j=1}^{2,000} \left(\mathcal{U}_{\hat{\varphi}_\tau^{(j)}} - \mathcal{L}_{\hat{\varphi}_\tau^{(j)}} \right),\ \tau = 1, 2, 3, 4,$$

and

$$\text{CP}_{\varphi_\tau}^{95\%} = \frac{1}{2,000} \sum\nolimits_{j=1}^{2,000} \mathbf{1}_{\left(\mathcal{L}_{\hat{\varphi}_\tau^{(j)}}, \mathcal{U}_{\hat{\varphi}_\tau^{(j)}}\right)} (\varphi_\tau),\ \tau = 1, 2, 3, 4,$$

respectively, where $\mathbf{1}(\cdot)$ is the indicator function and $(\mathcal{L}(\cdot), \mathcal{U}(\cdot))$ represents the (lower, upper) bounds of 95% asymptotic (or credible) interval of φ_τ. All numerical evaluations are performed via R software through the 'coda' package proposed by Plummer et al. [41] and the 'maxLik' package proposed by Henningsen and Toomet [27].

From the simulated results of α, λ, $R(t)$, and $h(t)$, obtained from the MLEs, MPSE, Bayes MCMC estimates via ML (MCMC-ML), Bayes MCMC estimates via MPS (MCMC-MPS), ACI-ML, ACI-MPS, HPD credible intervals via ML (HPD-ML), and HPD credible intervals via MPS (HPD-MPS), reported in Tables 1-8, some observations can be drawn and stated as:

- In general, all suggested estimates of the APE parameters α and λ or the reliability characteristics $R(t)$ and $h(t)$ perform satisfactorily in terms of the lowest RMSE, MRAB, and ACL values and the highest CP values.
- The Bayes MCMC estimates of all unknown parameters perform better than the classical estimates, as expected. This result is because the Bayes procedure combines the prior information of the model parameters with the censored data, while the classical estimates are obtained based on the censored data only.
- All Bayes point estimates and HPD credible intervals obtained from Prior-2 perform superior compared to those obtained from Prior-1 in terms of the smallest RMSEs, MRABs, and ACLs as well as highest CPs. It is also an expected observation due to the variance of Prior-2 is smaller than the variance of Prior-1.
- It can be seen that, in most cases, the CPs of all proposed interval estimates are close to the specified nominal level.
- Comparing the behavior of the proposed point estimation methods, in terms of minimum simulated RMSE and MRAB values, it is observed that the MPS and MCMC-MPS estimation methods of all unknown parameters perform better than the ML and MCMC-ML estimation methods in most cases.
- Comparing the behavior of the proposed interval estimation methods, in terms of the smallest ACL and highest CP values, it is noted that the ACIs (using MPSEs) of α and $R(t)$ perform better than others; the ACIs (using MLEs) of λ and $h(t)$ perform better than others; and the HPD credible intervals (using MPS) of all unknown parameters perform better than those obtained based on ML.
- As n increases, for all proposed tests, the RMSE, MRAB, and ACL values of all unknown parameters decrease while their CP values increase. A similar result is also observed when k and r are increased together.

- As T_i, $i = 1, 2$ (or at least one of them) increases, in most cases, it is seen that the RMSE and MRAB values of α, λ, and $R(t)$ decrease while the associated values of $h(t)$ increase as well as the ACL values of all unknown parameters narrow down while their CP values increase.

- To summarize, the Bayes MCMC paradigm using the product of spacing function is recommended to estimate the unknown parameter(s) of life in the presence of samples obtained from the unified hybrid censoring.

5. Real-Life Applications

To illustrate the importance and usefulness of the proposed estimation methods, two different real datasets from the engineering field are analyzed.

5.1. *Yarn data*

In this subsection, from Lawless [31], we shall use a dataset consisting of cycle-to-failure numbers for 25 (100 cm) specimens of yarn tested at a particular strain level. The failure times (in order) are 15, 20, 38, 42, 61, 76, 86, 98, 121, 146, 149, 157, 175, 176, 180, 180, 198, 220, 224, 251, 264, 282, 321, 325, and 653.

First, before calculating the theoretical results, we need to check whether the APE distribution is a suitable model to fit the yarn data or not. Thus, the MLEs $\hat{\alpha}$ and $\hat{\lambda}$ are computed in order to obtain the Kolmogorov-Smirnov (K-S) distance and its p-value. However, using the complete yarn dataset, the MLEs (with their standard errors (SEs)) of α and λ are 16.043 (19.804) and 0.0096 (0.0020), respectively, and the K-S distance is 0.115 with p-value 0.896. Since the p-value is quite high from the significance level of 5%, it means that the yarn dataset comes from the APE distribution.

From the yarn dataset, based on different combinations of T_i, $i = 1, 2$, k, and r, six different UHC samples are generated; see Table 9. From Table 9, the proposed Bayesian and non-Bayesian estimators with their SEs of α, λ, $R(t)$, and $h(t)$ (at time $t = 50$) are computed; see Table 10. Also, 95% ACI/HPD credible interval estimates with their lengths obtained by ML and MPS approaches are also listed in Table 11. Taking $\mathfrak{D} = 30,000$ and $\mathfrak{D}_0 = 5,000$, the Bayes point/interval estimates using likelihood and product of spacing functions are calculated based on the non-informative priors, i.e., $a_i = b_i = 0$, $i = 1, 2$. To run the MCMC sampler, the classical estimates of α and λ are used as initial values. Some important statistics of the MCMC samples, such as the mean, median, mode, standard deviation (SD), and skewness for MCMC outputs of α, λ, $R(t)$, and $h(t)$, are also computed; see Table 12.

Figure 3, using the generated sample of Case-I listed in Table 9, represents the trace plots for 25,000 MCMC variates of α, λ, $R(t)$, and $h(t)$. It exhibits that the MCMC procedure converges very well. Further, the histogram of the same unknown parameters is plotted in Figure 4. It indicates that the generated posteriors of all unknown parameters are fairly symmetrical. For specification, in each trace plot, the sample mean and two bounds of 95% HPD credible intervals are denoted as solid (—) and dashed (- - -) lines, respectively, while in each histogram plot, the sample mean is displayed with a dash-dot (-·-·-) line.

Table 1. The AEs (first-line), RMSEs (second-line), and MRABs (third-line) of α.

n	T_1	T_2	k	r	MLE	MPSE	MCMC-ML		MCMC-MPS	
Prior →							1	2	1	2
40	0.4	0.8	15	25	1.3468	1.0085	1.3043	1.2527	1.1783	1.1806
					1.2745	0.1916	0.1879	0.1389	0.0820	0.0503
					0.9968	0.1596	0.1246	0.0866	0.0693	0.0435
	0.8	1.2	25	35	1.0178	1.0099	1.1166	1.2457	1.1746	1.1903
					1.1959	0.1902	0.1502	0.1256	0.0731	0.0223
					0.9966	0.1584	0.0885	0.0780	0.0212	0.0081
	0.4	0.8	22	32	1.3126	1.0093	1.1112	1.2570	1.1927	1.2020
					1.1948	0.1908	0.1567	0.1208	0.0666	0.0338
					0.9956	0.1589	0.0917	0.0814	0.0189	0.0114
	0.8	1.5	15	25	0.9397	1.0086	1.3469	1.0449	1.1462	1.1920
					1.1962	0.1915	0.1674	0.1391	0.0710	0.0496
					0.9969	0.1595	0.1493	0.1217	0.0449	0.0185
	0.6	0.8	15	25	1.3468	1.0107	1.3032	1.2592	1.1908	1.1652
					1.2745	0.1895	0.1729	0.0934	0.0642	0.0514
					0.9968	0.1578	0.1179	0.0613	0.0374	0.0162
	0.4	1.3	15	25	0.9852	1.0092	1.4066	1.2772	1.1848	1.1684
					1.1961	0.1909	0.2769	0.1155	0.1198	0.0382
					0.9967	0.1590	0.2072	0.0848	0.0317	0.0265
	0.4	0.8	20	25	1.5507	1.0085	1.3587	1.3240	1.1937	1.1815
					1.2744	0.1912	0.1566	0.1358	0.0666	0.0425
					1.0025	0.1596	0.0921	0.0864	0.0196	0.0155
	0.4	0.8	15	35	1.3434	1.0107	1.3475	1.1565	1.2196	1.1822
					1.2746	0.1895	0.1713	0.1166	0.0664	0.0466
					0.9969	0.1578	0.1277	0.0692	0.0288	0.0149
80	0.4	0.8	30	60	1.4093	1.0090	1.1592	1.1886	1.1914	1.1556
					1.1944	0.1912	0.1283	0.0996	0.0640	0.0487
					0.9953	0.1592	0.0831	0.0547	0.0191	0.0165
	0.8	1.2	40	70	0.9074	1.0107	1.2109	1.1998	1.1972	1.1873
					1.1931	0.1895	0.1023	0.0539	0.0309	0.0201
					0.9937	0.1578	0.0731	0.0343	0.0130	0.0106
	0.4	0.8	45	65	1.1533	1.0107	1.1900	1.2536	1.1564	1.1913
					1.1927	0.1894	0.1233	0.0874	0.0566	0.0293
					0.9923	0.1577	0.0824	0.0503	0.0363	0.0114
	0.8	1.5	30	60	1.2596	1.0082	1.4717	1.1936	1.1778	1.1878
					1.1924	0.1912	0.1591	0.0697	0.0654	0.0241
					0.9932	0.1590	0.1224	0.0340	0.0240	0.0101
	0.6	0.8	30	60	1.4093	1.0133	1.3175	1.2381	1.1713	1.1805
					1.1944	0.1869	0.1431	0.0778	0.0477	0.0284
					0.9953	0.1556	0.1123	0.0527	0.0321	0.0163
	0.4	1.3	30	60	1.1831	1.0092	1.5403	1.1616	1.1668	1.1867
					1.1920	0.1900	0.2113	0.0864	0.0404	0.0233
					0.9930	0.1580	0.1997	0.0532	0.0278	0.0113
	0.4	0.8	42	60	1.3991	1.0133	1.2607	1.2043	1.1745	1.1860
					1.1940	0.1869	0.0943	0.0813	0.0645	0.0293
					0.9951	0.1556	0.0690	0.0550	0.0191	0.0143
	0.4	0.8	30	70	1.4093	1.0133	1.5593	1.1011	1.1912	1.1853
					1.1939	0.1869	0.1524	0.0922	0.0627	0.0279
					0.9950	0.1554	0.1083	0.0625	0.0186	0.0125

