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A New Biased Estimation Class to Combat the Multicollinearity in Regression Models: Modified Two–Parameter Liu Estimator

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Abstract: The multicollinearity problem occurrence of the explanatory variables affects the least-squares (LS) estimator seriously in the regression models. The multicollinearity adverse effects on the LS estimation are also investigated by many authors. Instead of the LS estimator, we propose a new modified two–parameter Liu (MTPL) estimator to handle the multicollinearity for the regression model based on two shrinkage parameters (k, d). Also, we give the necessary and sufficient conditions for the outperforming of the proposed MTPL estimator over the LS, ridge, Liu, Kibria-Lukman (KL), modified ridge type (MRT), and modified one–parameter Liu (MOPL) estimators by the scalar mean squared error (MSE) criterion. Optimal biasing parameters of the proposed MTPL estimator are derived. Simulation and real data are used to study the efficiency of the MTPL estimator. The results of the simulation study and two real-life applications show the superiority of the proposed estimator because the MSE of the proposed estimator is smaller than the other estimators.

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1. Introduction

In general, the linear regression model (LRM) is written as

$$y = X\gamma + \varepsilon, \tag{1.1}$$

where $y = (y_1, \dots, y_n)'$ is the response variable vector with order $n \times 1$, $X = \left\{x_{ij}\right\}; i = 1, \dots, n; j = 1, \dots, p$ is the design matrix with order $n \times p$ (where n is the sample size and p is the number of the regression coefficients in the model. Since the model contains an intercept then the number of the explanatory variables is $p - 1$ and $x_{i1} = 1$, $\gamma = \left(\gamma_1, \dots, \gamma_p\right)'$ is the vector of the regression coefficients with order $p \times 1$, and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)'$ is the vector of the error term of the model with order $n \times 1$; which $E(\varepsilon) = 0$ and $Cov(\varepsilon) = E\left(\varepsilon\varepsilon'\right) = \sigma^2 I_n$, where I_n is the identity matrix of order n. The ordinary least squares (OLS) estimator of γ in equation (1.1) is

$$\widehat{\gamma} = \left(X'X\right)^{-1} X'y = Z^{-1}X'y, \tag{1.2}$$

where $Z = X'X$. In the LRM, if the explanatory variables are correlated with each other, this means that there is a multicollinearity problem in the model. So, the OLS estimator does give inefficient (large standard error) estimates. In general, to handle the multicollinearity problem, Hoerl and Kennard [1] provided the ridge regression (RR) estimation class, it is one of the most popular methods to deal with the multicollinearity problem in the regression models. The RR technique is based on adding a biasing constant (k) to the equation (1.2). The ridge estimator is extended for several regression models in many papers, such as [2], [3], [4], [5], [6], [7], [8], [9], and [10].

Liu [11] proposed a new biased estimator to combat the multicollinearity problem in the LRM. Also, he showed that the Liu estimator is better than the ridge and OLS estimators in the presence of multicollinearity. The Liu estimator is also extended for several regression models in many papers, such as [12], [13], [14], [15], and [16].

Recently, Lukman, et al. [17] proposed a modified ridge type (MRT) estimator, and Kibria and Lukman [18] proposed a new version of the ridge type estimator called Kibria-Lukman (KL) estimator for the LRM. The MRT and KL estimators are also extended to several regression models in many papers, such as [19], [20],[21],[22],[23], and [24]. Furthermore, there are several two-parameter estimators suggested for handling the multicollinearity problem in the LRM such as [25], [26], [27], [28], [29], [54], [55], and [56], and in generalized linear models such as [23, 30, 31, 32, 33, 34, 57]

Also, Lukman, et al. [35] developed a new modified one–parameter Liu (MOPL) estimator for the LRM based on only one-parameter (d) as in the classical version of Liu estimator. They showed that the MOPL estimator was more efficient than the classical Liu estimator. Therefore, in this paper, we are interested in developing a new modified Liu estimator based on two shrinkage parameters (k, d) to realize the advantages of biased estimators with two parameters. As expected, our proposed estimator has many advantages as follows: First, its bias and variance are smaller than the others. Second, it is a general estimator as it includes both OLS and MOPL estimators. Third, since our proposed estimator contains two shrinkage parameters, it will be more flexible to obtain a better estimate of the regression parameters of the LRM.

This paper is organized as follows. Section 2 presents some existing estimators in the literature (ridge, Liu, KL, MRT, and MOPL estimators) and the proposed estimator. In section 3, theoretical comparisons between the proposed estimator and the other estimators are provided. In section 4, the optimal values of the biasing parameters of the estimators are presented. In section 5, we performed a simulation study to evaluate the performance of the proposed estimator with OLS, RR, Liu, KL, MRT, and MOPL estimators. Section 6 presents the results of two real-life applications. The conclusion is given in section 7.

2. Some Existing Estimators and the Proposed Estimator

Now, let $\Lambda = diag\left(\lambda_1, \ldots, \lambda_p\right)$ and $F = \left(f_1, \ldots, f_p\right)$ are the eigenvalues and eigenvectors matrices of Z, respectively, such that $F'X'XF = F'ZF = \Lambda$, where $\lambda_1 \geq \cdots \geq \lambda_p \geq 0$. Then the mean squared error matrix (MSEM) and the mean squared error (MSE) of the OLS estimator in equation (1.2) are defined as follows:

$$MSEM(\widehat{\gamma}) = E(\widehat{\gamma} - \gamma)(\widehat{\gamma} - \gamma)' = COV(\widehat{\gamma}) = \sigma^2 \Lambda^{-1}, \tag{2.1}$$

$$MSE(\widehat{\gamma}) = E(\widehat{\gamma} - \gamma)'(\widehat{\gamma} - \gamma) = \sigma^2\, tr\left(\Lambda^{-1}\right) = \sigma^2 \sum_{j=1}^{p} \frac{1}{\lambda_j}, \tag{2.2}$$

where $tr(\bullet)$ is the trace value.

- Hoerl and Kennard [1] proposed the RR estimator as follows:

$$\widehat{\gamma}_k = (Z + k_{Ridge} I_p)^{-1} Z\widehat{\gamma}; \quad k_{Ridge} > 0, \tag{2.3}$$

where I_p is the identity matrix of order p.
The bias vector and the MSEM of $\widehat{\gamma}_{\mathrm{k}}$ are

$$\text{Bias}\,(\widehat{\gamma}_{\mathrm{k}}) = -k_{Ridge}\mathrm{F}(\Lambda + k_{Ridge}\mathrm{I})^{-1}\alpha, \tag{2.4}$$

where $\alpha = F^{'}\gamma$.

$$\text{MSEM}\,(\widehat{\gamma}_{\mathrm{k}}) = \sigma^2\mathrm{F}(\Lambda + k_{Ridge}\mathrm{I})^{-1}\Lambda(\Lambda + k_{Ridge}\mathrm{I})^{-1}\mathrm{F}^{'} + k_{Ridge}^2\mathrm{F}(\Lambda + k_{Ridge}\mathrm{I})^{-1}\alpha\alpha^{'}(\Lambda + k_{Ridge}\mathrm{I})^{-1}F^{'}. \tag{2.5}$$

Then the MSE of $\widehat{\gamma}_{\mathrm{k}}$ is

$$MSE\,(\widehat{\gamma}_k) = tr\,(MSEM\,(\widehat{\gamma}_k)) = \sigma^2 \sum_{j=1}^{p} \frac{\lambda_j}{(\lambda_j + k_{Ridge})^2} + k_{Ridge}^2 \sum_{j=1}^{p} \frac{\alpha_j^2}{(\lambda_j + k_{Ridge})^2}. \tag{2.6}$$

- Liu [11] provided the Liu estimator for the LRM as follows:

$$\widehat{\gamma}_d = (Z + I_p)^{-1}\left(Z + d_{Liu} I_p\right)\widehat{\gamma}; \quad 0 < d_{Liu} < 1. \tag{2.7}$$

The bias vector and the MSEM of $\widehat{\gamma}_{\mathrm{d}}$ are

$$Bias\,(\widehat{\gamma}_d) = F(\Lambda + I_p)^{-1}\alpha(d_{Liu} - 1), \tag{2.8}$$

$$\text{MSEM}\,(\widehat{\gamma}_{\mathrm{d}}) = \sigma^2\mathrm{F}(\Lambda + I_p)^{-1}(\Lambda + d_{Liu}I_p)\Lambda^{-1}(\Lambda + d_{Liu}I_p)(\Lambda + I_p)^{-1}F^{'} + (d_{Liu} - 1)^2\mathrm{F}(\Lambda + I_p)^{-1}\alpha\alpha^{'}(\Lambda + I_p)^{-1}F^{'}. \tag{2.9}$$

Then the MSE of $\widehat{\gamma}_d$ is

$$MSE\,(\widehat{\gamma}_d) = tr\,(MSEM\,(\widehat{\gamma}_d)) = \sigma^2 \sum_{j=1}^{p} \frac{(\lambda_j + d_{Liu})^2}{\lambda_j(\lambda_j + 1)^2} + \sum_{j=1}^{p} \frac{\alpha_j^2(d_{Liu} - 1)^2}{(\lambda_j + 1)^2}. \tag{2.10}$$

- Lukman, et al. [17] provided the MRT estimator for the LRM as follows:

$$\widehat{\gamma}_{MRT} = (Z + k_{MRT}(1 + d_{MRT})I_p)^{-1} Z\widehat{\gamma}; \quad k_{MRT} > 0,\ 0 < d_{MRT} < 1. \tag{2.11}$$

The bias vector and the MSEM of $\widehat{\gamma}_{\mathrm{MRT}}$ are

$$Bias\,(\widehat{\gamma}_{MRT}) = F((\Lambda + k_{MRT}\,(1 + d_{MRT})\,I_p)^{-1}\Lambda - I_p)\alpha, \tag{2.12}$$

$$\begin{aligned} MSEM\,(\widehat{\gamma}_{MRT}) = {} & \sigma^2 F(\Lambda + k_{MRT}\,(1 + d_{MRT})\,I_p)^{-1}\Lambda\Lambda^{-1}\Lambda(\Lambda + k_{MRT}\,(1 + d_{MRT})\,I_p)^{-1}F^{'} + \\ & F((\Lambda + k_{MRT}\,(1 + d_{MRT})\,I_p)^{-1}\Lambda - I_p)\alpha\alpha^{'}((\Lambda + k_{MRT}\,(1 + d_{MRT})\,I_p)^{-1}\Lambda - I)F^{'} \end{aligned} \tag{2.13}$$

Then the MSE of $\widehat{\gamma}_{MRT}$ is

$$MSE\,(\widehat{\gamma}_{MRT}) = tr\,(MSEM\,(\widehat{\gamma}_{MRT})) = \sum_{j=1}^{p} \frac{\sigma^2\lambda_j}{(\lambda_j + k_{MRT}\,(1 + d_{MRT}))^2} + \sum_{j=1}^{p} \frac{k_{MRT}^2(1 + d_{MRT})^2\alpha_j^2}{(\lambda_j + k_{MRT}\,(1 + d_{MRT}))^2}. \tag{2.14}$$

- Kibria and Lukman [18] provided the KL estimator for the LRM as follows:

$$\widehat{\gamma}_{KL} = (Z + k_{KL}I_p)^{-1}(Z - k_{KL}I_p)\widehat{\gamma}; \quad k_{KL} > 0. \tag{2.15}$$

The bias vector and the MSEM of $\widehat{\gamma}_{\mathrm{KL}}$ are

$$\mathrm{Bias}\,(\widehat{\gamma}_{\mathrm{KL}}) = \mathrm{F}((\Lambda{+}k_{KL}I_p)^{-1}(\Lambda{-}k_{KL}I_p) - \mathrm{I})\alpha, \tag{2.16}$$

$$\begin{aligned}\mathrm{MSEM}\,(\widehat{\gamma}_{\mathrm{KL}}) =&\sigma^2\mathrm{F}(\Lambda{+}k_{KL}I_p)^{-1}\left(\Lambda{-}k_{KL}I_p\right)\Lambda^{-1}\left(\Lambda{-}k_{KL}I_p\right)(\Lambda{+}k_{KL}I_p)^{-1}F^{'}\\ &+\mathrm{F}((\Lambda{+}k_{KL}I_p)^{-1}(\Lambda{-}k_{KL}I_p) - \mathrm{I})\alpha\alpha^{'}((\Lambda{+}k_{KL}I_p)^{-1}(\Lambda{-}k_{KL}I_p) - \mathrm{I})\mathrm{F}^{'}.\end{aligned} \tag{2.17}$$

Then the MSE of $\widehat{\gamma}_{\mathrm{KL}}$ is

$$\mathrm{MSE}\,(\widehat{\gamma}_{KL}) = \mathrm{tr}\,(\mathrm{MSEM}\,(\widehat{\gamma}_{KL})) = \sum_{\mathrm{j}=1}^{p}\frac{\sigma^2(\lambda_{\mathrm{j}}{-}k_{KL})^2}{\lambda_{\mathrm{j}}(\lambda_{\mathrm{j}}{+}k_{KL})^2} + \sum_{\mathrm{j}=1}^{p}\frac{4k_{KL}^2\alpha_{\mathrm{j}}^2}{(\lambda_{\mathrm{j}}{+}k_{KL})^2}. \tag{2.18}$$

- Lukman, et al. [35] proposed a new one–parameter Liu estimator for the LRM called the modified one–parameter Liu (MOPL) estimator and is defined as

$$\widehat{\gamma}_{d_0} = (Z{+}I_p)^{-1}\left(Z{-}d_{MOPL}I_p\right)\widehat{\gamma}; \quad 0 < d_{MOPL} < 1. \tag{2.19}$$

The bias vector and the MSEM of $\widehat{\gamma}_{d_0}$ are

$$\mathrm{Bias}\,(\widehat{\gamma}_{d_0}) = -\mathrm{F}(\Lambda{+}I_p)^{-1}\alpha(d_{MOPL}{+}1), \tag{2.20}$$

$$\begin{aligned}\mathrm{MSEM}\,(\widehat{\gamma}_{d_0}) =&\sigma^2\mathrm{F}(\Lambda{+}I_p)^{-1}(\Lambda{-}d_{MOPL}I_p)\Lambda^{-1}(\Lambda{-}d_{MOPL}I_p)(\Lambda{+}I_p)^{-1}F^{'}\\ &+(d_{MOPL}{+}1)^2\mathrm{F}(\Lambda{+}I_p)^{-1}\alpha\alpha^{'}(\Lambda{+}I_p)^{-1}F^{'}.\end{aligned} \tag{2.21}$$

Then the MSE of $\widehat{\gamma}_{d_0}$ is

$$MSE\,(\widehat{\gamma}_{d_0}) = tr\,(MSEM\,(\widehat{\gamma}_{d_0})) = \sigma^2\sum_{j=1}^{p}\frac{(\lambda_j - d_{MOPL})^2}{\lambda_j(\lambda_j + 1)^2} + \sum_{j=1}^{p}\frac{\alpha_j^2(d_{MOPL} + 1)^2}{(\lambda_j + 1)^2}. \tag{2.22}$$

Recently, the MOPL estimator is developed for the Conway–Maxwell Poisson regression model by [16].