Table 2. The AEs (first-line), RMSEs (second-line), and MRABs (third-line) of λ.

n	T_1	T_2	k	r	MLE	MPSE	MCMC-ML		MCMC-MPS	
Prior →							1	2	1	2
40	0.4	0.8	15	25	0.9342	0.8031	1.1265	0.9447	1.2219	0.9691
					0.5908	0.4600	0.3841	0.1875	0.4404	0.1941
					0.6148	0.5149	0.4083	0.1862	0.4273	0.2116
	0.8	1.2	25	35	0.8869	0.8027	0.9526	0.9108	1.0546	0.9335
					0.4230	0.3767	0.1869	0.1580	0.2930	0.1564
					0.4385	0.4051	0.2020	0.1407	0.3183	0.1561
	0.4	0.8	22	32	0.9794	0.8018	0.9656	0.9215	1.0454	0.8897
					0.5419	0.4588	0.2463	0.1585	0.2854	0.1230
					0.5469	0.3681	0.2384	0.1589	0.3071	0.1136
	0.8	1.5	15	25	0.9359	0.8028	0.8022	0.8086	1.1425	0.9696
					0.4963	0.4433	0.2509	0.2196	0.3792	0.1968
					0.5000	0.4104	0.2534	0.1722	0.4281	0.2162
	0.6	0.8	15	25	0.9342	0.8031	1.0070	1.0482	1.1672	0.9821
					0.5908	0.4600	0.2839	0.2916	0.3989	0.2060
					0.6148	0.4394	0.3217	0.3103	0.4598	0.2276
	0.4	1.3	15	25	0.9173	0.8030	1.0592	0.9686	1.0806	0.9689
					0.5135	0.4344	0.2859	0.2477	0.2878	0.1985
					0.5200	0.4348	0.3257	0.2290	0.3507	0.2112
	0.4	0.8	20	25	0.9879	0.8020	1.0411	1.0306	1.2483	0.9065
					0.5727	0.4763	0.2676	0.2559	0.3245	0.1436
					0.5835	0.2848	0.3025	0.2883	0.2580	0.1398
	0.4	0.8	15	35	0.9288	0.8019	1.0486	0.8898	1.0861	0.9124
					0.5835	0.4105	0.2809	0.1204	0.3300	0.1372
					0.6097	0.4394	0.3107	0.1196	0.3577	0.1406
80	0.4	0.8	30	60	0.8359	0.8019	0.9057	0.7781	1.0001	0.8661
					0.4653	0.4106	0.1557	0.1497	0.2325	0.1131
					0.5056	0.2615	0.1429	0.1361	0.2521	0.0874
	0.8	1.2	40	70	0.7836	0.8047	0.7481	0.8712	0.9191	0.9139
					0.3594	0.2915	0.1646	0.1434	0.1640	0.1375
					0.3794	0.2296	0.1545	0.1191	0.1668	0.1424
	0.4	0.8	45	65	0.8815	0.8044	0.9057	0.8824	0.9283	0.8450
					0.3979	0.3709	0.1501	0.1429	0.1484	0.0994
					0.4056	0.2603	0.1527	0.1220	0.1606	0.0786
	0.8	1.5	30	60	0.7854	1.0794	0.9378	0.7920	0.8849	0.8571
					0.3279	0.2962	0.1636	0.1286	0.1177	0.1024
					0.3401	0.3492	0.1139	0.0931	0.1071	0.0879
	0.6	0.8	30	60	0.8359	0.8019	0.9278	0.8471	1.0665	0.8546
					0.4653	0.4106	0.2356	0.1583	0.2876	0.1007
					0.5056	0.2615	0.1970	0.1152	0.3332	0.0688
	0.4	1.3	30	60	0.7803	0.8025	0.8455	0.8551	0.9347	0.7243
					0.3478	0.2325	0.2264	0.1416	0.1932	0.1406
					0.3629	0.2499	0.1096	0.0859	0.1804	0.1458
	0.4	0.8	42	60	0.8843	0.8032	0.8808	0.9412	1.0064	0.7972
					0.4241	0.3550	0.2394	0.1999	0.2173	0.0979
					0.4375	0.2499	0.1894	0.1790	0.2204	0.0631
	0.4	0.8	30	70	0.8359	0.8031	1.0106	0.7885	0.9304	0.8584
					0.4653	0.3550	0.2619	0.0777	0.1539	0.0991
					0.5056	0.4194	0.2633	0.0720	0.1636	0.0735

Table 3. The AEs (first-line), RMSEs (second-line), and MRABs (third-line) of $R(t)$.

n	T_1	T_2	k	r	MLE	MPSE	MCMC-ML		MCMC-MPS	
Prior →							1	2	1	2
40	0.4	0.8	15	25	0.9357	0.9231	0.9144	0.9171	0.9050	0.9148
					0.0236	0.0192	0.0224	0.0163	0.0164	0.0149
					0.0203	0.0186	0.0198	0.0136	0.0153	0.0148
	0.8	1.2	25	35	0.9342	0.9233	0.9306	0.9199	0.9079	0.9193
					0.0223	0.0172	0.0191	0.0165	0.0158	0.0137
					0.0193	0.0166	0.0162	0.0107	0.0132	0.0118
	0.4	0.8	22	32	0.9358	0.9231	0.9135	0.9246	0.9088	0.9213
					0.0233	0.0191	0.0198	0.0171	0.0160	0.0138
					0.0201	0.0168	0.0171	0.0128	0.0152	0.0118
	0.8	1.5	15	25	0.9353	0.9232	0.9334	0.9107	0.9003	0.9146
					0.0222	0.0185	0.0211	0.0201	0.0173	0.0162
					0.0190	0.0167	0.0181	0.0159	0.0132	0.0112
	0.6	0.8	15	25	0.9357	0.9233	0.9131	0.9122	0.8974	0.9133
					0.0236	0.0172	0.0214	0.0192	0.0149	0.0133
					0.0203	0.0166	0.0186	0.0165	0.0145	0.0133
	0.4	1.3	15	25	0.9352	0.9232	0.9329	0.9059	0.9170	0.9347
					0.0219	0.0185	0.0201	0.0167	0.0175	0.0163
					0.0187	0.0167	0.0144	0.0114	0.0142	0.0127
	0.4	0.8	20	25	0.9361	0.9114	0.9249	0.9140	0.9232	0.9197
					0.0235	0.0191	0.0219	0.0177	0.0162	0.0136
					0.0206	0.0193	0.0166	0.0147	0.0167	0.0107
	0.4	0.8	15	35	0.9356	0.9233	0.9132	0.9183	0.8948	0.9192
					0.0237	0.0172	0.0199	0.0124	0.0153	0.0131
					0.0203	0.0166	0.0175	0.0121	0.0125	0.0110
80	0.4	0.8	30	60	0.9330	0.9233	0.9282	0.9213	0.9108	0.9233
					0.0165	0.0161	0.0159	0.0141	0.0147	0.0129
					0.0142	0.0162	0.0113	0.0101	0.0127	0.0102
	0.8	1.2	40	70	0.9316	0.9232	0.9158	0.9242	0.9173	0.9189
					0.0153	0.0141	0.0146	0.0133	0.0138	0.0124
					0.0131	0.0160	0.0114	0.0094	0.0130	0.0109
	0.4	0.8	45	65	0.9330	0.9233	0.9176	0.9218	0.9181	0.9242
					0.0160	0.0162	0.0153	0.0141	0.0139	0.0102
					0.0137	0.0161	0.0103	0.0089	0.0122	0.0097
	0.8	1.5	30	60	0.9246	0.9232	0.9173	0.9312	0.9213	0.9228
					0.0160	0.0158	0.0158	0.0150	0.0125	0.0112
					0.0140	0.0135	0.0126	0.0119	0.0118	0.0099
	0.6	0.8	30	60	0.9330	0.9233	0.9214	0.9271	0.9056	0.9243
					0.0165	0.0161	0.0150	0.0132	0.0147	0.0112
					0.0156	0.0163	0.0121	0.0098	0.0142	0.0096
	0.4	1.3	30	60	0.9232	0.9314	0.9201	0.9266	0.9045	0.9147
					0.0162	0.0152	0.0158	0.0142	0.0146	0.0115
					0.0131	0.0130	0.0120	0.0103	0.0126	0.0109
	0.4	0.8	42	60	0.9232	0.9331	0.9174	0.9110	0.8908	0.9285
					0.0177	0.0163	0.0168	0.0157	0.0141	0.0128
					0.0168	0.0140	0.0145	0.0100	0.0126	0.0101
	0.4	0.8	30	70	0.9330	0.9233	0.9132	0.9288	0.9179	0.9234
					0.0165	0.0161	0.0146	0.0101	0.0138	0.0116
					0.0162	0.0142	0.0142	0.0116	0.0125	0.0105

Table 4. The AEs (first-line), RMSEs (second-line), and MRABs (third-line) of $h(t)$.