- Following [35], we propose a new modified Liu estimator for the LRM based on $(k_{\mathrm{MTPL}}, d_{\mathrm{MTPL}})$ together. Our proposed estimator is obtained by augmenting $-(k_{\mathrm{MTPL}}{+}d_{\mathrm{MTPL}})\widehat{\gamma} = \gamma + \varepsilon$ to the LRM in equation (1.1) and then using the OLS estimator. In other words, the proposed estimator of γ is obtain by minimizing $(\gamma + (k_{\mathrm{MTPL}}{+}d_{\mathrm{MTPL}})\gamma^*)^{'}(\gamma + (k_{\mathrm{MTPL}}{+}d_{\mathrm{MTPL}})\gamma^*)$ subject to $(y - \mathrm{X}\gamma^*)^{'}(y - \mathrm{X}\gamma^*) = c$, where γ^* is an estimator, c is a constant, and $k_{\mathrm{MTPL}}, d_{\mathrm{MTPL}}$ are the Lagrangian multipliers. Then the proposed estimator of γ is given by

$$\widehat{\gamma}_{k,d_0} = (Z + I_p)^{-1}\left(Z - (k_{MTPL} + d_{MTPL})\,I_p\right)\widehat{\gamma}; \qquad k_{MTPL} > 0,\ 0 < d_{MTPL} < 1. \tag{2.23}$$

We will call the proposed estimator the "Modified Two–Parameter Liu'' (MTPL) estimator.
The bias vector and the MSEM of $\widehat{\gamma}_{k,d_0}$ are

$$Bias\,(\widehat{\gamma}_{k,d_0}) = -F(\Lambda + I_p)^{-1}\alpha(k_{MTPL} + d_{MTPL} + 1), \tag{2.24}$$

$$\begin{aligned}MSEM\,(\widehat{\gamma}_{k,d_0}) =&\sigma^2F(\Lambda + I_p)^{-1}(\Lambda - (k_{MTPL} + d_{MTPL})\,I_p)\Lambda^{-1}(\Lambda - (k_{MTPL} + d_{MTPL})\,I_p)(\Lambda + I_p)^{-1}F^{'} +\\ &(k_{MTPL} + d_{MTPL} + 1)^2F(\Lambda + I_p)^{-1}\alpha\alpha^{'}(\Lambda + I_p)^{-1}F^{'}.\end{aligned} \tag{2.25}$$

Then the MSE of $\widehat{\gamma}_{k,d_0}$ is

$$MSE\left(\widehat{\gamma}_{k,d_0}\right) = tr\left(MSEM\left(\widehat{\gamma}_{k,d_0}\right)\right) = \sigma^2 \sum_{j=1}^{p} \frac{(\lambda_j - k_{MTPL} - d_{MTPL})^2}{\lambda_j(\lambda_j+1)^2} + \sum_{j=1}^{p} \frac{\alpha_j^2(k_{MTPL} + d_{MTPL} + 1)^2}{(\lambda_j+1)^2}. \tag{2.26}$$

We can get the optimal k_{MTPL} for $\widehat{\gamma}_{k,d_0}$ by setting $\left(\partial \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right)/\partial k_{\mathrm{MTPL}}\right) = 0$, we have $\sigma^2\left(\lambda_j - k_{MTPL} - d_{MTPL}\right) = \lambda_j\alpha_j^2\left(k_{MTPL} + d_{MTPL} + 1\right)$, then the optimal k_{MTPL} is

$$k_{MTPL(opt)} = \frac{\lambda_j\left(\sigma^2 - \alpha_j^2\right)}{\sigma^2 + \lambda_j\alpha_j^2} - d_{MTPL}. \tag{2.27}$$

It notes that $k_{MTPL(opt)} > 0$ if and only if $\frac{\sigma^2(\lambda_j - d_{MTPL})}{\lambda_j(1+d_{MTPL})} > \alpha_j^2$. For this condition to be fulfilled, it must be $\sigma^2 > \alpha_j^2$.
Also, we can get the optimal d_{MTPL} for $\widehat{\gamma}_{k,d_0}$ by setting $\left(\partial \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right)/\partial d_{\mathrm{MTPL}}\right) = 0$:

$$d_{MTPL(opt)} = \frac{\sigma^2 - \alpha_{\mathrm{j}}^2}{\left(\sigma^2/\lambda_{\mathrm{j}}\right) + \alpha_{\mathrm{j}}^2} - k_{\mathrm{MTPL}}. \tag{2.28}$$

It notes that $0 < d_{MTPL(opt)} < 1$ if and only if $k_{\mathrm{MTPL}} < \frac{\sigma^2 - \alpha_{\mathrm{j}}^2}{(\sigma^2/\lambda_{\mathrm{j}}) + \alpha_{\mathrm{j}}^2} < (1 + k_{\mathrm{MTPL}})$. Also, for this condition to be fulfilled, it must be $\sigma^2 > \alpha_j^2$.

3. The Superiority of the Proposed Estimator

In this section, we will study and determine the main conditions to make the MTPL estimator perform well. The following six theorems demonstrate that our proposed MTPL estimator outperforms the OLS estimator, one-parameter estimators (RR, Liu, and KL), and two-parameter estimators (MRT and MOPL), respectively.

3.1. Comparison among OLS and MTPL estimators

Theorem 1. The MTPL estimator is better than the OLS estimator if $\sigma^2\left(A_j^2 - B_j^2\right) > (k_{\mathrm{MTPL}} + d_{\mathrm{MTPL}} + 1)^2\lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2$; for all j, $k_{\mathrm{MTPL}} > 0$, $0 < d_{\mathrm{MTPL}} < 1$, where $A = \left(\Lambda + I_p\right) = diag\left(A_j\right); A_j = \left(\lambda_{\mathrm{j}} + 1\right)$, $B = \Lambda - \left(k_{\mathrm{MTPL}} + d_{\mathrm{MTPL}}\right) I_p = diag\left(B_j\right); B_j = \left(\lambda_{\mathrm{j}} - k_{\mathrm{MTPL}} - d_{\mathrm{MTPL}}\right)$.

Proof.

The difference among the MSEM of the OLS and MTPL estimators is

$$\begin{aligned}\mathrm{MSEM}(\hat{\gamma}) - \mathrm{MSEM}(\hat{\gamma}_{k,d_0}) = \sigma^2\Lambda^{-1} & \\ & - \sigma^2 \mathrm{F}(\Lambda + I_p)^{-1}(\Lambda - (k_{\mathrm{MTPL}} + d_{\mathrm{MTPL}})I_p)\Lambda^{-1}(\Lambda - (k_{\mathrm{MTPL}} + d_{\mathrm{MTPL}})I_p)(\Lambda + I_p)^{-1}F' \\ & - (k_{\mathrm{MTPL}} + d_{\mathrm{MTPL}} + 1)^2 \mathrm{F}(\Lambda + I_p)^{-1}\alpha\alpha'(\Lambda + I_p)^{-1}F'.\end{aligned}$$

$$\mathrm{MSEM}\left(\widehat{\gamma}\right) - \mathrm{MSEM}\left(\widehat{\gamma}_{k,d_0}\right) = \sigma^2\Lambda^{-1} - \sigma^2 \mathrm{F}A^{-1}\mathrm{B}\,\Lambda^{-1}\mathrm{B}\,A^{-1}F' - (k_{\mathrm{MTPL}} + d_{\mathrm{MTPL}} + 1)^2 \mathrm{F}A^{-1}\alpha\alpha' A^{-1}F'.$$

where $A = \left(\Lambda + I_p\right) = diag\left(A_j\right); A_j = \left(\lambda_{\mathrm{j}} + 1\right)$, $B = \Lambda - \left(k_{\mathrm{MTPL}} + d_{\mathrm{MTPL}}\right) I_p = diag\left(B_j\right); B_j = \left(\lambda_{\mathrm{j}} - k_{\mathrm{MTPL}} - d_{\mathrm{MTPL}}\right)$. Then

$$\mathrm{MSE}\left(\widehat{\gamma}\right) - \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right) = \sum_{j=1}^{p}\left(\frac{\sigma^2}{\lambda_{\mathrm{j}}} - \frac{\sigma^2 B_j^2 + \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2(k_{\mathrm{MTPL}} + d_{\mathrm{MTPL}} + 1)^2}{\lambda_{\mathrm{j}}A_j^2}\right),$$

$\mathrm{MSE}\left(\widehat{\gamma}\right) - \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right) > 0$ if $\sigma^2 A_j^2 - \sigma^2 B_j^2 - \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2(k_{\mathrm{MTPL}} + d_{\mathrm{MTPL}} + 1)^2 > 0 \rightarrow$
$\sigma^2\left(A_j^2 - B_j^2\right) > \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2(k_{\mathrm{MTPL}} + d_{\mathrm{MTPL}} + 1)^2$. Then $\mathrm{MSE}\left(\widehat{\gamma}\right) > \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right)$ if $\sigma^2\left(A_j^2 - B_j^2\right) > \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2(k_{\mathrm{MTPL}} + d_{\mathrm{MTPL}} + 1)^2$, for all j.
The proof is completed.

3.2. Comparison among RR and MTPL estimators

Theorem 2. The MTPL estimator is better than the RR estimator if $\sigma^2\left(\lambda_j^2 A_j^2 - B_j^2(\lambda_\mathrm{j}+\mathrm{k})^2\right) > \lambda_\mathrm{j}\alpha_\mathrm{j}^2\left((\lambda_\mathrm{j}+\mathrm{k})^2(\mathrm{k}+d_{\mathrm{MTPL}}+1)^2 - k^2A_j^2\right)$; for all j, $k = k_{\mathrm{MTPL}} = k_{\mathrm{Ridge}} > 0,\ 0 < d_{\mathrm{MTPL}} < 1$.

Proof.

The difference among the MSEM of the RR and MTPL estimators under the assumption that $k_{Ridge} = k_{MTPL} = k$ is

$$\begin{aligned}\mathrm{MSEM}(\widehat{\gamma}_k) - \mathrm{MSEM}(\widehat{\gamma}_{k,d_0}) &= \sigma^2 F(\Lambda + kI_p)^{-1}\Lambda(\Lambda + kI_p)^{-1}F' \\ &\quad + k^2F(\Lambda + kI_p)^{-1}\alpha\alpha'(\Lambda + kI_p)^{-1}F' \\ &\quad - \sigma^2 FA^{-1}B\Lambda^{-1}BA^{-1}F' \\ &\quad - (k + d_{\mathrm{MTPL}} + 1)^2FA^{-1}\alpha\alpha'A^{-1}F'.\end{aligned}$$

Then

$$\mathrm{MSE}\left(\widehat{\gamma}_\mathrm{k}\right) - \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right) = \sum_{j=1}^{p}\left(\frac{\sigma^2\lambda_\mathrm{j} + \mathrm{k}^2\alpha_\mathrm{j}^2}{(\lambda_\mathrm{j}+\mathrm{k})^2} - \frac{\sigma^2\ B_j^2 + \lambda_\mathrm{j}\alpha_\mathrm{j}^2(\mathrm{k}+d_{\mathrm{MTPL}}+1)^2}{\lambda_\mathrm{j}A_j^2}\right),$$

$\mathrm{MSE}\left(\widehat{\gamma}_\mathrm{k}\right) - \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right) > 0$ if $\sigma^2\lambda_j^2A_j^2 + k^2\lambda_\mathrm{j}A_j^2\alpha_\mathrm{j}^2 - \sigma^2\ B_j^2(\lambda_\mathrm{j}+\mathrm{k})^2 - \lambda_\mathrm{j}\alpha_\mathrm{j}^2(\lambda_\mathrm{j}+\mathrm{k})^2(\mathrm{k}+d_{\mathrm{MTPL}}+1)^2 > 0 \rightarrow \sigma^2\left(\lambda_j^2A_j^2 - B_j^2(\lambda_\mathrm{j}+\mathrm{k})^2\right) > \lambda_\mathrm{j}\alpha_\mathrm{j}^2\left((\lambda_\mathrm{j}+\mathrm{k})^2(\mathrm{k}+d_{\mathrm{MTPL}}+1)^2 - k^2A_j^2\right)$. Then $\mathrm{MSE}\left(\widehat{\gamma}_k\right) > \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right)$ if $\sigma^2\left(\lambda_j^2A_j^2 - B_j^2(\lambda_\mathrm{j}+\mathrm{k})^2\right) > \lambda_\mathrm{j}\alpha_\mathrm{j}^2\left((\lambda_\mathrm{j}+\mathrm{k})^2(\mathrm{k}+d_{\mathrm{MTPL}}+1)^2 - k^2A_j^2\right)$, for all j, $k > 0$, and $0 < d_{\mathrm{MTPL}} < 1$. The proof is completed.

3.3. Comparison among Liu and MTPL estimators

Theorem 3.

Case (I): When $d_{\mathrm{MTPL}} = d_{\mathrm{Liu}} = d$, then the MTPL estimator is better than the Liu estimator if $\sigma^2\left((\lambda_\mathrm{j}+\mathrm{d})^2 - \left(\lambda_\mathrm{j} - k_{\mathrm{MTPL}} - d\right)^2\right) > \lambda_\mathrm{j}\alpha_\mathrm{j}^2\left(k_{\mathrm{MTPL}}\left(k_{\mathrm{MTPL}} + 2\right) + d\left(2k_{\mathrm{MTPL}} + 4\right)\right)$; for all j, $k_{\mathrm{MTPL}} > 0,\ 0 < d < 1$.

Case (II): When $d_{\mathrm{MTPL}} \neq d_{\mathrm{Liu}}$, then the MTPL estimator is better than the Liu estimator if $\sigma^2\left((\lambda_\mathrm{j}+d_{\mathrm{Liu}})^2 - B_j^2\right) > \lambda_\mathrm{j}\alpha_\mathrm{j}^2\left((k_{\mathrm{MTPL}}+d_{\mathrm{MTPL}}+1)^2 - (d_{\mathrm{Liu}}-1)^2\right)$; for all j, $k_{\mathrm{MTPL}} > 0,\ 0 < d_{\mathrm{MTPL}}, d_{\mathrm{Liu}} < 1$.

Proof.

The difference among the MSEM of the Liu and MTPL estimators is

$$\begin{aligned}\mathrm{MSEM}(\widehat{\gamma}_d) - \mathrm{MSEM}(\widehat{\gamma}_{k,d_0}) &= \sigma^2\mathrm{F}A^{-1}(\Lambda + d_{\mathrm{Liu}}I_p)\Lambda^{-1}(\Lambda + d_{\mathrm{Liu}}I_p)A^{-1}F' \\ &\quad + (d_{\mathrm{Liu}} - 1)^2\mathrm{F}A^{-1}\alpha\alpha'A^{-1}F' \\ &\quad - \sigma^2\mathrm{F}A^{-1}\mathrm{B}\,\Lambda^{-1}\mathrm{B}\,A^{-1}F' \\ &\quad - (k_{\mathrm{MTPL}} + d_{\mathrm{MTPL}} + 1)^2\mathrm{F}A^{-1}\alpha\alpha'A^{-1}F'.\end{aligned}$$

Case (I): If $d_{\mathrm{MTPL}} = d_{\mathrm{Liu}} = d$, then

$$\mathrm{MSE}\left(\widehat{\gamma}_\mathrm{d}\right) - \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right) = \sum_{j=1}^{p}\left(\frac{\sigma^2(\lambda_\mathrm{j}+\mathrm{d})^2 + (\mathrm{d}-1)^2\lambda_\mathrm{j}\alpha_\mathrm{j}^2}{\lambda_\mathrm{j}A_j^2} - \frac{\sigma^2\left(\lambda_\mathrm{j} - k_{\mathrm{MTPL}} - d\right)^2 + \lambda_\mathrm{j}\alpha_\mathrm{j}^2(k_{\mathrm{MTPL}}+d+1)^2}{\lambda_\mathrm{j}A_j^2}\right),$$

$\mathrm{MSE}\left(\widehat{\gamma}_\mathrm{d}\right) - \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right) > 0$ if $\sigma^2(\lambda_\mathrm{j}+\mathrm{d})^2 + (\mathrm{d}-1)^2\lambda_\mathrm{j}\alpha_\mathrm{j}^2 - \sigma^2\left(\lambda_\mathrm{j} - k_{\mathrm{MTPL}} - d\right)^2 - \lambda_\mathrm{j}\alpha_\mathrm{j}^2(k_{\mathrm{MTPL}}+d+1)^2 > 0 \rightarrow \sigma^2\left((\lambda_\mathrm{j}+\mathrm{d})^2 - \left(\lambda_\mathrm{j} - k_{\mathrm{MTPL}} - d\right)^2\right) > \lambda_\mathrm{j}\alpha_\mathrm{j}^2\left((k_{\mathrm{MTPL}}+d+1)^2 - (\mathrm{d}-1)^2\right) \rightarrow \sigma^2\left((\lambda_\mathrm{j}+\mathrm{d})^2 - \left(\lambda_\mathrm{j} - k_{\mathrm{MTPL}} - d\right)^2\right) > \lambda_\mathrm{j}\alpha_\mathrm{j}^2\left(k_{\mathrm{MTPL}}\left(k_{\mathrm{MTPL}} + 2\right) + d\left(2k_{\mathrm{MTPL}} + 4\right)\right)$. Then $\mathrm{MSE}\left(\widehat{\gamma}_\mathrm{d}\right) > \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right)$ if $\sigma^2\left((\lambda_\mathrm{j}+\mathrm{d})^2 - \left(\lambda_\mathrm{j} - k_{\mathrm{MTPL}} - d\right)^2\right) > \lambda_\mathrm{j}\alpha_\mathrm{j}^2\left(k_{\mathrm{MTPL}}\left(k_{\mathrm{MTPL}} + 2\right) + d\left(2k_{\mathrm{MTPL}} + 4\right)\right)$, for all j, $k_{\mathrm{MTPL}} > 0,\ 0 < d < 1$.

Case (II): If $d_{\text{MTPL}} \neq d_{\text{Liu}}$, then

$$\text{MSE}\left(\widehat{\gamma}_{\text{d}}\right) - \text{MSE}\left(\widehat{\gamma}_{k,d_0}\right) = \sum_{j=1}^{p}\left(\frac{\sigma^2(\lambda_{\text{j}}+d_{\text{Liu}})^2 + (d_{\text{Liu}}-1)^2\lambda_{\text{j}}\alpha_{\text{j}}^2}{\lambda_{\text{j}}A_j^2} - \frac{\sigma^2\ B_j^2 + \lambda_{\text{j}}\alpha_{\text{j}}^2(k_{\text{MTPL}}+d_{\text{MTPL}}+1)^2}{\lambda_{\text{j}}A_j^2}\right),$$

$\text{MSE}\left(\widehat{\gamma}_{\text{d}}\right) - \text{MSE}\left(\widehat{\gamma}_{k,d_0}\right) > 0$ if $\sigma^2(\lambda_{\text{j}}+d_{\text{Liu}})^2 + (d_{\text{Liu}}-1)^2\lambda_{\text{j}}\alpha_{\text{j}}^2 - \sigma^2 B_j^2 - \lambda_{\text{j}}\alpha_{\text{j}}^2(k_{\text{MTPL}}+d_{\text{MTPL}}+1)^2 > 0 \rightarrow \sigma^2\left((\lambda_{\text{j}}+d_{\text{Liu}})^2 - B_j^2\right) > \lambda_{\text{j}}\alpha_{\text{j}}^2\left((k_{\text{MTPL}}+d_{\text{MTPL}}+1)^2 - (d_{\text{Liu}}-1)^2\right)$. Then $\text{MSE}\left(\widehat{\gamma}_{\text{d}}\right) > \text{MSE}\left(\widehat{\gamma}_{k,d_0}\right)$ if $\sigma^2\left((\lambda_{\text{j}}+d_{\text{Liu}})^2 - B_j^2\right) > \lambda_{\text{j}}\alpha_{\text{j}}^2\left((k_{\text{MTPL}}+d_{\text{MTPL}}+1)^2 - (d_{\text{Liu}}-1)^2\right)$, for all j, $k_{\text{MTPL}} > 0,\ 0 < d_{\text{MTPL}}, d_{\text{Liu}} < 1$. The proof is completed.

3.4. Comparison among KL and MTPL estimators

Theorem 4.
The MTPL estimator is better than the KL estimator if $\sigma^2\left(A_j^2(\lambda_{\text{j}}-\text{k})^2 - B_j^2(\lambda_{\text{j}}+\text{k})^2\right) > \lambda_{\text{j}}\alpha_{\text{j}}^2\left((\lambda_{\text{j}}+\text{k})^2(\text{k}+d_{\text{MTPL}}+1)^2 - 4k^2A_j^2\right)$; for all j, $k_{KL} = k_{MTPL} = k > 0,\ 0 < d_{\text{MTPL}} < 1$.

Proof.
The difference among the MSEM of the KL and MTPL estimators under the assumption that $k_{KL} = k_{MTPL} = k$ is

$$\begin{aligned}\text{MSEM}(\hat{\gamma}_{\text{KL}}) - \text{MSEM}(\hat{\gamma}_{k,d_0}) &= \sigma^2\text{F}\left([\Lambda + \text{k}I_p]^{-1}(\Lambda - \text{k}I_p)\Lambda^{-1}(\Lambda - \text{k}I_p)(\Lambda + \text{k}I_p)^{-1}\right)F' \\ &+ \text{F}\left((\Lambda + \text{k}I_p)^{-1}(\Lambda - \text{k}I_p) - I_p\right)\alpha\alpha'\left((\Lambda + \text{k}I_p)^{-1}(\Lambda - \text{k}I_p) - I_p\right)\text{F}' \\ &- \sigma^2\text{F}A^{-1}\text{B}\Lambda^{-1}\text{B}A^{-1}F' \\ &- (\text{k} + d_{\text{MTPL}} + 1)^2\text{F}A^{-1}\alpha\alpha'A^{-1}F'.\end{aligned}$$

Then

$$\text{MSE}\left(\widehat{\gamma}_{KL}\right) - \text{MSE}\left(\widehat{\gamma}_{k,d_0}\right) = \sum_{j=1}^{p}\left(\frac{\sigma^2(\lambda_{\text{j}}-\text{k})^2 + 4\text{k}^2\lambda_{\text{j}}\alpha_{\text{j}}^2}{\lambda_{\text{j}}(\lambda_{\text{j}}+\text{k})^2} - \frac{\sigma^2\ B_j^2 + \lambda_{\text{j}}\alpha_{\text{j}}^2(\text{k}+d_{\text{MTPL}}+1)^2}{\lambda_{\text{j}}A_j^2}\right),$$

$\text{MSE}\left(\widehat{\gamma}_{\text{KL}}\right) - \text{MSE}\left(\widehat{\gamma}_{k,d_0}\right) > 0$ if $\sigma^2A_j^2(\lambda_{\text{j}}-\text{k})^2 + 4\text{k}^2A_j^2\lambda_{\text{j}}\alpha_{\text{j}}^2 - \sigma^2\ (\lambda_{\text{j}}+\text{k})^2B_j^2 - \lambda_{\text{j}}\alpha_{\text{j}}^2(\lambda_{\text{j}}+\text{k})^2(\text{k}+d_{\text{MTPL}}+1)^2 > 0 \rightarrow \sigma^2\left(A_j^2(\lambda_{\text{j}}-\text{k})^2 - B_j^2(\lambda_{\text{j}}+\text{k})^2\right) > \lambda_{\text{j}}\alpha_{\text{j}}^2\left((\lambda_{\text{j}}+\text{k})^2(\text{k}+d_{\text{MTPL}}+1)^2 - 4k^2A_j^2\right)$. Then $\text{MSE}\left(\widehat{\gamma}_{KL}\right) > \text{MSE}\left(\widehat{\gamma}_{k,d_0}\right)$ if $\sigma^2\left(A_j^2(\lambda_{\text{j}}-\text{k})^2 - B_j^2(\lambda_{\text{j}}+\text{k})^2\right) > \lambda_{\text{j}}\alpha_{\text{j}}^2\left((\lambda_{\text{j}}+\text{k})^2(\text{k}+d_{\text{MTPL}}+1)^2 - 4k^2A_j^2\right)$, for all j, $k > 0,\ 0 < d_{\text{MTPL}} < 1$. The proof is completed.

3.5. Comparison among MRT and MTPL estimators

Theorem 5.
Case (I): When $d_{\text{MTPL}} = d_{\text{MRT}} = d$ and $k_{\text{MTPL}} = k_{\text{MRT}} = k$, then the MTPL estimator is better than the MRT estimator if $\sigma^2\left(\lambda_j^2A_j^2 - \left(\lambda_{\text{j}}-\text{k}-d\right)^2(\lambda_{\text{j}}+\text{k}\,(1+\text{d})\,)^2\right) > \lambda_{\text{j}}\alpha_{\text{j}}^2\left((\lambda_{\text{j}}+\text{k}\,(1+\text{d})\,)^2(\text{k}+d+1)^2 - \text{k}^2A_j^2(1+\text{d})^2\right)$; for all j, $k > 0,\ 0 < d < 1$.
Case (II): When $d_{\text{MTPL}} \neq d_{\text{MRT}}$ and $k_{\text{MTPL}} = k_{\text{MRT}} = k$, then the MTPL estimator is better than the MRT estimator if $\sigma^2\left(\lambda_j^2A_j^2 - B_j^2(\lambda_{\text{j}}+\text{k}\,(1+d_{\text{MRT}})\,)^2\right) > \lambda_{\text{j}}\alpha_{\text{j}}^2\left((\lambda_{\text{j}}+\text{k}\,(1+d_{\text{MRT}})\,)^2(\text{k}+d_{\text{MTPL}}+1)^2 - \text{k}^2A_j^2(1+d_{\text{MRT}})^2\right)$; for all j, $k > 0,\ 0 < d_{\text{MTPL}}, d_{\text{MRT}} < 1$.