n	T_1	T_2	k	r	MLE	MPSE	MCMC-ML		MCMC-MPS	
Prior →							1	2	1	2
40	0.4	0.8	15	25	0.6828	0.8720	0.9103	0.8242	0.7979	0.8447
					0.2335	0.2142	0.2201	0.1769	0.1635	0.1521
					0.2538	0.2233	0.2394	0.1562	0.1863	0.1500
	0.8	1.2	25	35	0.6940	0.8950	0.7221	0.7925	0.8662	0.7996
					0.2235	0.1856	0.2073	0.1813	0.1611	0.1091
					0.2445	0.2185	0.1704	0.1479	0.1793	0.1586
	0.4	0.8	22	32	0.6822	0.9616	0.7481	0.8634	0.7984	0.8228
					0.2315	0.1934	0.1952	0.1670	0.1640	0.1399
					0.2524	0.2095	0.1736	0.1468	0.1869	0.1217
	0.8	1.5	15	25	0.6972	0.7996	0.6858	0.7855	0.8964	0.8062
					0.3220	0.1988	0.2218	0.1925	0.1895	0.1211
					0.2330	0.1886	0.1825	0.1259	0.1562	0.1286
	0.6	0.8	15	25	0.9957	0.6828	0.9261	0.7053	0.9107	0.7908
					0.2819	0.2335	0.2461	0.1616	0.2007	0.1227
					0.3555	0.2538	0.2609	0.1756	0.2398	0.1271
	0.4	1.3	15	25	0.9931	0.7996	0.8408	0.6867	0.8709	0.6781
					0.2929	0.1988	0.2605	0.2193	0.1921	0.1874
					0.3546	0.1962	0.2371	0.2016	0.1886	0.1728
	0.4	0.8	20	25	0.7889	0.9313	0.6806	0.9060	0.7988	0.7449
					0.3332	0.2087	0.2345	0.1977	0.1645	0.1392
					0.2575	0.2678	0.2333	0.2045	0.1875	0.1285
	0.4	0.8	15	35	0.9205	0.7980	0.9159	0.8538	0.8608	0.7997
					0.2586	0.1635	0.2222	0.1553	0.1515	0.1262
					0.2533	0.1863	0.2469	0.1648	0.1725	0.1087
80	0.4	0.8	30	60	0.8697	0.7982	0.8391	0.7053	0.9377	0.8012
					0.1795	0.1785	0.1781	0.1616	0.1374	0.1313
					0.1887	0.1867	0.1650	0.1407	0.1686	0.1262
	0.8	1.2	40	70	0.8838	0.9533	0.9005	0.7166	0.8495	0.8011
					0.1940	0.1592	0.1679	0.1527	0.1521	0.1030
					0.2229	0.1672	0.1550	0.1297	0.1614	0.1475
	0.4	0.8	45	65	0.7891	0.8005	0.7056	0.8188	0.8579	0.7910
					0.1881	0.1637	0.1574	0.1517	0.1524	0.1111
					0.1705	0.1590	0.1429	0.1229	0.1681	0.1014
	0.8	1.5	30	60	0.9429	1.0564	0.8671	0.7198	0.8228	0.7992
					0.2168	0.1580	0.1729	0.1503	0.1363	0.1082
					0.2136	0.1382	0.1804	0.1635	0.1210	0.1088
	0.6	0.8	30	60	0.9150	1.0884	0.8249	0.7616	0.7979	0.7982
					0.2580	0.2159	0.2236	0.1459	0.1635	0.1079
					0.3182	0.2241	0.1703	0.1112	0.1863	0.1087
	0.4	1.3	30	60	0.7035	1.0078	0.7671	0.7182	0.8962	0.7992
					0.2687	0.1803	0.1571	0.1507	0.1629	0.1376
					0.1948	0.1720	0.1689	0.1407	0.1655	0.1088
	0.4	0.8	42	60	0.9386	1.1632	0.8666	0.7049	0.8406	0.7999
					0.2304	0.1564	0.1963	0.1597	0.1481	0.1085
					0.2191	0.1584	0.2006	0.1734	0.1410	0.1109
	0.4	0.8	30	70	0.6833	1.1187	0.7053	0.7413	0.8459	0.7982
					0.2342	0.1527	0.1616	0.1081	0.1434	0.1079
					0.2544	0.1490	0.1756	0.0973	0.1317	0.0891

Table 5. The ACLs(CPs) for 95% ACI/HPD credible intervals of α.

n	T_1	T_2	k	r	ACI-ML ACI-MPS	HPD-ML HPD-MPS	
Prior →						1	2
40	0.4	0.8	15	25	1.9852(0.908)	0.7231(0.945)	0.4348(0.962)
					0.8771(0.939)	0.2153(0.980)	0.1304(0.989)
	0.8	1.2	25	35	1.3774(0.935)	0.4082(0.973)	0.2582(0.981)
					0.8542(0.944)	0.1408(0.988)	0.1106(0.991)
	0.4	0.8	22	32	1.9028(0.910)	0.4179(0.970)	0.2755(0.976)
					0.8390(0.948)	0.1287(0.986)	0.0928(0.995)
	0.8	1.5	15	25	1.3683(0.937)	0.6943(0.951)	0.3841(0.967)
					0.7381(0.957)	0.1214(0.989)	0.1175(0.991)
	0.6	0.8	15	25	1.8980(0.911)	0.4439(0.965)	0.3590(0.971)
					0.8668(0.941)	0.1168(0.991)	0.0412(0.993)
	0.4	1.3	15	25	1.4816(0.928)	0.5668(0.958)	0.2603(0.978)
					0.8467(0.946)	0.1254(0.990)	0.0666(0.995)
	0.4	0.8	20	25	1.8330(0.919)	0.4187(0.969)	0.2277(0.983)
					0.8274(0.950)	0.1683(0.986)	0.0820(0.997)
	0.4	0.8	15	35	1.8757(0.916)	0.3854(0.972)	0.2967(0.976)
					0.8653(0.941)	0.1373(0.987)	0.0867(0.996)
80	0.4	0.8	30	60	1.1394(0.932)	0.6813(0.953)	0.3184(0.975)
					0.7751(0.946)	0.2056(0.984)	0.1194(0.994)
	0.8	1.2	40	70	0.7911(0.948)	0.3324(0.977)	0.2188(0.986)
					0.5638(0.967)	0.0691(0.996)	0.0340(0.999)
	0.4	0.8	45	65	1.0534(0.941)	0.3742(0.975)	0.2478(0.980)
					0.7018(0.955)	0.1087(0.993)	0.0843(0.997)
	0.8	1.5	30	60	0.7935(0.948)	0.5431(0.959)	0.2100(0.988)
					0.6230(0.962)	0.1189(0.993)	0.0462(0.998)
	0.6	0.8	30	60	1.1394(0.932)	0.4051(0.969)	0.2494(0.979)
					0.7651(0.949)	0.1094(0.992)	0.0546(0.998)
	0.4	1.3	30	60	0.7921(0.947)	0.4800(0.965)	0.2533(0.978)
					0.6452(0.960)	0.0794(0.994)	0.0488(0.999)
	0.4	0.8	42	60	1.1295(0.935)	0.2515(0.982)	0.2077(0.986)
					0.6716(0.958)	0.1050(0.992)	0.0789(0.997)
	0.4	0.8	30	70	1.1337(0.934)	0.3029(0.978)	0.2028(0.987)
					0.7351(0.953)	0.1311(0.990)	0.0579(0.998)

Table 6. The ACLs(CPs) for 95% ACI/HPD credible intervals of λ.

n	T_1	T_2	k	r	ACI-ML ACI-MPS	HPD-ML HPD-MPS	
Prior →						1	2
40	0.4	0.8	15	25	0.7056(0.911)	0.6860(0.913)	0.5252(0.927)
					0.7655(0.896)	0.5114(0.917)	0.3960(0.937)
	0.8	1.2	25	35	0.4950(0.926)	0.4802(0.931)	0.3758(0.941)
					0.6339(0.915)	0.4395(0.936)	0.3369(0.942)
	0.4	0.8	22	32	0.5179(0.928)	0.5072(0.934)	0.3405(0.946)
					0.6840(0.909)	0.4301(0.938)	0.2455(0.957)
	0.8	1.5	15	25	0.5039(0.933)	0.3813(0.940)	0.3562(0.943)
					0.6490(0.913)	0.4939(0.928)	0.2415(0.959)
	0.6	0.8	15	25	0.6014(0.920)	0.5136(0.927)	0.4637(0.934)
					0.7555(0.902)	0.4915(0.930)	0.2982(0.948)
	0.4	1.3	15	25	0.5401(0.924)	0.4720(0.934)	0.4115(0.937)
					0.6628(0.912)	0.3837(0.942)	0.2895(0.951)
	0.4	0.8	20	25	0.6261(0.917)	0.5226(0.925)	0.4064(0.938)
					0.7135(0.907)	0.5026(0.927)	0.3001(0.948)
	0.4	0.8	15	35	0.6466(0.915)	0.4702(0.933)	0.2668(0.953)
					0.7533(0.903)	0.4593(0.934)	0.2458(0.957)
80	0.4	0.8	30	60	0.5383(0.930)	0.5020(0.932)	0.3317(0.943)
					0.5344(0.915)	0.3575(0.935)	0.2742(0.950)
	0.8	1.2	40	70	0.4045(0.940)	0.3419(0.946)	0.3126(0.949)
					0.4637(0.928)	0.3070(0.945)	0.2723(0.952)
	0.4	0.8	45	65	0.4090(0.941)	0.3671(0.944)	0.3105(0.950)
					0.4755(0.926)	0.2843(0.948)	0.2341(0.958)
	0.8	1.5	30	60	0.3711(0.943)	0.3533(0.947)	0.2705(0.953)
					0.4323(0.932)	0.2689(0.953)	0.2230(0.963)
	0.6	0.8	30	60	0.5952(0.928)	0.4110(0.938)	0.3167(0.948)
					0.5314(0.920)	0.3480(0.938)	0.2157(0.965)
	0.4	1.3	30	60	0.4944(0.937)	0.3522(0.944)	0.3256(0.946)
					0.4511(0.930)	0.3219(0.946)	0.2283(0.961)
	0.4	0.8	42	60	0.4792(0.939)	0.4023(0.941)	0.3261(0.946)
					0.4911(0.923)	0.2703(0.953)	0.2253(0.963)
	0.4	0.8	30	70	0.4998(0.935)	0.3167(0.949)	0.2620(0.955)
					0.5244(0.918)	0.2806(0.950)	0.2014(0.967)

Table 7. The ACLs(CPs) for 95% ACI/HPD credible intervals of $R(t)$.