Proof.
The difference among the MSEM of the MRT and MTPL estimators is

$$\begin{aligned}\text{MSEM}(\hat{\gamma}_{\text{MRT}}) - \text{MSEM}(\hat{\gamma}_{k,d_0}) &= \sigma^2F(\Lambda + k_{\text{MRT}}(1 + d_{\text{MRT}})I_p)^{-1}\Lambda\Lambda^{-1}\Lambda(\Lambda + k_{\text{MRT}}(1 + d_{\text{MRT}})I_p)^{-1}F' \\ &+ F((\Lambda + k_{\text{MRT}}[(1 + d_{\text{MRT}})I_p])^{-1}\Lambda - I_p)\alpha\alpha'((\Lambda + k_{\text{MRT}}(1 + d_{\text{MRT}})I_p)^{-1}\Lambda - I_p)F' \\ &- \sigma^2FA^{-1}B\Lambda^{-1}B\Lambda^{-1}A^{-1}F' \\ &- (k_{\text{MTPL}} + d_{\text{MTPL}} + 1)^2FA^{-1}\alpha\alpha'A^{-1}F'\end{aligned}$$

Case (I): If $d_{\mathrm{MTPL}} = d_{\mathrm{MRT}} = d$ and $k_{\mathrm{MTPL}} = k_{\mathrm{MRT}} = k$, then

$$\mathrm{MSE}\left(\widehat{\gamma}_{\mathrm{MRT}}\right) - \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right) = \sum_{j=1}^{p}\left(\frac{\sigma^2\lambda_{\mathrm{j}} + \mathrm{k}^2(1+\mathrm{d})^2\alpha_{\mathrm{j}}^2}{(\lambda_{\mathrm{j}}+\mathrm{k}\,(1+\mathrm{d}))^2} - \frac{\sigma^2\left(\lambda_{\mathrm{j}}-\mathrm{k}-d\right)^2 + \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2(\mathrm{k}+d+1)^2}{\lambda_{\mathrm{j}}A_j^2}\right),$$

$\mathrm{MSE}\left(\widehat{\gamma}_{\mathrm{MRT}}\right) - \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right) > 0$ if $\sigma^2\lambda_j^2A_j^2 + \mathrm{k}^2(1+\mathrm{d})^2\alpha_{\mathrm{j}}^2\lambda_{\mathrm{j}}A_j^2 - \sigma^2\left(\lambda_{\mathrm{j}}-\mathrm{k}-d\right)^2(\lambda_{\mathrm{j}}+\mathrm{k}\,(1+\mathrm{d}))^2 - \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2(\lambda_{\mathrm{j}}+\mathrm{k}\,(1+\mathrm{d}))^2(\mathrm{k}+d+1)^2 > 0 \rightarrow \sigma^2\left(\lambda_j^2A_j^2 - \left(\lambda_{\mathrm{j}}-\mathrm{k}-d\right)^2(\lambda_{\mathrm{j}}+\mathrm{k}\,(1+\mathrm{d}))^2\right) > \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2\left((\lambda_{\mathrm{j}}+\mathrm{k}\,(1+\mathrm{d}))^2(\mathrm{k}+d+1)^2 - \mathrm{k}^2A_j^2(1+\mathrm{d})^2\right)$. Then $\mathrm{MSE}\left(\widehat{\gamma}_{\mathrm{MRT}}\right) > \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right)$ if $\sigma^2\left(\lambda_j^2A_j^2 - \left(\lambda_{\mathrm{j}}-\mathrm{k}-d\right)^2(\lambda_{\mathrm{j}}+\mathrm{k}\,(1+\mathrm{d}))^2\right) > \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2\left((\lambda_{\mathrm{j}}+\mathrm{k}\,(1+\mathrm{d}))^2(\mathrm{k}+d+1)^2 - \mathrm{k}^2A_j^2(1+\mathrm{d})^2\right)$, for all $j,\ k > 0,\ 0 < d < 1$.

Case (II): If $d_{\mathrm{MTPL}} \neq d_{\mathrm{MRT}}$ and $k_{\mathrm{MTPL}} = k_{\mathrm{MRT}} = k$, then

$$\mathrm{MSE}\left(\widehat{\gamma}_{\mathrm{MRT}}\right) - \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right) = \sum_{j=1}^{p}\left(\frac{\sigma^2\lambda_{\mathrm{j}} + \mathrm{k}^2(1+d_{\mathrm{MRT}})^2\alpha_{\mathrm{j}}^2}{(\lambda_{\mathrm{j}}+\mathrm{k}\,(1+d_{\mathrm{MRT}}))^2} - \frac{\sigma^2\,B_j^2 + \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2(\mathrm{k}+d_{\mathrm{MTPL}}+1)^2}{\lambda_{\mathrm{j}}A_j^2}\right),$$

$\mathrm{MSE}\left(\widehat{\gamma}_{\mathrm{MRT}}\right) - \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right) > 0$ if
$\sigma^2\lambda_j^2A_j^2 + \mathrm{k}^2(1+d_{\mathrm{MRT}})^2\alpha_{\mathrm{j}}^2\lambda_{\mathrm{j}}A_j^2 - \sigma^2B_j^2(\lambda_{\mathrm{j}}+\mathrm{k}\,(1+d_{\mathrm{MRT}}))^2 - \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2(\lambda_{\mathrm{j}}+\mathrm{k}\,(1+d_{\mathrm{MRT}}))^2(\mathrm{k}+d_{\mathrm{MTPL}}+1)^2 > 0 \rightarrow$
$\sigma^2\left(\lambda_j^2A_j^2 - B_j^2(\lambda_{\mathrm{j}}+\mathrm{k}\,(1+d_{\mathrm{MRT}}))^2\right) > \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2\left((\lambda_{\mathrm{j}}+\mathrm{k}\,(1+d_{\mathrm{MRT}}))^2(\mathrm{k}+d_{\mathrm{MTPL}}+1)^2 - \mathrm{k}^2A_j^2(1+d_{\mathrm{MRT}})^2\right)$. Then $\mathrm{MSE}\left(\widehat{\gamma}_{\mathrm{MRT}}\right) > \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right)$ if $\sigma^2\left(\lambda_j^2A_j^2 - B_j^2(\lambda_{\mathrm{j}}+\mathrm{k}\,(1+d_{\mathrm{MRT}}))^2\right) > \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2\left((\lambda_{\mathrm{j}}+\mathrm{k}\,(1+d_{\mathrm{MRT}}))^2(\mathrm{k}+d_{\mathrm{MTPL}}+1)^2 - \mathrm{k}^2A_j^2(1+d_{\mathrm{MRT}})^2\right)$, for all $j,\ k > 0,\ 0 < d_{\mathrm{MTPL}}, d_{\mathrm{MRT}} < 1$. The proof is completed.

3.6. Comparison among MOPL and MTPL estimators

Theorem 6.
The MTPL estimator is better than the MOPL estimator, under the assumption that $d_{MTPL} = d_{MOPL} = d$, if $\sigma^2\left((\lambda_{\mathrm{j}}-d)^2 - B_j^2\right) > \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2k_{\mathrm{MTPL}}\left(k_{\mathrm{MTPL}} + 2\,(d+1)\right)$; for all $j,\ k_{\mathrm{MTPL}} > 0,\ 0 < d < 1$.

Proof.
The difference among the MSEM of the MOPL and MTPL estimators under the assumption that $d_{MTPL} = d_{MOPL} = d$ is

$$\begin{aligned}\mathrm{MSEM}(\hat{\gamma}_{d_0}) - \mathrm{MSEM}(\hat{\gamma}_{k,d_0}) &= \sigma^2\mathrm{F}A^{-1}(\Lambda - dI_p)\Lambda^{-1}(\Lambda - dI_p)A^{-1}F' \\ &+ (d+1)^2\mathrm{F}A^{-1}\alpha\alpha' A^{-1}F' \\ &- \sigma^2\mathrm{F}A^{-1}\mathrm{B}\,\Lambda^{-1}\mathrm{B}\,A^{-1}F' \\ &- (k_{\mathrm{MTPL}} + d + 1)^2\mathrm{F}A^{-1}\alpha\alpha' A^{-1}F'.\end{aligned}$$

Then

$$\mathrm{MSE}\left(\widehat{\gamma}_{d_0}\right) - \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right) = \sum_{j=1}^{p}\left(\frac{\sigma^2(\lambda_{\mathrm{j}}-d)^2 + \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2(d+1)^2}{\lambda_{\mathrm{j}}A_j^2} - \frac{\sigma^2B_j^2 + \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2(k_{\mathrm{MTPL}}+d+1)^2}{\lambda_{\mathrm{j}}A_j^2}\right),$$

$\mathrm{MSE}\left(\widehat{\gamma}_{d_0}\right) - \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right) > 0$ if $\sigma^2(\lambda_{\mathrm{j}}-d)^2 + \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2(d+1)^2 - \sigma^2B_j^2 - \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2(k_{\mathrm{MTPL}}+d+1)^2 > 0 \rightarrow \sigma^2\left((\lambda_{\mathrm{j}}-d)^2 - B_j^2\right) > \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2k_{\mathrm{MTPL}}\left(k_{\mathrm{MTPL}} + 2\,(d+1)\right)$. Then $\mathrm{MSE}\left(\widehat{\gamma}_{d_0}\right) > \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right)$ if $\sigma^2\left((\lambda_{\mathrm{j}}-d)^2 - B_j^2\right) > \lambda_{\mathrm{j}}\alpha_{\mathrm{j}}^2k_{\mathrm{MTPL}}\left(k_{\mathrm{MTPL}} + 2\,(d+1)\right)$, for all $j,\ k_{\mathrm{MTPL}} > 0,\ 0 < d < 1$. The proof is completed.

4. Selecting *k* and *d* Parameters

In this section, we present the best estimators of the shrinkage parameters (k, d) for each biased estimator (RR, Liu, KL, MRT and MOPL) based on previous studies. In addition, we suggest the best estimators of the k and d parameters for our proposed estimator.

1. For the RR estimator, we use the best estimator for k_{Ridge} according to [36]:

$$\hat{k}_{Ridge}=\max_{j}\left(1/\sqrt{\frac{\widehat{\sigma}^2}{\lambda_j\widehat{\alpha}_j^2}}\right),\tag{4.1}$$

where $\widehat{\sigma}^2=\frac{y'y-\widehat{\gamma}'X'y}{n-p}$, and $\widehat{\alpha}_j$ is the jth element of $\widehat{\alpha}=F'\widehat{\gamma}$.

2. For the Liu estimator, we use the best estimator for d_{Liu} according to [37]:

$$\hat{d}_{Liu}=max\left[0,\min_{j}\left(\frac{\widehat{\alpha}_j^2-\widehat{\sigma}^2}{max_j\left(\widehat{\sigma}^2/\lambda_j\right)+max_j\left(\widehat{\alpha}_j^2\right)}\right)\right].\tag{4.2}$$

3. For the KL estimator, we use the best estimator for k_{KL} according to [18]:

$$\hat{k}_{KL}=\min_{j}\left(\frac{\lambda_j\widehat{\sigma}^2}{\widehat{\sigma}^2+2\lambda_j\widehat{\alpha}_j^2}\right).\tag{4.3}$$

4. Following [17],we use the following best estimators for k_{MRT} and d_{MRT} in the MRT estimator according to their algorithm:

$$\widehat{\mathrm{k}}_{MRT}=\frac{p\widehat{\sigma}^2}{\sum_{j=1}^{p}(1+d)\widehat{\alpha}_j^2};\widehat{\mathrm{d}}_{MRT}=\frac{p}{\sum_{j=1}^{p}\left(\frac{k\widehat{\alpha}_j^2}{\widehat{\sigma}^2-k\widehat{\alpha}_j^2}\right)}.\tag{4.4}$$

5. Following [35], we suggest use the following estimator of d_{MOPL} parameter in the MOPL estimator:

$$\hat{d}_{MOPL}=\max\left(0,\frac{1}{p}\sum_{j=1}^{p}\left(\frac{\lambda_j\left(\widehat{\sigma}^2-\widehat{\alpha}_j^2\right)}{\widehat{\sigma}^2+\lambda_j\widehat{\alpha}_j^2}\right)\right).\tag{4.5}$$

6. For the proposed MTPL estimator, we suggest use $\hat{d}_{MOPL}$ as an estimator of d_{MTPL} parameter and the following estimator for k_{MTPL} parameter, based on $k_{MTPL(opt)}$ defined in equation (2.27):

$$\hat{k}_{MTPL}=max\left(0,\left(\frac{\lambda_{min}\left(\widehat{\sigma}^2-\widehat{\alpha}_{min}^2\right)}{\widehat{\sigma}^2+\lambda_{min}\widehat{\alpha}_{max}^2}-\hat{d}_{Liu}\right)\right);\tag{4.6}$$

where $\lambda_{min}=min_j\left(\lambda_j\right)$, $\widehat{\alpha}_{max}^2=max_j\left(\widehat{\alpha}_j^2\right)$, and $\widehat{\alpha}_{min}^2=min_j\left(\widehat{\alpha}_j^2\right)$.

We will use the suggested biasing k and d estimators mentioned in equations (4.1) to (4.6) in the following simulation study and applications.

5. Monte Carlo Simulation Study

In this section, the simulation study is conducted to evaluate the performance of the proposed MTPL estimator and the other estimators under different simulation factors (see Table 1).