n	T_1	T_2	k	r	ACI-ML ACI-MPS	HPD-ML HPD-MPS	
Prior →						1	2
40	0.4	0.8	15	25	0.0907(0.913)	0.0530(0.930)	0.0444(0.952)
					0.0703(0.924)	0.0451(0.934)	0.0353(0.958)
	0.8	1.2	25	35	0.0583(0.931)	0.0452(0.937)	0.0432(0.954)
					0.0555(0.936)	0.0442(0.938)	0.0297(0.963)
	0.4	0.8	22	32	0.0629(0.926)	0.0376(0.943)	0.0322(0.960)
					0.0461(0.940)	0.0371(0.943)	0.0262(0.967)
	0.8	1.5	15	25	0.0631(0.926)	0.0479(0.935)	0.0391(0.956)
					0.0596(0.933)	0.0397(0.941)	0.0270(0.967)
	0.6	0.8	15	25	0.0523(0.935)	0.0485(0.937)	0.0388(0.957)
					0.0701(0.925)	0.0429(0.940)	0.0272(0.967)
	0.4	1.3	15	25	0.0635(0.925)	0.0446(0.939)	0.0432(0.954)
					0.0609(0.932)	0.0365(0.945)	0.0241(0.968)
	0.4	0.8	20	25	0.0769(0.920)	0.0458(0.937)	0.0321(0.961)
					0.0656(0.928)	0.0379(0.942)	0.0287(0.965)
	0.4	0.8	15	35	0.0487(0.937)	0.0381(0.942)	0.0305(0.963)
					0.0700(0.925)	0.0380(0.942)	0.0248(0.968)
80	0.4	0.8	30	60	0.0551(0.932)	0.0451(0.947)	0.0349(0.954)
					0.0490(0.940)	0.0333(0.958)	0.0290(0.968)
	0.8	1.2	40	70	0.0423(0.940)	0.0374(0.950)	0.0335(0.958)
					0.0426(0.948)	0.0327(0.960)	0.0275(0.970)
	0.4	0.8	45	65	0.0450(0.938)	0.0363(0.951)	0.0318(0.960)
					0.0437(0.946)	0.0275(0.965)	0.0237(0.975)
	0.8	1.5	30	60	0.0356(0.946)	0.0322(0.955)	0.0291(0.965)
					0.0423(0.950)	0.0303(0.963)	0.0258(0.973)
	0.6	0.8	30	60	0.0514(0.935)	0.0371(0.950)	0.0341(0.958)
					0.0489(0.941)	0.0321(0.960)	0.0235(0.975)
	0.4	1.3	30	60	0.0528(0.934)	0.0338(0.953)	0.0332(0.958)
					0.0415(0.953)	0.0290(0.964)	0.0202(0.978)
	0.4	0.8	42	60	0.0472(0.937)	0.0372(0.950)	0.0303(0.964)
					0.0453(0.944)	0.0315(0.962)	0.0221(0.976)
	0.4	0.8	30	70	0.0458(0.938)	0.0349(0.952)	0.0286(0.965)
					0.0490(0.940)	0.0261(0.967)	0.0231(0.976)

Table 8. The ACLs(CPs) for 95% ACI/HPD credible intervals of $h(t)$.

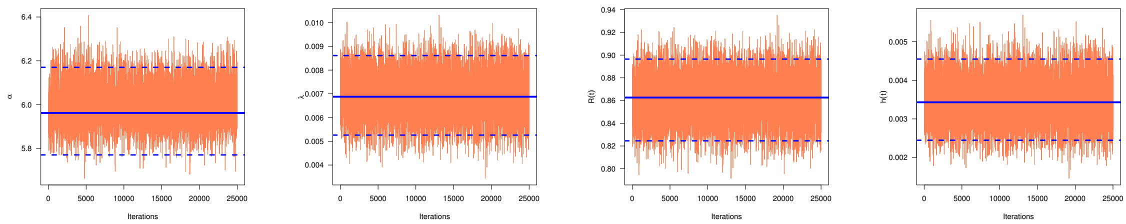
n	T_1	T_2	k	r	ACI-ML ACI-MPS	HPD-ML HPD-MPS	
Prior →						1	2
40	0.4	0.8	15	25	0.8489(0.891)	0.5739(0.913)	0.4802(0.931)
					0.8957(0.887)	0.5068(0.931)	0.3955(0.938)
	0.8	1.2	25	35	0.6009(0.905)	0.4923(0.924)	0.4713(0.934)
					0.6317(0.902)	0.4917(0.931)	0.3023(0.954)
	0.4	0.8	22	32	0.5041(0.914)	0.4216(0.931)	0.3481(0.940)
					0.6813(0.910)	0.4164(0.940)	0.2860(0.957)
	0.8	1.5	15	25	0.6467(0.901)	0.5198(0.918)	0.4199(0.938)
					0.6973(0.908)	0.4731(0.934)	0.3327(0.951)
	0.6	0.8	15	25	0.5729(0.907)	0.5298(0.916)	0.4274(0.937)
					0.7621(0.899)	0.4769(0.934)	0.2988(0.955)
	0.4	1.3	15	25	0.7000(0.898)	0.4877(0.925)	0.4733(0.934)
					0.6602(0.905)	0.3846(0.942)	0.3165(0.953)
	0.4	0.8	20	25	0.7110(0.896)	0.5025(0.920)	0.3473(0.941)
					0.7621(0.899)	0.4201(0.938)	0.3422(0.949)
	0.4	0.8	15	35	0.5312(0.910)	0.4262(0.929)	0.3282(0.944)
					0.7619(0.900)	0.4270(0.938)	0.2709(0.960)
80	0.4	0.8	30	60	0.6018(0.900)	0.5021(0.921)	0.3772(0.942)
					0.6114(0.898)	0.3647(0.943)	0.3190(0.951)
	0.8	1.2	40	70	0.4597(0.920)	0.4093(0.934)	0.3614(0.944)
					0.4618(0.918)	0.3591(0.945)	0.2658(0.960)
	0.4	0.8	45	65	0.4734(0.917)	0.4112(0.933)	0.3479(0.945)
					0.4934(0.915)	0.2999(0.955)	0.2590(0.962)
	0.8	1.5	30	60	0.3852(0.926)	0.3576(0.937)	0.3189(0.948)
					0.4308(0.922)	0.2964(0.955)	0.2819(0.957)
	0.6	0.8	30	60	0.5314(0.912)	0.3994(0.934)	0.3770(0.943)
					0.5703(0.909)	0.3543(0.945)	0.2562(0.964)
	0.4	1.3	30	60	0.4494(0.922)	0.3700(0.936)	0.3643(0.944)
					0.5862(0.910)	0.3029(0.953)	0.2235(0.968)
	0.4	0.8	42	60	0.4892(0.916)	0.4117(0.933)	0.3394(0.947)
					0.5170(0.913)	0.3141(0.951)	0.2423(0.965)
	0.4	0.8	30	70	0.5186(0.914)	0.3772(0.938)	0.3105(0.950)
					0.5314(0.910)	0.2860(0.957)	0.2518(0.964)

Table 9. Six UHC samples from yarn data.

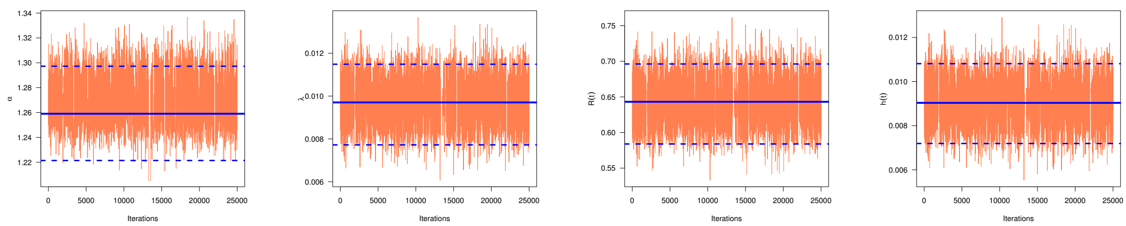
Case	$T_1(d_1)$	$T_2(d_2)$	k	r	ω	υ
I	150(11)	250(19)	8	10	11	150
II	150(11)	250(19)	8	15	15	180
III	150(11)	250(19)	8	22	19	250
IV	50(4)	200(17)	8	13	13	175
V	50(4)	200(17)	8	20	17	200
VI	50(4)	90(7)	8	15	8	98

Table 10. Point estimates (SEs) of α, λ, $R(t)$, and $h(t)$ from yarn data.

Case	Parameter	MLE	MPSE	MCMC-ML	MCMC-MPS
I	α	6.0641($1.42\times10^{+1}$)	1.2862(1.25×10^{-1})	5.9620(6.48×10^{-4})	1.2590(1.25×10^{-4})
	λ	0.0070(4.78×10^{-3})	0.0098(3.50×10^{-3})	0.0069(5.47×10^{-6})	0.0097(6.14×10^{-6})
	$R(50)$	0.8620(6.95×10^{-2})	0.6417(1.07×10^{-1})	0.8626(1.17×10^{-4})	0.6432(1.83×10^{-4})
	$h(50)$	0.0034(1.20×10^{-3})	0.0091(3.44×10^{-3})	0.0034(3.44×10^{-6})	0.0090(5.90×10^{-6})
II	α	14.155($0.66\times10^{+1}$)	1.1069(1.03×10^{-1})	14.0522(6.48×10^{-4})	1.0301(4.98×10^{-5})
	λ	0.0092(1.77×10^{-3})	0.0087(2.33×10^{-3})	0.0092(5.32×10^{-6})	0.0087(2.23×10^{-6})
	$R(50)$	0.8737(3.12×10^{-2})	0.6593(7.52×10^{-2})	0.8739(9.00×10^{-5})	0.6498(7.37×10^{-5})
	$h(50)$	0.0036(9.49×10^{-4})	0.0084(2.30×10^{-3})	0.0036(3.06×10^{-6})	0.0082(2.24×10^{-6})
III	α	14.022($1.04\times10^{+1}$)	1.3378(1.71×10^{-1})	13.918(6.57×10^{-4})	1.3241(1.28×10^{-4})
	λ	0.0092(1.79×10^{-3})	0.0076(1.79×10^{-3})	0.0092(5.18×10^{-6})	0.0076(5.56×10^{-6})
	$R(50)$	0.8728(3.70×10^{-2})	0.7135(5.89×10^{-2})	0.8728(8.80×10^{-5})	0.7156(1.81×10^{-4})
	$h(50)$	0.0036(9.49×10^{-4})	0.0069(1.70×10^{-3})	0.0036(3.00×10^{-6})	0.0069(5.24×10^{-6})
IV	α	7.2053($0.76\times10^{+1}$)	1.1569(1.07×10^{-1})	7.1035(6.49×10^{-4})	1.0995(1.64×10^{-4})
	λ	0.0074(2.40×10^{-3})	0.0094(2.79×10^{-3})	0.0073(5.27×10^{-6})	0.0093(5.03×10^{-6})
	$R(50)$	0.8644(4.44×10^{-2})	0.6419(8.68×10^{-2})	0.8651(1.07×10^{-4})	0.6385(1.56×10^{-4})
	$h(50)$	0.0035(1.02×10^{-3})	0.0090(2.75×10^{-3})	0.0034(3.23×10^{-6})	0.0091(4.94×10^{-6})
V	α	17.316($0.63\times10^{+1}$)	1.1299(1.23×10^{-1})	17.211(6.81×10^{-4})	1.0841(1.43×10^{-4})
	λ	0.0098(1.66×10^{-3})	0.0081(2.02×10^{-3})	0.0097(5.24×10^{-6})	0.0081(5.12×10^{-6})
	$R(50)$	0.8762(2.83×10^{-2})	0.6805(6.67×10^{-2})	0.8768(8.46×10^{-5})	0.6765(1.72×10^{-4})
	$h(50)$	0.0036(9.14×10^{-4})	0.0078(1.99×10^{-3})	0.0036(2.99×10^{-6})	0.0079(5.09×10^{-6})
VI	α	14.194($1.07\times10^{+1}$)	1.0686(6.80×10^{-2})	14.091(6.39×10^{-4})	1.0321(2.76×10^{-5})
	λ	0.0099(3.06×10^{-3})	0.0141(5.57×10^{-3})	0.0098(5.90×10^{-6})	0.0145(4.57×10^{-6})
	$R(50)$	0.8617(4.45×10^{-2})	0.5012(1.38×10^{-1})	0.8626(1.02×10^{-4})	0.4895(1.12×10^{-4})
	$h(50)$	0.0040(1.45×10^{-3})	0.0139(5.57×10^{-3})	0.0040(3.55×10^{-6})	0.0144(4.59×10^{-6})