The dependent variable (y_i) is given by equation (1.1) with $\varepsilon_i\sim iidN(0,\sigma^2)$; where $iidN$ stands for independent and identically normal distributed. The values of the regression coefficients including intercept are provided as $\gamma'\gamma=1$, and $\gamma_1=\cdots=\gamma_p$, as in [23, 38, 39]. Following [2, 38, 40], the correlated explanatory variables are generated as

$$x_{ij}=\rho\omega_{ip}+\omega_{ij}\left(1-\rho^2\right)^{0.5},\quad i=1,\ldots,n\ j=2,\ldots,p,\tag{5.1}$$

Table 1. Assumed values for different simulation factors

Factor	**Notation**	**Values**
Sample size	n	30, 50, 75, 100, 150, 200
Number of regression coefficients	p	4, 8, 12
Correlation degree	ρ	0.85, 0.90, 0.95, 0.99, 0.999
Standard deviation of the error	σ	0.5, 1, 5, 10

where ω_{ij} are the independent standard normal pseudo–random numbers. For the different combinations of σ, n, ρ, and p. the output data are repeated 1000 times. To evaluate the performance of different estimators, the estimated MSE criterion is calculated as follows:

$$MSE(\widehat{\gamma}^{*}) = \frac{1}{1000}\sum_{l=1}^{1000}(\widehat{\gamma}_l^{*} - \gamma)'(\widehat{\gamma}_l^{*} - \gamma), \tag{5.2}$$

where γ is the $p \times 1$ vector of the true coefficients, while $\widehat{\gamma}_l^{*}$ is the vector of the estimated coefficients at l^{th} replication. The suggested k and d estimators in equations (4.1) to (4.6) are used in our simulation study.

Tables 2 – 13 describe the MSE values for all mixtures of σ, n, ρ, and p. In each table, the smallest MSE value in each row is highlighted in bold. From the simulation results of Tables 2 – 13, we can summarize the following conclusions:

1. From Tables 2 – 13, in terms of MSE, the MTPL estimator has the best performance because it has the minimum MSE, while the OLS estimator has the weakest performance because it has the maximum MSE, which is influenced by multicollinearity. We also noticed that the second-best estimator after the proposed MTPL estimator is MOPL because it has a smaller MSE than OLS, RR, Liu, KL, and MRT estimators.
2. Based on the simulation results, we find that variations in the sample size (n) have an impact on the values of MSE; it is thought that increasing n reduces the MSE values of all estimators. The values of MSE grow as the number of regression coefficients (p) increases. MSEs are directly affected by the standard deviation of the error (σ); the MSEs grow as σ increases.
3. In all the cases evaluated, the MTPL estimator performs the best. Furthermore, the MTPL estimator performs better for larger values of ρ.

Another measure of comparable performance is the relative efficiency (RE), which is derived by the estimated MSEs from equation (5.2) as follows [10, 39]:

$$\mathrm{RE}(\widehat{\gamma}_a) = \frac{MSE(\widehat{\gamma})}{MSE(\widehat{\gamma}_a)}, \tag{5.3}$$

where $\widehat{\gamma}_a$ stands for the RR, Liu, KL, MRT, MOPL, or MTPL estimators. Moreover, the root mean squared error (RMSE) has long been used as a standard statistical tool for assessing the performance of each estimator. The RMSE is calculated by

$$\mathrm{RMSE} = \sqrt{\mathrm{MSE}(\widehat{\gamma}_a)}. \tag{5.4}$$

Table 2. MSE values when $\sigma = 0.5$, $p = 4$

n	ρ	OLS	RR	Liu	KL	MRT	MOPL	MTPL
30	**0.85**	0.62838	0.60550	0.54820	0.60528	0.57953	0.38089	**0.38023**
	0.90	0.84700	0.81219	0.71287	0.81185	0.77193	0.44155	**0.44075**
	0.95	1.77823	1.67265	1.31305	1.67151	1.54141	0.52056	**0.51970**
	0.99	8.37060	7.75481	5.62271	7.75148	6.89996	0.39099	**0.39058**
	0.999	86.91904	80.72409	54.54662	80.71746	70.95554	0.10221	**0.10218**
50	**0.85**	0.35519	0.34658	0.32633	0.34653	0.33400	0.25278	**0.25258**
	0.90	0.54681	0.52802	0.47437	0.52785	0.50291	0.32770	**0.32715**
	0.95	1.33195	1.25557	1.04298	1.25466	1.16398	0.43938	**0.43861**
	0.99	4.86316	4.51902	3.30919	4.51562	4.03919	0.44018	**0.43953**
	0.999	61.63427	57.02981	38.54797	57.02613	49.63586	0.09532	**0.09529**
75	**0.85**	0.19230	0.18989	0.18352	0.18988	0.18536	0.16152	**0.16146**
	0.90	0.31489	0.30813	0.29220	0.30809	0.29816	0.23473	**0.23453**
	0.95	0.63186	0.60860	0.55014	0.60840	0.58005	0.36024	**0.35975**
	0.99	2.67634	2.50512	1.94135	2.50348	2.28797	0.48758	**0.48667**
	0.999	36.04726	33.38185	22.83933	33.37739	29.14740	0.12910	**0.12903**
100	**0.85**	0.15703	0.15553	0.15186	0.15552	0.15266	0.13731	**0.13726**
	0.90	0.24915	0.24479	0.23384	0.24476	0.23773	0.19556	**0.19542**
	0.95	0.51057	0.49353	0.45356	0.49339	0.47119	0.30825	**0.30797**
	0.99	2.42957	2.28247	1.83730	2.28104	2.09503	0.50812	**0.50729**
	0.999	26.20675	24.36339	16.33969	24.35990	21.53599	0.16352	**0.16343**
150	**0.85**	0.11164	0.11081	0.10849	0.11081	0.10906	0.10039	**0.10036**
	0.90	0.15661	0.15479	0.15006	0.15479	0.15121	0.13310	**0.13306**
	0.95	0.30202	0.29530	0.27916	0.29526	0.28536	0.22195	**0.22175**
	0.99	1.63547	1.53676	1.26590	1.53582	1.41966	0.44946	**0.44887**
	0.999	17.10337	15.86849	11.35648	15.86540	13.95225	0.22978	**0.22958**
200	**0.85**	0.07112	0.07080	0.06993	0.07080	0.07009	0.06674	**0.06673**
	0.90	0.09222	0.09160	0.09001	0.09160	0.09032	0.08393	**0.08391**
	0.95	0.23534	0.23114	0.22214	0.23112	0.22422	0.18387	**0.18377**
	0.99	1.28039	1.21910	1.04860	1.21855	1.14299	0.47471	**0.47405**
	0.999	15.21938	14.12710	9.86232	14.12287	12.39496	0.26894	**0.26873**

Table 3. MSE values when $\sigma = 0.5$, $p = 8$

n	ρ	**OLS**	**RR**	**Liu**	**KL**	**MRT**	**MOPL**	**MTPL**
30	**0.85**	1.98059	1.90101	1.31205	1.90025	1.72648	1.01042	**1.00585**
	0.90	3.84058	3.63351	1.91984	3.63131	3.14048	1.16368	**1.15874**
	0.95	5.35505	5.09795	2.62441	5.09662	4.53785	1.45762	**1.45125**
	0.99	29.73529	28.06849	9.32343	28.06599	23.85135	1.20810	**1.20615**
	0.999	409.93378	384.63814	95.38602	384.57380	314.76272	0.27711	**0.27706**
50	**0.85**	0.95346	0.92896	0.75651	0.92878	0.86325	0.66291	**0.66037**
	0.90	1.62245	1.56696	1.13191	1.56650	1.43199	0.90856	**0.90441**
	0.95	3.20623	3.04070	1.78233	3.03910	2.66418	1.13427	**1.12952**
	0.99	17.76844	16.78533	5.47525	16.78115	14.22568	1.33130	**1.32848**
	0.999	173.70880	162.79285	37.36113	162.76087	133.06277	0.37766	**0.37749**
75	**0.85**	0.62808	0.61550	0.52259	0.61542	0.57501	0.47409	**0.47258**
	0.90	0.92862	0.90366	0.72746	0.90347	0.83312	0.61853	**0.61640**
	0.95	1.81467	1.74265	1.19953	1.74197	1.57255	0.91873	**0.91468**
	0.99	10.40413	9.82280	3.77717	9.81980	8.33234	1.37440	**1.37123**
	0.999	109.90923	103.57962	25.67299	103.55806	85.67212	0.50128	**0.50083**
100	**0.85**	0.46828	0.46120	0.40606	0.46116	0.43559	0.37661	**0.37566**
	0.90	0.67054	0.65746	0.56671	0.65739	0.61677	0.50935	**0.50783**
	0.95	1.34425	1.29330	0.92338	1.29281	1.16356	0.73882	**0.73559**
	0.99	8.20402	7.72654	3.23198	7.72354	6.55013	1.38885	**1.38448**
	0.999	81.53791	77.00704	19.76056	76.99100	64.64225	0.59131	**0.59064**
150	**0.85**	0.28150	0.27936	0.26247	0.27935	0.26966	0.25205	**0.25175**
	0.90	0.43237	0.42657	0.38318	0.42654	0.40506	0.35757	**0.35677**
	0.95	0.87248	0.84909	0.67900	0.84891	0.78262	0.58870	**0.58654**
	0.99	5.30280	5.01934	2.48082	5.01732	4.31183	1.28228	**1.27760**
	0.999	53.48380	50.53265	11.98461	50.52313	42.42235	0.73795	**0.73722**
200	**0.85**	0.22581	0.22425	0.21213	0.22424	0.21697	0.20446	**0.20429**
	0.90	0.30259	0.29954	0.27697	0.29953	0.28696	0.26259	**0.26226**
	0.95	0.67844	0.66397	0.56537	0.66388	0.62000	0.49973	**0.49845**
	0.99	3.57510	3.41075	1.96876	3.40968	3.03167	1.25528	**1.24932**
	0.999	41.45914	39.17773	11.22550	39.16945	32.79491	0.98234	**0.98129**

Table 4. MSE values when $\sigma = 0.5$, $p = 12$

n	ρ	OLS	RR	Liu	KL	MRT	MOPL	MTPL
30	**0.85**	3.06905	2.94697	1.76742	2.94590	2.56630	1.48616	**1.47769**
	0.90	5.90268	5.63342	2.59139	5.63208	4.80967	1.91538	**1.90441**
	0.95	12.70772	12.02907	3.65961	12.02524	9.64689	2.20834	**2.20077**
	0.99	93.29749	87.40621	14.00670	87.40255	65.83900	2.07213	**2.07055**
	0.999	642.71600	608.05508	77.26317	607.93974	486.15167	0.50139	**0.50126**
50	**0.85**	1.89362	1.82425	1.23479	1.82352	1.58819	1.06905	**1.06259**
	0.90	2.88639	2.76877	1.59488	2.76771	2.38546	1.36360	**1.35407**
	0.95	6.12732	5.84543	2.56159	5.84392	4.92643	1.86797	**1.85804**
	0.99	31.63995	30.09319	6.63660	30.08691	24.53974	2.34502	**2.34077**
	0.999	366.65671	345.39536	42.11239	345.32551	267.15830	0.67136	**0.67097**
75	**0.85**	1.15720	1.12856	0.87443	1.12834	1.01231	0.81006	**0.80595**
	0.90	1.80473	1.74449	1.20211	1.74393	1.52814	1.05344	**1.04740**
	0.95	3.50447	3.35350	1.83229	3.35228	2.86900	1.51476	**1.50552**
	0.99	18.63990	17.64379	4.72577	17.63993	14.15576	2.36705	**2.36122**
	0.999	258.62622	243.74438	30.93642	243.68402	186.10331	0.84378	**0.84328**
100	**0.85**	0.76244	0.75130	0.64708	0.75124	0.69709	0.62258	**0.62011**
	0.90	1.21246	1.18164	0.89886	1.18141	1.05465	0.82340	**0.81942**
	0.95	2.73383	2.62554	1.52679	2.62457	2.24498	1.28275	**1.27561**
	0.99	15.19796	14.43809	4.27785	14.43452	11.70338	2.31849	**2.31168**
	0.999	142.90587	135.65007	16.69861	135.61749	109.04380	1.04200	**1.04089**
150	**0.85**	0.47248	0.46741	0.41910	0.46739	0.43763	0.40675	**0.40558**
	0.90	0.75182	0.73868	0.61678	0.73860	0.67577	0.58500	**0.58277**
	0.95	1.61015	1.56182	1.10732	1.56141	1.37972	0.98659	**0.98139**
	0.99	8.93762	8.45391	3.00940	8.45090	6.83611	2.03559	**2.02835**
	0.999	87.55287	82.96893	10.55341	82.94930	66.75832	1.37145	**1.36984**
200	**0.85**	0.40904	0.40487	0.36510	0.40486	0.37971	0.35398	**0.35306**
	0.90	0.53849	0.53190	0.47028	0.53187	0.49614	0.45372	**0.45240**
	0.95	1.13489	1.10696	0.84719	1.10677	0.99017	0.78455	**0.78027**
	0.99	6.45887	6.14763	2.70112	6.14551	5.10483	1.93117	**1.92160**
	0.999	65.80040	62.56729	9.01952	62.54941	50.63084	1.58032	**1.57788**

Table 5. MSE values when $\sigma = 1$, $p = 4$

n	$\boldsymbol{\rho}$	**OLS**	**RR**	**Liu**	**KL**	**MRT**	**MOPL**	**MTPL**
30	**0.85**	2.51352	2.03073	1.36053	2.02180	1.60418	0.59928	**0.58616**
	0.90	3.38799	2.72955	1.61248	2.72097	2.13911	0.60617	**0.59281**
	0.95	7.11290	5.63174	2.70141	5.62987	4.14178	0.55170	**0.54501**
	0.99	33.48241	26.17030	11.99741	26.19299	18.57346	0.22412	**0.22311**
	0.999	347.67616	274.79224	115.23950	274.97772	195.91112	0.11414	**0.11411**
50	**0.85**	1.42075	1.17134	0.88562	1.16512	0.94704	0.50758	**0.49671**
	0.90	2.18726	1.76262	1.11356	1.75388	1.37772	0.55142	**0.53853**
	0.95	5.32778	4.16870	2.27484	4.16405	3.05675	0.45184	**0.44530**
	0.99	19.45262	15.22214	6.75720	15.23872	10.67740	0.28963	**0.28756**
	0.999	246.53710	191.99876	85.74347	192.19886	131.86559	0.07069	**0.07066**
75	**0.85**	0.76922	0.66323	0.56711	0.66028	0.56672	0.42094	**0.41021**
	0.90	1.25954	1.04853	0.80938	1.04285	0.85748	0.50214	**0.48985**
	0.95	2.52746	2.02475	1.36563	2.01552	1.57186	0.52706	**0.51434**
	0.99	10.70536	8.41168	4.13491	8.41239	6.05612	0.32564	**0.32199**
	0.999	144.18904	112.24996	53.48422	112.31772	77.03284	0.07021	**0.07014**
100	**0.85**	0.62812	0.55664	0.50670	0.55484	0.48628	0.38832	**0.38174**
	0.90	0.99660	0.83919	0.69830	0.83478	0.69503	0.45316	**0.44415**
	0.95	2.04228	1.64188	1.17741	1.63182	1.27095	0.50097	**0.49124**
	0.99	9.71829	7.69184	4.35304	7.69008	5.57507	0.36257	**0.35915**
	0.999	104.82701	82.43703	38.84841	82.46390	57.65619	0.07028	**0.07019**
150	**0.85**	0.44657	0.40260	0.37339	0.40159	0.35356	0.30433	**0.29948**
	0.90	0.62642	0.54340	0.48566	0.54121	0.46063	0.35785	**0.35082**
	0.95	1.20807	0.99839	0.80983	0.99278	0.80719	0.45897	**0.44907**
	0.99	6.54187	5.09115	3.05804	5.08777	3.67596	0.36694	**0.36310**
	0.999	68.41349	53.35339	27.47205	53.38407	36.67176	0.08837	**0.08814**
200	**0.85**	0.28446	0.26588	0.25387	0.26557	0.24087	0.22373	**0.22136**
	0.90	0.36887	0.33502	0.31546	0.33425	0.29568	0.26026	**0.25695**
	0.95	0.94136	0.78821	0.68850	0.78391	0.64902	0.41975	**0.41206**
	0.99	5.12158	4.12464	2.61818	4.11736	3.10248	0.47004	**0.46435**
	0.999	60.87751	47.78818	24.56343	47.84325	32.85005	0.10768	**0.10743**

Table 6. MSE values when $\sigma = 1$, $p = 8$

n	ρ	OLS	RR	Liu	KL	MRT	MOPL	MTPL
30	**0.85**	7.92235	6.63508	2.29113	6.63745	4.65926	1.56239	**1.51844**
	0.90	15.36232	12.65629	2.90881	12.67838	8.25262	1.51614	**1.49210**
	0.95	21.42019	17.94125	3.50170	17.95304	12.36114	1.36491	**1.33941**
	0.99	118.94114	98.30370	12.72517	98.53798	63.71314	0.59457	**0.59130**
	0.999	1639.73514	1334.66526	124.38220	1338.50123	813.65217	0.12121	**0.12115**
50	**0.85**	3.81383	3.21685	1.65006	3.20913	2.33798	1.35371	**1.29087**
	0.90	6.48980	5.46922	2.08642	5.46539	3.82240	1.50622	**1.45071**
	0.95	12.82494	10.51956	2.66840	10.53699	6.86021	1.43979	**1.41006**
	0.99	71.07377	58.68080	6.22085	58.75231	37.13916	0.73836	**0.73207**
	0.999	694.83520	561.83417	46.67365	563.22086	339.30452	0.13580	**0.13557**
75	**0.85**	2.51232	2.14580	1.27822	2.13855	1.56165	1.10988	**1.05345**
	0.90	3.71450	3.13392	1.60119	3.12632	2.21104	1.26397	**1.21070**
	0.95	7.25867	6.03422	2.09185	6.03280	4.08967	1.43753	**1.39201**
	0.99	41.61651	34.20541	4.13358	34.25344	21.43571	0.92792	**0.91872**
	0.999	439.63694	360.40184	33.24511	360.92462	218.62324	0.16935	**0.16886**
100	**0.85**	1.87313	1.62886	1.09070	1.62335	1.21807	0.96768	**0.92276**
	0.90	2.68215	2.28914	1.38999	2.28279	1.67966	1.16712	**1.10917**
	0.95	5.37701	4.42076	1.76917	4.41470	2.95859	1.29610	**1.25119**
	0.99	32.81606	26.79039	4.00993	26.82804	16.72791	1.10773	**1.09470**
	0.999	326.15165	268.57085	24.43369	268.80747	166.99257	0.19775	**0.19695**
150	**0.85**	1.12600	1.01885	0.81721	1.01654	0.80403	0.76318	**0.73032**
	0.90	1.72949	1.51119	1.05980	1.50631	1.13739	0.93993	**0.89498**
	0.95	3.48993	2.92344	1.50207	2.91635	2.05360	1.21941	**1.16726**
	0.99	21.21119	17.45403	3.33141	17.46701	10.96405	1.22976	**1.20985**
	0.999	213.93521	176.20048	14.18891	176.36447	108.53312	0.25622	**0.25510**
200	**0.85**	0.90324	0.82448	0.67833	0.82277	0.65356	0.63849	**0.61431**
	0.90	1.21036	1.07957	0.82756	1.07648	0.83012	0.75651	**0.72557**
	0.95	2.71377	2.31464	1.38448	2.30743	1.66966	1.14037	**1.09458**
	0.99	14.30039	11.95699	2.77654	11.96384	7.98388	1.38135	**1.34740**
	0.999	165.83655	136.86997	14.54629	137.05211	84.27755	0.37057	**0.36882**

Table 7. MSE values when $\sigma = 1$, $p = 12$

n	ρ	OLS	RR	Liu	KL	MRT	MOPL	MTPL
30	**0.85**	12.27619	10.45821	2.97653	10.46788	6.77485	2.45340	**2.38424**
	0.90	23.61071	20.01674	3.38144	20.04318	12.65860	2.37015	**2.32156**
	0.95	50.83088	42.53073	4.01504	42.69958	24.60535	2.28038	**2.25894**
	0.99	373.18995	305.29536	13.11842	306.70163	161.96856	1.03695	**1.03483**
	0.999	2570.86400	2147.86971	79.67884	2153.99363	1264.64898	0.18635	**0.18620**
50	**0.85**	7.57448	6.37082	2.43701	6.37606	4.06647	2.08267	**2.00624**
	0.90	11.54558	9.75736	2.63919	9.76934	6.08402	2.25196	**2.18188**
	0.95	24.50928	20.75625	3.22225	20.78855	12.63689	2.25196	**2.20579**
	0.99	126.55981	106.97324	5.52733	107.14990	63.36857	1.33614	**1.32730**
	0.999	1466.62685	1210.04138	36.71391	1214.82729	658.95640	0.22050	**0.22011**
75	**0.85**	4.62878	3.95702	1.99077	3.95135	2.62655	1.82795	**1.74043**
	0.90	7.21890	6.12503	2.40324	6.12495	3.89641	2.07205	**1.99471**
	0.95	14.01786	11.82824	2.85043	11.83992	7.26659	2.34792	**2.28172**
	0.99	74.55960	62.13935	4.28531	62.26420	35.74059	1.67610	**1.65919**
	0.999	1034.50487	853.19404	30.40773	855.57593	448.88162	0.26800	**0.26748**
100	**0.85**	3.04974	2.67705	1.71048	2.67063	1.91263	1.62740	**1.53126**
	0.90	4.84982	4.15142	2.02090	4.14513	2.73270	1.84241	**1.75709**
	0.95	10.93532	9.25759	2.69965	9.26283	5.65637	2.24479	**2.17777**
	0.99	60.79182	51.09635	3.98015	51.17806	29.46267	1.82978	**1.80859**
	0.999	571.62350	478.84739	15.01746	479.36235	276.54729	0.33599	**0.33463**
150	**0.85**	1.88993	1.68422	1.22313	1.67972	1.20258	1.17419	**1.10877**
	0.90	3.00726	2.61911	1.58449	2.61212	1.79092	1.48807	**1.40984**
	0.95	6.44061	5.48644	2.30067	5.48472	3.54142	2.02169	**1.93517**
	0.99	35.75050	29.67342	3.33723	29.72758	17.02405	2.10795	**2.08004**
	0.999	350.21147	292.44557	8.50419	292.97189	169.18509	0.47668	**0.47452**
200	**0.85**	1.63618	1.47136	1.08325	1.46755	1.04868	1.04102	**0.98963**
	0.90	2.15396	1.91368	1.32945	1.90806	1.36193	1.27275	**1.20598**
	0.95	4.53956	3.89857	1.93669	3.89305	2.56433	1.78517	**1.69965**
	0.99	25.83547	21.71749	3.42113	21.74966	12.90953	2.26531	**2.22124**
	0.999	263.20161	221.75721	7.02535	221.97378	128.27497	0.58105	**0.57721**

Table 8. MSE values when $\sigma = 5$, $p = 4$

n	ρ	OLS	RR	Liu	KL	MRT	MOPL	MTPL
30	**0.85**	10.05407	5.31264	1.79865	5.32540	2.78885	0.52481	**0.49257**
	0.90	13.55195	7.30081	2.07238	7.35073	3.92928	0.52059	**0.49836**
	0.95	28.45160	14.89986	3.48551	15.17392	7.30381	0.46450	**0.45453**
	0.99	133.92963	68.48416	13.33873	69.29913	33.18133	0.25356	**0.25251**
	0.999	1390.70462	733.96119	135.57503	744.47119	359.90322	0.26432	**0.26429**
50	**0.85**	5.68299	2.95813	1.24201	2.99617	1.56726	0.54889	**0.50092**
	0.90	8.74903	4.55718	1.47393	4.64742	2.37496	0.55770	**0.52472**
	0.95	21.31114	10.84003	2.56841	11.04405	5.16376	0.34347	**0.33319**
	0.99	77.81049	39.58179	7.89452	40.37908	17.90869	0.26136	**0.25967**
	0.999	986.14839	493.67533	96.19289	502.65837	222.14224	0.15936	**0.15933**
75	**0.85**	3.07688	1.61364	0.95365	1.60689	0.92923	0.57891	**0.48798**
	0.90	5.03817	2.64455	1.13739	2.67122	1.42705	0.56975	**0.51172**
	0.95	10.10983	5.18093	1.64319	5.22133	2.58020	0.40870	**0.37897**
	0.99	42.82144	21.76379	5.20059	21.96296	10.05192	0.23561	**0.23132**
	0.999	576.75615	286.09682	63.33853	289.87429	127.11935	0.13228	**0.13221**
100	**0.85**	2.51248	1.38691	0.95155	1.37529	0.84159	0.60001	**0.49919**
	0.90	3.98640	2.07895	1.10603	2.09463	1.13193	0.55052	**0.48386**
	0.95	8.16912	4.17275	1.44276	4.24465	2.08080	0.44222	**0.41086**
	0.99	38.87314	19.96824	4.98068	20.16906	9.25609	0.20607	**0.20149**
	0.999	419.30805	211.36208	42.79881	213.18702	94.67316	0.09880	**0.09870**
150	**0.85**	1.78627	0.97562	0.79691	0.95974	0.61100	0.55128	**0.45450**
	0.90	2.50569	1.29313	0.89489	1.28948	0.73111	0.53334	**0.45564**
	0.95	4.83228	2.45859	1.13779	2.47713	1.25332	0.46837	**0.41766**
	0.99	26.16748	12.87242	3.74624	13.05184	5.85454	0.19943	**0.19421**
	0.999	273.65395	134.92716	34.40513	136.60156	59.26597	0.08045	**0.08025**
200	**0.85**	1.13784	0.65888	0.63722	0.63538	0.45424	0.50283	**0.39837**
	0.90	1.47547	0.78663	0.69359	0.76421	0.50114	0.48424	**0.38712**
	0.95	3.76544	1.90723	1.11484	1.91602	0.99998	0.47866	**0.42608**
	0.99	20.48630	10.95334	3.27848	11.06448	5.25307	0.26197	**0.25211**
	0.999	243.51002	124.05606	36.45595	126.40972	54.83958	0.07466	**0.07448**