(a) MCMC-ML

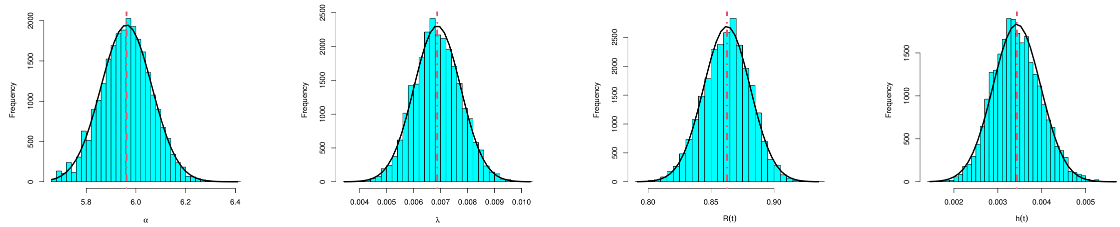


(b) MCMC-MPS

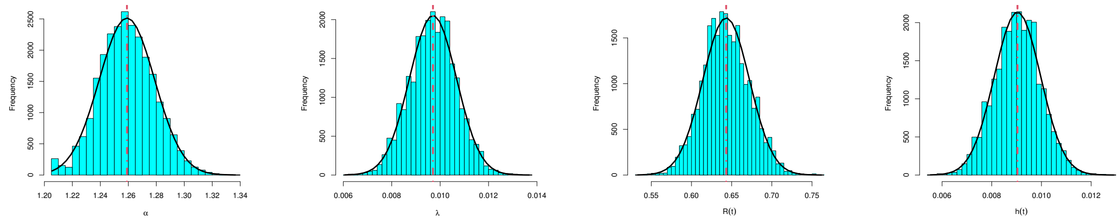
Figure 3. Trace plots for MCMC draws of α, λ, $R(t)$, and $h(t)$ from yarn data.

Table 11. Interval estimates of α, λ, $R(t)$, and $h(t)$ from yarn data.

Case	Parameter	ACI-ML			ACI-MPS			HPD-ML			HPD-MPS		
		Low.	Upp.	Len.	Low.	Upp.	Len.	Low.	Upp.	Len.	Low.	Upp.	Len.
I	α	0.0001	33.917	55.707	1.0404	1.5320	0.4916	5.7708	6.1703	0.3995	1.2214	1.2972	0.0759
	λ	0.0001	0.0163	0.0187	0.0030	0.0167	0.0137	0.0053	0.0086	0.0034	0.0077	0.0115	0.0038
	$R(50)$	0.7257	0.9983	0.2726	0.4315	0.8519	0.4204	0.8245	0.8965	0.0720	0.5839	0.6962	0.1123
	$h(50)$	0.0011	0.0058	0.0047	0.0023	0.0158	0.0135	0.0024	0.0046	0.0021	0.0072	0.0108	0.0036
II	α	1.3073	27.002	25.695	0.9048	1.3090	0.4042	13.843	14.242	0.3935	1.0262	1.0394	0.0133
	λ	0.0058	0.0127	0.0069	0.0041	0.0133	0.0091	0.0075	0.0108	0.0033	0.0075	0.0089	0.0014
	$R(50)$	0.8125	0.9349	0.1224	0.5120	0.8066	0.2946	0.8475	0.9030	0.0555	0.6450	0.6905	0.0455
	$h(50)$	0.0017	0.0054	0.0037	0.0039	0.0129	0.0090	0.0026	0.0045	0.0019	0.0074	0.0088	0.0014
III	α	0.0001	34.377	40.710	1.0028	1.6728	0.6700	13.722	14.131	0.4091	1.2845	1.3645	0.0800
	λ	0.0057	0.0128	0.0070	0.0041	0.0111	0.0070	0.0076	0.0108	0.0032	0.0058	0.0092	0.0034
	$R(50)$	0.8003	0.9452	0.1449	0.5980	0.8289	0.2309	0.8445	0.8989	0.0544	0.6613	0.7726	0.1113
	$h(50)$	0.0017	0.0055	0.0037	0.0036	0.0102	0.0067	0.0027	0.0046	0.0018	0.0052	0.0085	0.0032
IV	α	0.0001	22.077	29.744	0.9478	1.3660	0.4183	6.8973	7.2984	0.4011	1.0650	1.1429	0.0779
	λ	0.0027	0.0121	0.0094	0.0039	0.0149	0.0109	0.0057	0.0090	0.0033	0.0079	0.0111	0.0032
	$R(50)$	0.7774	0.9515	0.1741	0.4718	0.8119	0.3401	0.8328	0.8989	0.0661	0.5898	0.6892	0.0994
	$h(50)$	0.0015	0.0055	0.0040	0.0036	0.0144	0.0108	0.0025	0.0044	0.0020	0.0074	0.0106	0.0031
V	α	4.9967	29.636	24.639	0.8892	1.3705	0.4813	17.006	17.425	0.4196	1.0523	1.1240	0.0717
	λ	0.0066	0.0130	0.0065	0.0041	0.0121	0.0079	0.0082	0.0114	0.0033	0.0065	0.0096	0.0031
	$R(50)$	0.8208	0.9317	0.1110	0.5497	0.8113	0.2616	0.8489	0.9016	0.0527	0.6266	0.7298	0.1032
	$h(50)$	0.0018	0.0054	0.0036	0.0039	0.0117	0.0078	0.0027	0.0046	0.0019	0.0062	0.0093	0.0031
VI	α	0.0001	35.178	41.968	0.9353	1.2019	0.2666	13.891	14.292	0.4011	1.0263	1.0381	0.0118
	λ	0.0039	0.0159	0.0120	0.0032	0.0251	0.0218	0.0080	0.0117	0.0036	0.0135	0.0160	0.0025
	$R(50)$	0.7746	0.9489	0.1743	0.2298	0.7726	0.5427	0.8323	0.8950	0.0627	0.4540	0.5136	0.0596
	$h(50)$	0.0011	0.0068	0.0057	0.0030	0.0248	0.0218	0.0029	0.0050	0.0022	0.0134	0.0159	0.0025



(a) MCMC-ML



(b) MCMC-MPS

Figure 4. Histograms for MCMC draws of α, λ, $R(t)$, and $h(t)$ from yarn data.

Table 12. Statistics for MCMC outputs of α, λ, $R(t)$, and $h(t)$ from yarn data.

Case	Parameter	Mean	Median	Mode	SD	Skewness
			MCMC-ML			
I	α	5.96200	5.96415	5.68276	0.10242	-0.03386
	λ	0.00688	0.00685	0.00594	0.00086	0.07210
	$R(50)$	0.86261	0.86331	0.87992	0.01854	-0.10466
	$h(50)$	0.00343	0.00340	0.00291	0.00054	0.21110
II	α	14.0522	14.0527	13.7750	0.10241	-0.02918
	λ	0.00917	0.00918	0.00981	0.00084	0.01263
	$R(50)$	0.87393	0.87392	0.86195	0.01424	-0.09516
	$h(50)$	0.00356	0.00355	0.00395	0.00048	0.18261
III	α	13.9181	13.9204	13.6209	0.10381	-0.13705
	λ	0.00920	0.00920	0.00900	0.00082	0.04498
	$R(50)$	0.87280	0.87294	0.87507	0.01391	-0.12374
	$h(50)$	0.00359	0.00358	0.00350	0.00047	0.21513
IV	α	7.10347	7.10554	6.76605	0.10268	-0.18571
	λ	0.00731	0.00728	0.00656	0.00083	0.05451
	$R(50)$	0.86513	0.86569	0.87732	0.01692	-0.09746
	$h(50)$	0.00344	0.00342	0.00305	0.00051	0.20268
V	α	17.2108	17.2132	16.8454	0.10775	-0.37345
	λ	0.00973	0.00971	0.00951	0.00083	0.08093
	$R(50)$	0.87682	0.87750	0.87942	0.01338	-0.17024
	$h(50)$	0.00360	0.00357	0.00349	0.00047	0.25527
VI	α	14.0913	14.0911	13.8397	0.10101	0.02334
	λ	0.00984	0.00983	0.00876	0.00093	0.06221
	$R(50)$	0.86259	0.86295	0.88017	0.01609	-0.15323
	$h(50)$	0.00396	0.00393	0.00334	0.00056	0.24651
			MCMC-MPS			
I	α	1.25902	1.25863	1.20549	0.01983	0.06643
	λ	0.00970	0.00972	0.00821	0.00097	0.00283
	$R(50)$	0.64315	0.64206	0.68395	0.02900	0.12467
	$h(50)$	0.00903	0.00904	0.00771	0.00093	0.01748
II	α	1.03008	1.02616	1.02616	0.00788	3.54786
	λ	0.00873	0.00878	0.00878	0.00035	-1.40850
	$R(50)$	0.64976	0.64775	0.64775	0.01166	1.83158
	$h(50)$	0.00865	0.00870	0.00870	0.00035	-1.62380
III	α	1.32408	1.32382	1.27622	0.02030	-0.03122
	λ	0.00755	0.00754	0.00764	0.00088	0.07522
	$R(50)$	0.71563	0.71546	0.70844	0.02854	0.02929
	$h(50)$	0.00685	0.00684	0.00702	0.00083	0.09414
IV	α	1.09954	1.10242	1.06747	0.02598	0.16978
	λ	0.00934	0.00949	0.00949	0.00079	0.01131
	$R(50)$	0.63845	0.63349	0.62992	0.02466	0.28924
	$h(50)$	0.00906	0.00921	0.00930	0.00078	-0.10329
V	α	1.09954	1.10242	1.06747	0.02598	0.16978
	λ	0.00934	0.00949	0.00949	0.00079	0.01131
	$R(50)$	0.63845	0.63349	0.62992	0.02466	0.28924
	$h(50)$	0.00906	0.00921	0.00930	0.00078	-0.10329
VI	α	1.03212	1.03176	1.02633	0.00436	1.79177
	λ	0.01446	0.01443	0.01502	0.00072	0.16626
	$R(50)$	0.48945	0.49007	0.47509	0.01766	-0.01434
	$h(50)$	0.01435	0.01432	0.01493	0.00073	0.13978