Table 9. MSE values when $\sigma = 5$, $p = 8$

n	ρ	OLS	RR	Liu	KL	MRT	MOPL	MTPL
30	**0.85**	31.68939	18.94519	2.12946	19.43572	8.46442	1.35048	**1.26596**
	0.90	61.44928	36.08411	2.46878	37.68927	15.04098	1.15153	**1.11722**
	0.95	85.68078	50.85986	2.72371	51.84976	22.18691	0.80431	**0.77202**
	0.99	475.76456	278.45007	10.86722	290.12661	114.15252	0.35046	**0.34770**
	0.999	6558.94054	3739.59794	104.49941	3913.58812	1445.78347	0.17510	**0.17506**
50	**0.85**	15.25532	8.86513	1.89716	9.05396	3.85865	1.50601	**1.32105**
	0.90	25.95920	15.43304	1.91211	15.82270	6.44619	1.29711	**1.18041**
	0.95	51.29975	29.10517	1.94159	30.40137	11.45452	0.97831	**0.93438**
	0.99	284.29507	164.28938	3.78194	170.75956	63.22822	0.34713	**0.34135**
	0.999	2779.34082	1547.91324	28.17313	1631.44543	575.80114	0.13176	**0.13159**
75	**0.85**	10.04930	5.85444	1.76499	6.00764	2.49371	1.51232	**1.29405**
	0.90	14.85798	8.70873	1.85813	9.00780	3.61007	1.45459	**1.30041**
	0.95	29.03467	16.77720	1.78297	17.33668	6.68350	1.15403	**1.06257**
	0.99	166.46605	94.56719	2.23258	98.80326	35.03546	0.40787	**0.39810**
	0.999	1758.54774	996.57426	22.44147	1041.81158	358.50443	0.11532	**0.11497**
100	**0.85**	7.49250	4.41987	1.72952	4.54047	1.95949	1.51119	**1.27829**
	0.90	10.72859	6.20880	1.78980	6.33337	2.64547	1.44950	**1.22942**
	0.95	21.50804	12.03991	1.69381	12.55133	4.72938	1.22160	**1.11693**
	0.99	131.26425	73.89235	2.37291	77.71968	27.69863	0.49919	**0.48507**
	0.999	1304.60661	742.01833	14.24099	773.26624	273.27981	0.11035	**0.10974**
150	**0.85**	4.50401	2.65924	1.56685	2.67098	1.26963	1.43483	**1.10419**
	0.90	6.91797	4.05102	1.65638	4.12018	1.74202	1.43565	**1.18225**
	0.95	13.95974	7.96920	1.76668	8.24815	3.24274	1.36796	**1.20734**
	0.99	84.84474	47.99470	2.15878	50.01035	17.53995	0.59394	**0.56953**
	0.999	855.74083	485.23825	9.51691	504.50501	172.66440	0.10840	**0.10746**
200	**0.85**	3.61296	2.16751	1.42831	2.17460	1.05119	1.32385	**1.03736**
	0.90	4.84142	2.83208	1.50320	2.87511	1.28542	1.36499	**1.09596**
	0.95	10.85507	6.33763	1.84192	6.53181	2.64407	1.47294	**1.29028**
	0.99	57.20155	33.38054	1.76394	34.36581	12.96285	0.72935	**0.68034**
	0.999	663.34621	380.74891	9.61242	396.83255	137.95685	0.13525	**0.13385**

Table 10. MSE values when $\sigma = 5$, $p = 12$

n	$\boldsymbol{\rho}$	**OLS**	**RR**	**Liu**	**KL**	**MRT**	**MOPL**	**MTPL**
30	**0.85**	49.10476	30.95603	2.78603	32.07401	12.26105	2.28430	**2.15108**
	0.90	94.44282	58.66970	2.48265	60.25767	22.67356	1.71832	**1.64633**
	0.95	203.32353	124.65235	2.48845	131.28524	44.23857	1.34308	**1.32079**
	0.99	1492.75982	885.56352	7.19194	928.51774	287.62125	0.46698	**0.46549**
	0.999	10283.45598	6307.43360	44.93245	6614.36452	2319.19211	0.19141	**0.19131**
50	**0.85**	30.29790	18.37618	2.64271	19.32130	6.93654	2.30786	**2.13716**
	0.90	46.18230	28.32047	2.38079	29.47883	10.30555	2.06670	**1.93370**
	0.95	98.03712	60.46761	2.14491	62.77669	21.62685	1.50567	**1.44204**
	0.99	506.23924	313.44456	2.43160	327.62257	110.70769	0.55102	**0.54330**
	0.999	5866.50739	3466.05001	22.55416	3666.46960	1130.33302	0.15329	**0.15305**
75	**0.85**	18.51513	11.32779	2.66619	11.74222	4.27632	2.46326	**2.19150**
	0.90	28.87560	17.73004	2.62765	18.46710	6.49550	2.31716	**2.13198**
	0.95	56.07145	34.13091	2.18442	35.71132	12.11885	1.82423	**1.70682**
	0.99	298.23840	178.51474	2.26839	188.73548	60.20414	0.69459	**0.67819**
	0.999	4138.01948	2430.45777	14.94358	2563.31125	747.10713	0.12133	**0.12098**
100	**0.85**	12.19897	7.61332	2.61567	7.78175	3.02520	2.47903	**2.05892**
	0.90	19.39930	11.91004	2.62012	12.37277	4.41562	2.42599	**2.16039**
	0.95	43.74128	26.88528	2.45651	28.12434	9.30573	2.05581	**1.91917**
	0.99	243.16729	147.46500	1.78004	154.77576	49.20295	0.80076	**0.77855**
	0.999	2286.49399	1373.06736	8.11991	1442.21533	462.67073	0.13267	**0.13170**
150	**0.85**	7.55971	4.62191	2.30448	4.75357	1.85824	2.20446	**1.80868**
	0.90	12.02906	7.43563	2.52716	7.69801	2.82841	2.38484	**2.05982**
	0.95	25.76244	15.85276	2.56472	16.54145	5.79914	2.27917	**2.05221**
	0.99	143.00199	84.84414	1.66406	90.06221	28.35074	1.03708	**1.00374**
	0.999	1400.84589	838.08692	4.06913	888.02755	283.27378	0.16450	**0.16293**
200	**0.85**	6.54471	4.09718	2.20029	4.22039	1.67692	2.12449	**1.77897**
	0.90	8.61583	5.37597	2.40383	5.52387	2.12757	2.31065	**1.93370**
	0.95	18.15822	11.21108	2.55702	11.61455	4.09193	2.39263	**2.11819**
	0.99	103.34188	62.43017	1.96060	65.70888	21.32741	1.22570	**1.16739**
	0.999	1052.80643	639.81826	3.02145	672.44749	211.06912	0.18827	**0.18537**

Table 11. MSE values when σ = 10, p = 4

n	ρ	OLS	RR	Liu	KL	MRT	MOPL	MTPL
30	**0.85**	40.21629	8.97668	1.45375	9.22176	2.87297	0.47034	**0.46525**
	0.90	54.20782	13.08743	1.64117	13.81640	4.40928	0.43819	**0.43310**
	0.95	113.80642	25.76227	2.60030	30.00318	7.81059	0.38903	**0.38683**
	0.99	535.71853	116.60212	8.01883	130.80226	35.67990	0.39674	**0.39668**
	0.999	5562.81848	1285.69150	95.27943	1445.43621	377.83335	0.44709	**0.44709**
50	**0.85**	22.73196	4.85248	1.00443	5.57697	1.51876	0.41036	**0.39821**
	0.90	34.99613	7.79338	1.17838	9.30679	2.47702	0.41673	**0.40793**
	0.95	85.24455	18.13116	1.84464	21.22833	5.27453	0.37385	**0.37311**
	0.99	311.24197	64.12760	5.29221	75.49326	16.58017	0.36223	**0.36217**
	0.999	3944.59355	796.18638	53.71907	930.36508	208.17614	0.33228	**0.33218**
75	**0.85**	12.30750	2.57713	0.82547	2.75112	0.84634	0.40402	**0.38716**
	0.90	20.15267	4.36686	0.92065	4.98423	1.38500	0.40684	**0.39038**
	0.95	40.43931	8.31619	1.21558	9.08767	2.39201	0.36534	**0.36393**
	0.99	171.28577	34.49887	3.49134	37.36241	8.95819	0.33374	**0.33327**
	0.999	2307.02460	446.52957	33.09007	511.49976	109.72303	0.30256	**0.30227**
100	**0.85**	10.04990	2.32929	0.86098	2.44024	0.80012	0.40671	**0.37709**
	0.90	15.94558	3.39796	0.96140	3.88441	1.07612	0.38546	**0.37120**
	0.95	32.67647	6.86522	1.01686	7.98669	2.00886	0.34803	**0.34361**
	0.99	155.49258	31.75960	2.81379	35.54385	8.35405	0.26700	**0.26681**
	0.999	1677.23221	324.92999	24.58158	357.71817	78.97513	0.23781	**0.23183**
150	**0.85**	7.14507	1.56068	0.81171	1.72758	0.54486	0.40071	**0.36403**
	0.90	10.02277	1.98192	0.77821	2.28807	0.61210	0.36523	**0.34456**
	0.95	19.32911	3.82742	0.81807	4.30656	1.08234	0.32170	**0.31845**
	0.99	104.66991	19.76650	2.25480	22.74012	5.06916	0.24020	**0.23094**
	0.999	1094.61581	204.80039	19.10822	234.54890	49.43523	0.19304	**0.19108**
200	**0.85**	4.55137	0.99173	0.79112	1.07137	0.37956	0.38517	**0.31294**
	0.90	5.90190	1.19806	0.71714	1.29526	0.42332	0.34079	**0.32289**
	0.95	15.06176	2.98825	0.82434	3.37470	0.87032	0.31822	**0.31783**
	0.99	81.94520	18.22366	1.92741	20.25187	4.86295	0.24367	**0.24362**
	0.999	974.04008	199.74878	19.51359	233.49236	48.84287	0.16753	**0.16654**

Table 12. MSE values when $\sigma = 10$, $p = 8$

n	ρ	OLS	RR	Liu	KL	MRT	MOPL	MTPL
30	**0.85**	126.75757	37.38525	1.20785	46.93852	9.44314	0.75209	**0.69542**
	0.90	245.79712	73.06801	1.41858	97.14448	17.69256	0.68037	**0.66208**
	0.95	342.72310	97.60752	1.67700	116.28621	23.69858	0.47304	**0.45608**
	0.99	1903.05825	546.08532	5.91667	724.66109	123.82337	0.40487	**0.40364**
	0.999	26235.76216	7380.62169	57.34153	9797.03497	1655.77790	0.35334	**0.35332**
50	**0.85**	61.02129	15.91832	1.20004	20.14783	3.61803	0.86175	**0.72131**
	0.90	103.83678	28.68722	1.03533	37.21221	6.20553	0.66733	**0.59433**
	0.95	205.19901	53.60112	0.95284	75.94471	11.07333	0.53187	**0.50923**
	0.99	1137.18028	309.63010	1.54812	424.23701	63.08920	0.32937	**0.32667**
	0.999	11117.36328	2912.15204	13.54189	4176.59076	587.94672	0.26809	**0.26802**
75	**0.85**	40.19718	10.64308	1.27871	14.25276	2.27213	0.96320	**0.77821**
	0.90	59.43193	16.19790	1.18640	22.10301	3.41772	0.83995	**0.72564**
	0.95	116.13869	30.54576	0.91696	42.18003	6.09594	0.58981	**0.53607**
	0.99	665.86421	172.19909	0.97652	246.85944	32.29654	0.29934	**0.29529**
	0.999	7034.19097	1825.65406	9.37941	2581.98836	345.61394	0.22718	**0.22704**
100	**0.85**	29.97001	8.08257	1.40025	11.06797	1.75000	1.04602	**0.82141**
	0.90	42.91434	10.88071	1.20615	14.02289	2.29361	0.85651	**0.67976**
	0.95	86.03217	21.57896	0.91490	31.24425	4.37584	0.62930	**0.56675**
	0.99	525.05699	135.19680	0.94212	200.65856	26.17361	0.30719	**0.30149**
	0.999	5218.42644	1340.55319	4.64457	1908.15287	248.29463	0.20122	**0.20099**
150	**0.85**	18.01605	4.54568	1.46397	5.56395	1.01273	1.08995	**0.69026**
	0.90	27.67189	7.09051	1.26439	9.19830	1.43300	0.92620	**0.67895**
	0.95	55.83894	14.12299	1.08143	20.22979	2.84166	0.72065	**0.60938**
	0.99	339.37896	85.44602	0.83585	124.17611	15.80059	0.27201	**0.26089**
	0.999	3422.96333	863.72185	3.08448	1233.73793	149.45134	0.15248	**0.15207**
200	**0.85**	14.45186	3.77013	1.56008	4.88428	0.87623	1.19210	**0.79959**
	0.90	19.36569	4.91385	1.35997	6.63751	1.03836	1.01913	**0.71195**
	0.95	43.42028	11.35027	1.17641	15.66426	2.28405	0.81087	**0.65570**
	0.99	228.80621	59.66931	0.61964	81.89310	10.94415	0.30030	**0.27506**
	0.999	2653.38484	697.36641	3.74541	987.47260	128.66126	0.14040	**0.13981**