5.2. Electronic data

In this application, we analyze a real dataset from an accelerated life test experiment that represents the failure times (in minutes) for a sample of 15 electronic components. This dataset was reported by Lawless [31] and is: 1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2. First, using the K-S test and its p-value, we fit the APE distribution to the complete electronic components data. As a result, the MLEs (SEs) of α and λ are 5.8455(8.6629) and 0.0527(0.0166), respectively, as well as the K-S(p-value) statistic is 0.095(0.997). This result shows that the APE model is very suitable for the given data.

Table 13. Six UHC samples from electronic components data.

Case	$T_1(d_1)$	$T_2(d_2)$	k	r	ω	υ
I	40(11)	55(13)	7	10	11	40
II	40(11)	55(13)	7	12	12	46.3
III	40(11)	55(13)	7	14	13	55
IV	15(5)	60(14)	7	13	13	53.9
V	15(5)	50(12)	7	14	12	50
VI	15(5)	19(6)	7	14	7	19.7

From the complete electronic data, for various combinations of k, r, and T_i, $i = 1, 2$, six different UHC samples are generated and reported in Table 13. Using Table 13, the classical (MLEs and MPSEs) estimates and Bayes (MCMC-ML and MCMC-MPS) estimates of α, λ, $R(t)$, and $h(t)$ (at $t = 5$) are calculated and reported in Table 14. Moreover, 95% ACI/HPD credible intervals developed by ML and MPS functions of the same unknown parameters are obtained and provided in Table 15. To calculate the Bayes estimates using the M-H algorithm based on the improper gamma priors, we simulate $\mathfrak{D} = 30,000$ and $\mathfrak{D}_0 = 2,000$ MCMC samples from α, λ, $R(t)$, and $h(t)$. Tables 14 and 16 showed that the Bayes estimates and associated HPD credible interval estimates obtained from the MPS method perform better than those obtained from the ML method in terms of minimum SEs and lengths.Furthermore, some vital statistics based on 25,000 MCMC outputs of all unknown parameters are obtained; see Table 16. To judge the convergence of simulated MCMC variates, trace plots of α, λ, $R(t)$, and $h(t)$ are plotted in Figure 5. It indicates that the MCMC iterations of all unknown parameters converge very well. Also, the histogram of the same unknown parameters is displayed in Figure 6. It is clear from the estimates that all the generated posteriors of α, λ, $R(t)$, or $h(t)$ are almost symmetrical. To summarize, we conclude that the proposed methodologies, using both yarn and electronic datasets, provide a good explanation of the unknown parameters of the life of the alpha power exponential distribution in the presence of unified hybrid censored samples.

6. Concluding Remarks

This article explores different classical and Bayesian estimation approaches to derive both point and interval estimators of the model parameter and reliability characteristics of the alpha-power exponential distribution in the presence of samples collected from unified hybrid censoring. Specifically, in the

Table 14. Point estimates (SEs) of α, λ, $R(t)$, and $h(t)$ from electronic components data.

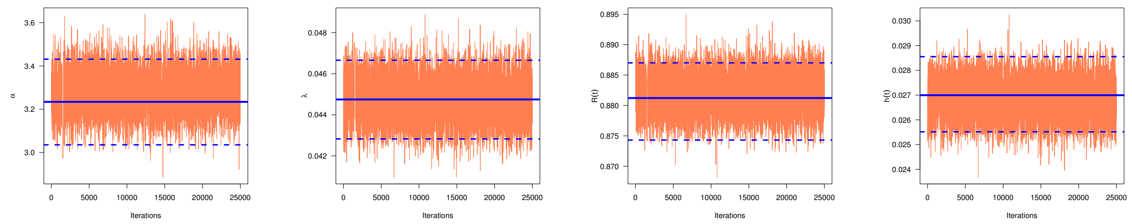
Case	Parameter	MLE	MPSE	MCMC-ML	MCMC-MPS
I	α	3.3373(0.55×10^{+1})	1.3089(2.01×10^{-1})	3.2342(6.38×10^{-4})	1.2974(1.29×10^{-4})
	λ	0.0448(2.12×10^{-2})	0.0509(1.62×10^{-2})	0.0447(6.30×10^{-6})	0.0508(6.35×10^{-6})
	$R(5)$	0.8830(5.73×10^{-2})	0.7982(5.93×10^{-2})	0.8812(2.04×10^{-5})	0.7976(2.42×10^{-5})
	$h(5)$	0.0266(1.14×10^{-2})	0.0457(1.52×10^{-2})	0.0270(4.93×10^{-6})	0.0459(6.14×10^{-6})
II	α	3.3518(0.52×10^{+1})	1.4538(2.32×10^{-1})	3.2507(6.27×10^{-4})	1.4451(1.27×10^{-4})
	λ	0.0447(1.91×10^{-2})	0.0457(1.38×10^{-2})	0.0447(6.33×10^{-6})	0.0457(6.32×10^{-6})
	$R(5)$	0.8834(5.62×10^{-2})	0.8248(5.03×10^{-2})	0.8816(2.05×10^{-5})	0.8245(2.33×10^{-5})
	$h(5)$	0.0265(1.13×10^{-2})	0.0393(1.25×10^{-2})	0.0269(4.97×10^{-6})	0.0393(5.80×10^{-6})
III	α	3.3141(0.56×10^{+1})	1.6086(2.51×10^{-1})	3.2075(6.47×10^{-4})	1.6015(1.27×10^{-4})
	λ	0.0444(1.92×10^{-2})	0.0417(1.19×10^{-2})	0.0444(6.20×10^{-6})	0.0417(6.31×10^{-6})
	$R(5)$	0.8835(6.19×10^{-2})	0.8461(4.28×10^{-2})	0.8817(2.03×10^{-5})	0.8458(2.26×10^{-5})
	$h(5)$	0.0265(1.23×10^{-2})	0.0342(1.05×10^{-2})	0.0269(4.90×10^{-6})	0.0343(5.53×10^{-6})
IV	α	3.4957(0.55×10^{+1})	1.6121(2.53×10^{-1})	3.3925(6.34×10^{-4})	1.6049(1.27×10^{-4})
	λ	0.0452(1.82×10^{-2})	0.0417(1.18×10^{-2})	0.0452(6.22×10^{-6})	0.0417(6.32×10^{-6})
	$R(5)$	0.8846(5.84×10^{-2})	0.8464(4.27×10^{-2})	0.8829(1.96×10^{-5})	0.8462(2.26×10^{-5})
	$h(5)$	0.0263(1.17×10^{-2})	0.0341(1.05×10^{-2})	0.0267(4.76×10^{-6})	0.0342(5.53×10^{-6})
V	α	2.4213(0.44×10^{+1})	1.4443(2.25×10^{-1})	2.3186(6.33×10^{-4})	1.4354(1.30×10^{-4})
	λ	0.0401(2.07×10^{-2})	0.0459(1.38×10^{-2})	0.0401(6.32×10^{-6})	0.0459(6.31×10^{-6})
	$R(5)$	0.8773(6.09×10^{-2})	0.8237(5.03×10^{-2})	0.8747(2.55×10^{-5})	0.8233(2.35×10^{-5})
	$h(5)$	0.0273(1.21×10^{-2})	0.0395(1.26×10^{-2})	0.0279(5.93×10^{-6})	0.0396(5.84×10^{-6})
VI	α	6.8855(0.11×10^{+2})	1.1261(1.24×10^{-1})	6.7847(6.26×10^{-4})	1.0694(1.60×10^{-4})
	λ	0.0575(2.87×10^{-2})	0.0709(2.99×10^{-2})	0.0574(6.26×10^{-6})	0.0708(4.98×10^{-6})
	$R(5)$	0.8948(5.22×10^{-2})	0.7140(1.04×10^{-1})	0.8941(1.35×10^{-5})	0.7089(2.07×10^{-5})
	$h(5)$	0.0256(1.09×10^{-2})	0.0680(2.95×10^{-2})	0.0257(3.68×10^{-6})	0.0691(5.48×10^{-6})

Table 15. Interval estimates of α, λ, $R(t)$, and $h(t)$ from electronic components data.

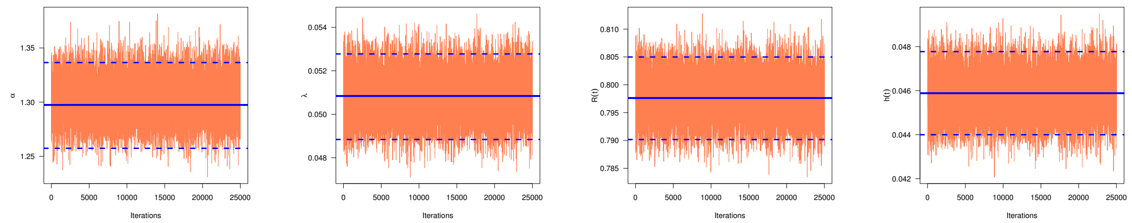
Case	Parameter	ACI-ML			ACI-MPS			HPD-ML			HPD-MPS		
		Low.	Upp.	Len.	Low.	Upp.	Len.	Low.	Upp.	Len.	Low.	Upp.	Len.
I	α	0.0001	14.193	14.193	0.9145	1.7033	0.7887	3.0356	3.4313	0.3957	1.2574	1.3365	0.0792
	λ	0.0032	0.0863	0.0831	0.0192	0.0826	0.0634	0.0428	0.0467	0.0038	0.0488	0.0528	0.0039
	$R(5)$	0.7707	0.9952	0.2245	0.6821	0.9144	0.2323	0.8743	0.8870	0.0127	0.7902	0.8050	0.0148
	$h(5)$	0.0043	0.0489	0.0446	0.0160	0.0755	0.0595	0.0255	0.0286	0.0030	0.0440	0.0478	0.0038
II	α	0.0001	13.555	13.555	1.0001	1.9076	0.9075	3.0591	3.4412	0.3821	1.4064	1.4852	0.0788
	λ	0.0074	0.0820	0.0747	0.0188	0.0727	0.0540	0.0427	0.0466	0.0039	0.0437	0.0476	0.0039
	$R(5)$	0.7734	0.9935	0.2202	0.7263	0.9233	0.1970	0.8754	0.8878	0.0124	0.8172	0.8315	0.0143
	$h(5)$	0.0044	0.0486	0.0442	0.0147	0.0638	0.0491	0.0254	0.0284	0.0030	0.0375	0.0411	0.0036
III	α	0.0001	14.371	14.371	1.1175	2.0997	0.9822	2.9960	3.3934	0.3974	1.5629	1.6417	0.0788
	λ	0.0068	0.0820	0.0752	0.0185	0.0650	0.0465	0.0424	0.0462	0.0038	0.0398	0.0437	0.0039
	$R(5)$	0.7621	1.0049	0.2428	0.7622	0.9299	0.1676	0.8755	0.8882	0.0127	0.8391	0.8529	0.0139
	$h(5)$	0.0024	0.0505	0.0481	0.0137	0.0547	0.0410	0.0253	0.0284	0.0031	0.0326	0.0360	0.0034
IV	α	0.0001	14.273	14.273	1.1163	2.1078	0.9915	3.2031	3.5925	0.3894	1.5645	1.6433	0.0788
	λ	0.0094	0.0809	0.0715	0.0185	0.0649	0.0465	0.0433	0.0472	0.0039	0.0397	0.0437	0.0039
	$R(5)$	0.7702	0.9990	0.2288	0.7626	0.9302	0.1676	0.8771	0.8892	0.0121	0.8391	0.8531	0.0140
	$h(5)$	0.0034	0.0492	0.0458	0.0137	0.0546	0.0410	0.0252	0.0281	0.0030	0.0325	0.0359	0.0034
V	α	0.0001	11.018	11.018	1.0033	1.8852	0.8819	2.1327	2.5264	0.3937	1.3939	1.4738	0.0799
	λ	0.0001	0.0807	0.0807	0.0188	0.0730	0.0542	0.0381	0.0419	0.0039	0.0439	0.0478	0.0039
	$R(5)$	0.7579	0.9967	0.2389	0.7250	0.9223	0.1973	0.8663	0.8819	0.0156	0.8161	0.8304	0.0144
	$h(5)$	0.0037	0.0510	0.0473	0.0149	0.0642	0.0493	0.0262	0.0298	0.0036	0.0378	0.0414	0.0036
VI	α	0.0001	28.621	28.621	0.8836	1.3686	0.4849	6.5977	6.9798	0.3820	1.0425	1.1141	0.0716
	λ	0.0012	0.1137	0.1125	0.0123	0.1295	0.1172	0.0556	0.0594	0.0039	0.0695	0.0727	0.0032
	$R(5)$	0.7926	0.9971	0.2045	0.5102	0.9178	0.4076	0.8900	0.8983	0.0083	0.7033	0.7157	0.0124
	$h(5)$	0.0042	0.0469	0.0427	0.0102	0.1257	0.1155	0.0246	0.0269	0.0023	0.0673	0.0707	0.0035

Table 16. Statistics for MCMC outputs of α, λ, $R(t)$, and $h(t)$ from electronic components data.

Case	Parameter	Mean	Median	Mode	SD	Skewness
			MCMC-ML			
I	α	3.23417	3.23542	2.97544	0.10083	-0.06327
	λ	0.04475	0.04474	0.04548	0.00100	-0.02116
	$R(5)$	0.88122	0.88130	0.87430	0.00322	-0.06150
	$h(5)$	0.02700	0.02698	0.02855	0.00078	0.04737
II	α	3.25070	3.25010	3.02437	0.09910	0.05559
	λ	0.04470	0.04470	0.04483	0.00100	-0.02570
	$R(5)$	0.88165	0.88165	0.87706	0.00325	-0.01061
	$h(5)$	0.02690	0.02690	0.02789	0.00079	0.00883
III	α	3.20746	3.20930	2.91728	0.10222	-0.02780
	λ	0.04440	0.04441	0.04455	0.00098	-0.01703
	$R(5)$	0.88165	0.88171	0.87567	0.00322	-0.05747
	$h(5)$	0.02687	0.02685	0.02816	0.00078	0.05092
IV	α	3.39245	3.39319	3.15932	0.10024	0.00448
	λ	0.04518	0.04518	0.04503	0.00098	-0.06120
	$R(5)$	0.88287	0.88285	0.87912	0.00310	0.01885
	$h(5)$	0.02669	0.02670	0.02747	0.00075	-0.02122
V	α	2.31858	2.32007	2.10406	0.10005	-0.00049
	λ	0.04011	0.04010	0.04223	0.00100	-0.01042
	$R(5)$	0.87472	0.87483	0.86223	0.00403	-0.10420
	$h(5)$	0.02789	0.02787	0.03078	0.00094	0.09847
VI	α	6.78473	6.78508	6.49470	0.09904	0.02414
	λ	0.05743	0.05744	0.05734	0.00099	-0.01477
	$R(5)$	0.89407	0.89406	0.89187	0.00213	0.02833
	$h(5)$	0.02573	0.02574	0.02620	0.00058	-0.01477
			MCMC-MPS			
I	α	1.29740	1.29741	1.24055	0.02046	-0.01103
	λ	0.05084	0.05084	0.05080	0.00100	0.01642
	$R(5)$	0.79762	0.79764	0.79406	0.00382	-0.04013
	$h(5)$	0.04589	0.04588	0.04667	0.00097	0.04895
II	α	1.44508	1.44496	1.40640	0.02010	0.02644
	λ	0.04571	0.04571	0.04435	0.00100	0.00041
	$R(5)$	0.82448	0.82449	0.82581	0.00369	0.01088
	$h(5)$	0.03934	0.03934	0.03857	0.00092	0.00175
III	α	1.60150	1.60168	1.54291	0.02007	0.01078
	λ	0.04173	0.04173	0.04154	0.00100	-0.01206
	$R(5)$	0.84583	0.84583	0.84392	0.00357	0.01756
	$h(5)$	0.03426	0.03426	0.03465	0.00087	-0.00471
IV	α	1.60495	1.60495	1.56799	0.02009	-0.00442
	λ	0.04167	0.04168	0.04187	0.00100	-0.01237
	$R(5)$	0.84616	0.84617	0.84387	0.00357	0.00276
	$h(5)$	0.03418	0.03418	0.03470	0.00087	0.00833
V	α	1.43538	1.43552	1.38924	0.02049	-0.02012
	λ	0.04589	0.04589	0.04666	0.00100	-0.01763
	$R(5)$	0.82331	0.82332	0.81809	0.00371	0.01035
	$h(5)$	0.03962	0.03961	0.04085	0.00092	0.00006
VI	α	1.06943	1.06513	1.04249	0.02529	0.55714
	λ	0.07079	0.07048	0.07048	0.00079	0.54419
	$R(5)$	0.70886	0.70793	0.70732	0.00327	0.91814
	$h(5)$	0.06915	0.06946	0.06946	0.00087	-0.77920

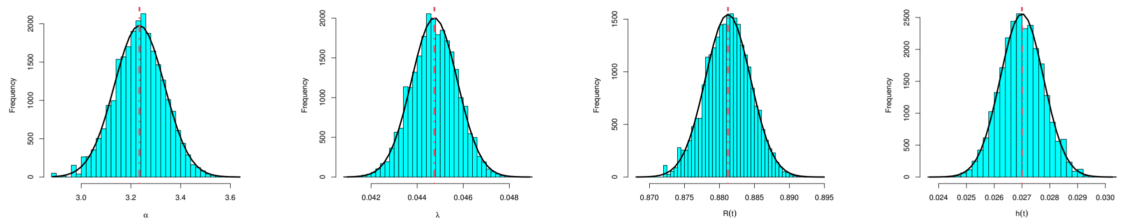


(a) MCMC-ML

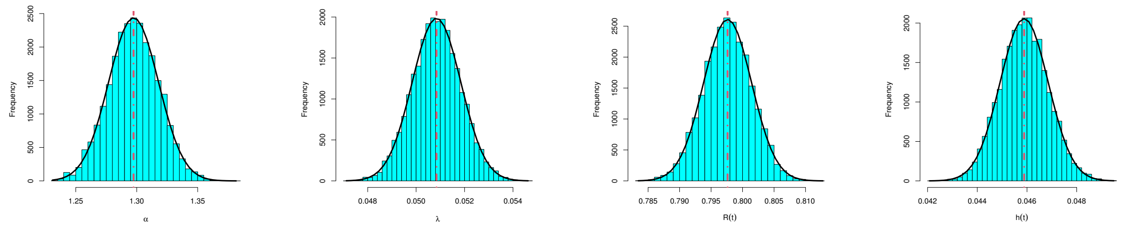


(b) MCMC-MPS

Figure 5. Trace plots for MCMC draws of α, λ, $R(t)$, and $h(t)$ from electronic components data.



(a) MCMC-ML



(b) MCMC-MPS

Figure 6. Histograms for MCMC draws of α, λ, $R(t)$, and $h(t)$ from electronic components data.

classical setup, we have focused on the likelihood and product of spacing estimates, along with estimates of their corresponding confidence intervals. Using independent gamma priors, the Metropolis-Hastings sampler has been used to approximate the point and credible interval estimates in a Bayesian model based on the likelihood and spacing functions. Various Monte Carlo simulations have been conducted to assess the performance of the proposed methods through four criteria of precision called root mean squared error, mean relative absolute bias, average confidence length, and coverage percentage. To highlight the applicability of the presented estimators, two engineering applications have been analyzed based on observed failure times on samples of yarn and electronic components. To sum up, the numerical results recommended that the acquired estimators, according to the Bayes approach via the product of spacing functions, perform well compared to others.

References

1. Abo-Kasem, O. E., Ibrahim, O., Aljohani, H. M., Hussam, E., Kilai, M., & Aldallal, R. (2022). Statistical analysis based on progressive Type-I censored scheme from alpha power exponential distribution with engineering and medical applications. Journal of Mathematics, 2022.
2. Almetwally, E. M., Almongy, H. M., Rastogi, M. K., & Ibrahim, M. (2020). Maximum product spacing estimation of Weibull distribution under adaptive type-II progressive censoring schemes. Annals of Data Science, 7, 257-279.
3. Alotaibi, R., Elshahhat, A., Rezk, H., & Nassar, M. (2022). Inferences for alpha power exponential distribution using adaptive progressively Type-II hybrid censored data with applications. Symmetry, 14, 651.
4. Alrumayh, A., Weera, W., Khogeer, H. A., & Almetwally, E. M. (2023). Optimal analysis of adaptive type-II progressive censored for new unit-lindley model. Journal of King Saud University-Science, 35(2), 102462.
5. Alshenawy, R., Sabry, M. A., Almetwally, E. M., & Almongy, H. M. (2021). Product spacing of stress–strength under progressive hybrid censored for exponentiated-gumbel distribution. Computers, Materials and Continua, 66(3), 2973-2995.
6. Anatolyev, S., & Kosenok, G. (2005). An alternative to maximum likelihood based on spacings. Econometric Theory, 21(2), 472-476.
7. Ateya, S. F. (2017). Estimation under inverse Weibull distribution based on Balakrishnan's unified hybrid censored scheme. Communications in Statistics-Simulation & Computation, 46(5), 3645-3666.
8. Balakrishnan, N., Rasouli, A., & Farsipour, N. S. (2008). Exact likelihood inference based on an unified hybrid censored sample from the exponential distribution. Journal of Statistical Computation and Simulation, 78, 475-488.
9. Basu, S., Singh, S. K., & Singh, U. (2018). Bayesian inference using product of spacings function for progressive hybrid Type-I censoring scheme. Statistics, 52, 345-363.
10. Chandrasekar, B., Childs, A., & Balakrishnan, N. (2004). Exact likelihood inference for the exponential distribution under generalized Type-I and Type-II hybrid censoring. Naval Research Logistics, 51, 994-1004.

11. Chen, M., & Shao, Q. (1999). Monte Carlo estimation of Bayesian credible and HPD intervals. Journal of Computational and Graphical Statistics, 8, 69–92.

12. Cheng, R. C. H., & Amin, N. A. K. (1983). Estimating parameters in continuous univariate distributions with a shifted origin. Journal of the Royal Statistical Society: Series B, 45, 394-403.

13. Cheng, R. C. H., & Iles, T. C. (1987). Corrected maximum likelihood in non-regular problems. Journal of the Royal Statistical Society: Series B (Methodological), 49(1), 95-101.

14. Cheng, R. C. H., & Traylor, L. (1995). Non-regular maximum likelihood problems. Journal of the Royal Statistical Society: Series B (Methodological), 57(1), 3-24.

15. Childs, A., Chandrasekar, B., Balakrishnan, N., & Kundu, D. (2003). Exact likelihood inference based on Type-I and Type-II hybrid censored samples from the exponential distribution. Annals of the Institute of Statistical Mathematics, 55, 319-330.

16. Coolen, F. P. A., & Newby, M. J. (1990). A note on the use of the product of spacings in Bayesian inference. Department of Mathematics and Computing Science, University of Technology.

17. Dey, S., Elshahhat, A., & Nassar, M. (2023). Analysis of progressive type-II censored gamma distribution. Computational Statistics, 38(1), 481-508.

18. Dutta, S., & Kayal, S. (2022). Bayesian and non-Bayesian inference of Weibull lifetime model based on partially observed competing risks data under unified hybrid censoring scheme. Quality and Reliability Engineering International, 38(7), 3867-3891.

19. Dutta, S., Lio, Y., & Kayal, S. (2024). Parametric inferences using dependent competing risks data with partially observed failure causes from MOBK distribution under unified hybrid censoring. Journal of Statistical Computation and Simulation, 94(2), 376-399.

20. Dutta, S., Ng, H. K. T., & Kayal, S. (2023). Inference for a general family of inverted exponentiated distributions under unified hybrid censoring with partially observed competing risks data. Journal of Computational and Applied Mathematics, 422, 114934.

21. EL-Sagheer, R. M., Mahmoud, M. A. W., & Hasaballah, H. M. (2019). Bayesian estimations using MCMC approach under three-parameter Burr-XII distribution based on unified hybrid censored scheme. Journal of Statistical Theory and Practice, 13, 65.

22. El-Sherpieny, E. S. A., Almetwally, E. M., & Muhammed, H. Z. (2020). Progressive Type-II hybrid censored schemes based on maximum product spacing with application to power Lomax distribution. Physica A: Statistical Mechanics and Its Applications, 553, 124251.

23. Gelman, A., Carlin, J. B., Stern, H. S., Dunson, D. B., Vehtari, A., & Rubin, D. B. (2004). Bayesian data analysis, 2nd ed. Chapman and Hall/CRC.

24. Ghazal, M. & Shihab, Q. (2018). Exponentiated Pareto distribution: A Bayes study utilizing MCMC technique under unified hybrid censoring scheme. Journal of the Egyptian Mathematical Society, 26, 376-394.

25. Ghazal, M. G. M. & Hasaballah, H. M. (2017). Bayesian estimations using MCMC approach under exponentiated Rayleigh distribution based on unified hybrid censored scheme. Journal of Statistics Applications and Probability, 6, 329-344.

26. Greene, W. H. (2012). Econometric analysis, 4th Ed. Prentice Hall, New York, USA.

27. Henningsen, A., & Toomet, O. (2011). maxLik: A package for maximum likelihood estimation in R. Computational Statistics, 26(3), 443-458.

28. Haj Ahmad, H., Almetwally, E. M., Rabaiah, A., & Ramadan, D. A. (2023). Statistical analysis of alpha power inverse Weibull distribution under hybrid censored scheme with applications to ball bearings technology and biomedical data. Symmetry, 15(1), 161.

29. Jeon, Y. E. & Kang, S.-B. (2020). Estimation of the Rayleigh distribution under unified hybrid censoring. Austrian Journal of Statistics, 50, 59-73.

30. Kundu D. (2008). Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring. Technometrics, 50, 144-154.

31. Lawless, J. F. (2011). Statistical models and methods for lifetime data, 2nd Ed.; John Wiley and Sons: Hoboken, NJ, USA.

32. Mahdavi, A., & Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution. Communications in Statistics-Theory and Methods, 46, 6543-6557.

33. Mahmouda, M. A. W., Diab, L., & Ghazal, M. G. M. (2019). On study of exponentiated gamma distribution based on unified hybrid censored data. Al-Azhar Bulletin of Science, 30, 13-27.

34. Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H., & Teller, E. (1953). Equation of state calculations by fast computing machines. The Journal of Chemical Physics, 21(6), 1087-1092.

35. Nassar, M., Afify, A. Z., & Shakhatreh, M. (2020). Estimation methods of alpha power exponential distribution with applications to engineering and medical data. Pakistan Journal of Statistics and Operation Research, 16, 149-166.

36. Nassar, M., Alotaibi, R., & Elshahhat, A. (2024). E-Bayesian Estimation Using Spacing Function for Inverse Lindley Adaptive Type-I Progressively Censored Samples: Comparative Study with Applications. Applied Bionics and Biomechanics, 2024(1), 5567457.

37. Nassar, M., Dey, S., Wang, L., & Elshahhat, A. (2024). Estimation of Lindley constant-stress model via product of spacing with Type-II censored accelerated life data. Communications in Statistics-Simulation and Computation, 53(1), 288-314.

38. Ng, H. K. T., Luo, L., Hu, Y., & Duan, F. (2012). Parameter estimation of three-parameter Weibull distribution based on progressively Type-II censored samples. Journal of Statistical Computation and Simulation, 82, 1661-1678.

39. Panahi, H. (2017). Estimation of the Burr Type-III distribution with application in unified hybrid censored sample of fracture toughness. Journal of Applied Statistics, 44, 2575-2592.

40. Panahi, H., & Sayyareh, A. (2016). Estimation and prediction for a unified hybrid censored Burr Type-XII distribution. Pakistan Journal of Statistics and Operation Research, 86, 55-73.

41. Plummer, M., Best, N., Cowles, K., & Vines, K. (2006). CODA: convergence diagnosis and output analysis for MCMC. R News, 6(1), 7-11.

42. Rabie, A., & Li, J. (2019). Inferences for Burr-X model based on unified hybrid censored data. International Journal of Applied Mathematics, 49, 1-7.

43. Ranneby, B. (1984). The maximum spacing method. An estimation method related to the maximum likelihood method. Scandinavian Journal of Statistics, 93-112.

44. Salah, M. M. (2020). On Progressive Type-II Censored Samples from Alpha Power Exponential Distribution. Journal of Mathematics, 2020(1), 2584184.

45. Salah, M. M., Ahmed, E. A., Alhussain, Z. A., Ahmed, H. H., El-Morshedy, M., and Eliwa, M. S. (2021). Statistical inferences for Yype-II hybrid censoring data from the alpha power exponential distribution. Plos One, 16, e0244316.