Table 13. MSE values when $\sigma = 10$, $p = 12$

n	ρ	OLS	RR	Liu	KL	MRT	MOPL	MTPL
30	**0.85**	196.41902	65.66187	1.58473	86.05973	13.94164	1.23523	**1.13751**
	0.90	377.77129	122.60292	1.21370	152.52384	25.52568	0.88194	**0.83491**
	0.95	813.29413	265.37261	1.16375	358.18610	53.20116	0.69096	**0.67996**
	0.99	5971.03926	1900.32277	3.15147	2420.18841	372.65712	0.38780	**0.38722**
	0.999	41133.82393	13628.17050	22.95670	18252.90006	2827.55739	0.32482	**0.32477**
50	**0.85**	121.19161	37.12305	1.58124	54.01188	7.13169	1.33458	**1.20729**
	0.90	184.72922	57.27996	1.29521	78.85367	10.56951	1.09368	**1.00368**
	0.95	392.14848	123.46627	1.05306	166.72695	22.44283	0.75417	**0.71789**
	0.99	2024.95694	651.19285	1.06244	904.30439	118.42820	0.37011	**0.36657**
	0.999	23466.02956	7055.56488	9.45525	9841.91772	1257.36108	0.26747	**0.26738**
75	**0.85**	74.06051	22.05818	1.86140	31.13502	4.07551	1.61510	**1.37235**
	0.90	115.50241	35.39179	1.57450	50.41350	6.43891	1.33081	**1.19200**
	0.95	224.28581	67.40761	1.03945	98.46326	11.62547	0.83878	**0.77005**
	0.99	1192.95361	355.85779	0.97720	524.74471	60.63485	0.37484	**0.36784**
	0.999	16552.07792	4856.45105	4.12735	6688.97259	826.73463	0.18645	**0.18631**
100	**0.85**	48.79587	14.32621	1.94537	19.01196	2.55633	1.68315	**1.25084**
	0.90	77.59719	23.16036	1.73082	33.83582	4.00313	1.49398	**1.27104**
	0.95	174.96511	53.77196	1.24665	77.57670	8.93204	0.99776	**0.91004**
	0.99	972.66917	295.02897	0.72970	424.59947	48.68911	0.40334	**0.39326**
	0.999	9145.97595	2717.21429	2.81104	3989.32292	444.74429	0.16744	**0.16704**
150	**0.85**	30.23883	8.57640	2.19383	12.17767	1.51200	1.89503	**1.40727**
	0.90	48.11623	14.40554	2.00706	20.76577	2.48373	1.74501	**1.40923**
	0.95	103.04978	31.21180	1.44182	45.83122	5.26012	1.19323	**1.02747**
	0.99	572.00794	168.63286	0.63329	252.52873	27.85378	0.40618	**0.39052**
	0.999	5603.38357	1654.85686	1.10265	2488.60530	273.02938	0.15362	**0.15297**
200	**0.85**	26.17884	7.95184	2.41504	11.29446	1.44735	2.12290	**1.64400**
	0.90	34.46331	10.22999	2.18113	14.50086	1.75663	1.90212	**1.45188**
	0.95	72.63288	21.80168	1.63813	31.68636	3.62866	1.41023	**1.17341**
	0.99	413.36752	123.02816	0.75546	183.30267	19.62992	0.45599	**0.42706**
	0.999	4211.22570	1258.00468	1.07581	1861.28011	197.48937	0.13790	**0.13670**

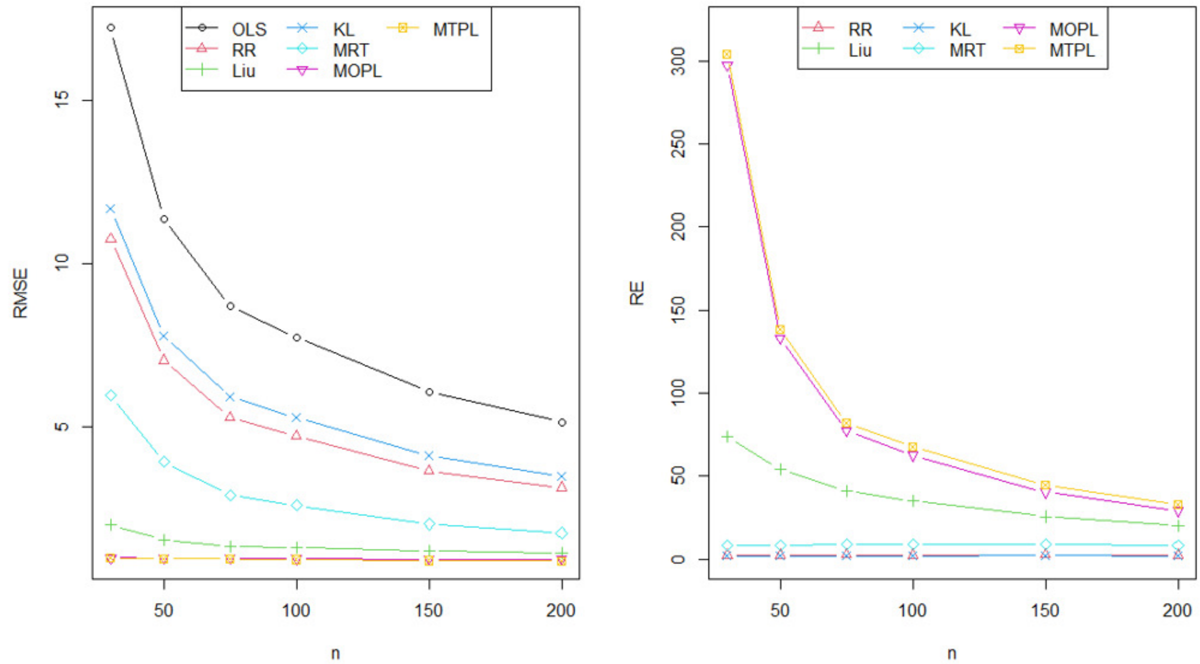

Figure 1. RMSE and RE of different estimators categorized by n.

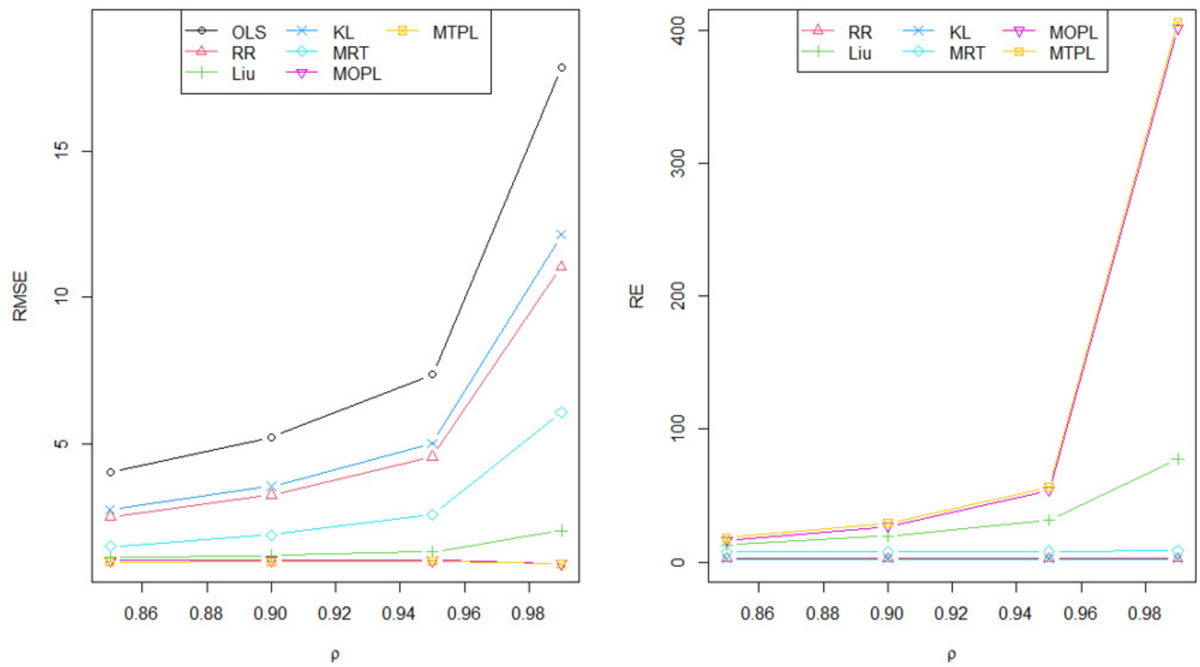

Figure 2. RMSE and RE of different estimators categorized by ρ.

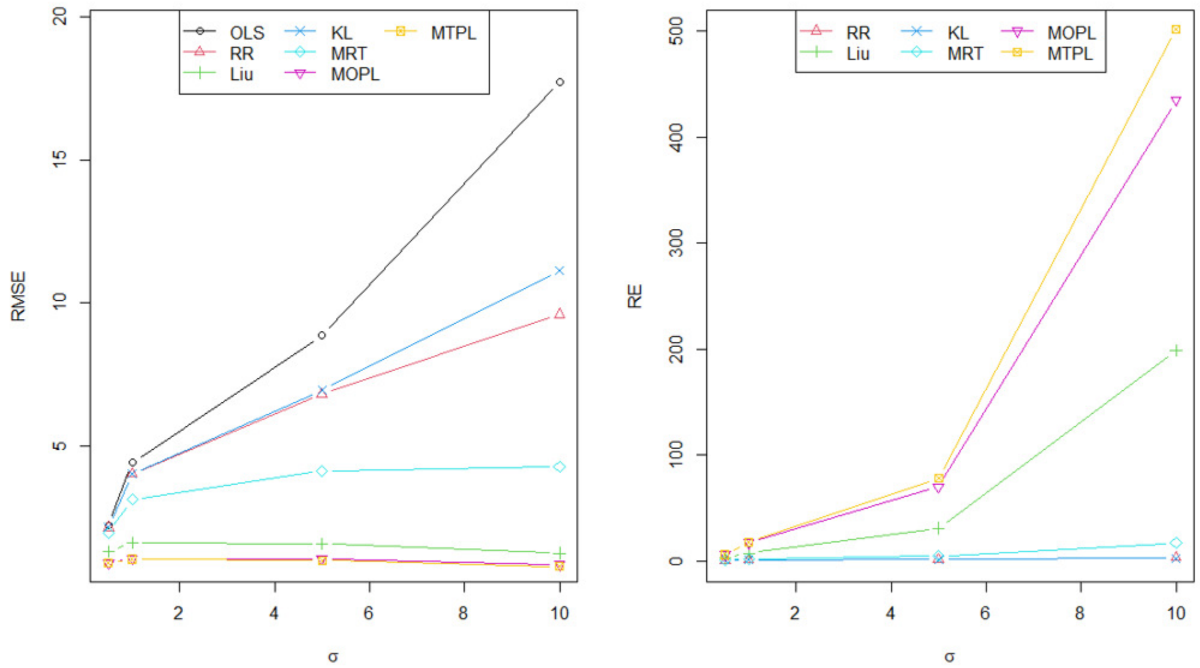

Figure 3. RMSE and RE of different estimators categorized by σ.

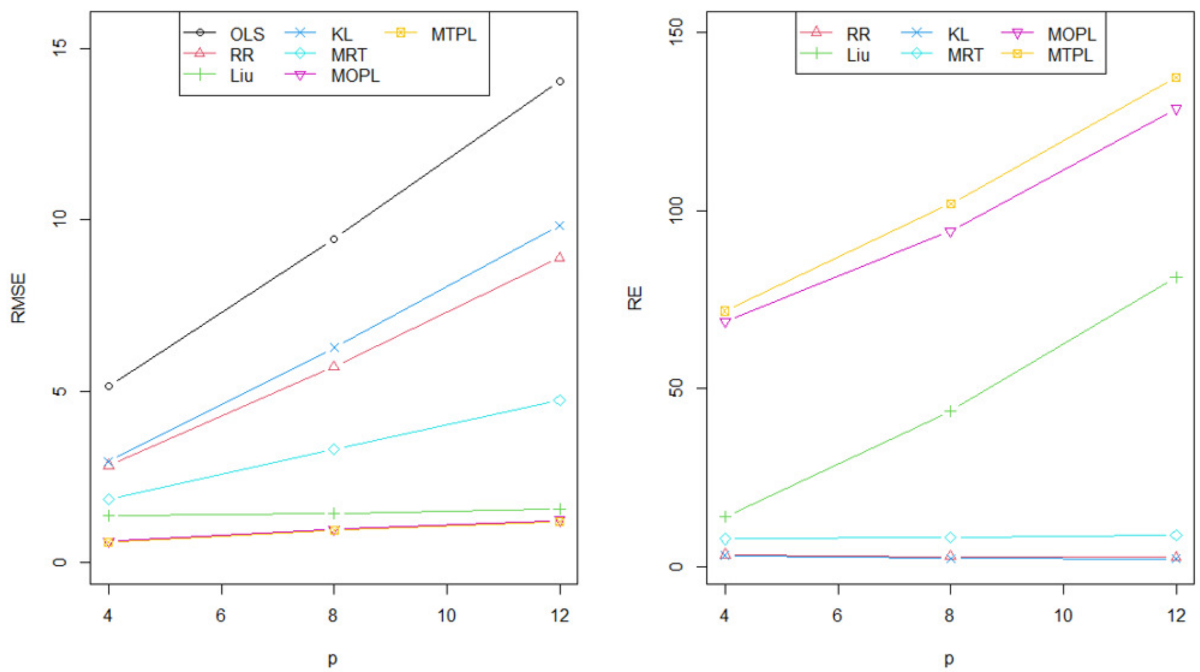

Figure 4. RMSE and RE of different estimators categorized by p.

Figures 1 – 4 present the RMSE of the seven estimators (OLS, RR, Liu, KL, MRT, MOPL, and MTPL) and the RE of the six estimators (RR, Liu, KL, MRT, MOPL, and MTPL) for different values of n, ρ, σ, and p, respectively. Figures 1 – 4 indicate that the OLS estimator has the largest value of RMSE, while the MTPL estimator has the smallest value of RMSE.

For RE, also the MTPL estimator has higher RE than the RR, Liu, KL, MRT, and MOPL estimators for different levels of the sample size (n), levels of the correlation degree between explanatory variables (ρ), levels of the standard deviation of the error (σ), and the number of regression coefficients (p).

6. Real-life Applications

6.1. Application I: Portland Cement Data

To explore the MTPL estimator performance and other estimators, the Portland cement data is used that originally adopted by [41]. This data was applied in several studies, such as [17, 18, 35, 42]. The model of this data is considered as follows

$$y_i = \gamma_1 x_{i1} + \gamma_2 x_{i2} + \gamma_3 x_{i3} + \gamma_4 x_{i4} + \varepsilon_i;\ i = 1, \ldots, n, \tag{6.1}$$

where y is the heat evolved after 180 days of curing measured in calories per gram of cement, x_1 is tricalcium aluminate, x_2 is tricalcium silicate, x_3 is tetracalcium aluminoferrite, and x_4 is β-dicalcium silicate. The eigenvalues of Z matrix are 44676.2, 5965.4, 810.0, and 105.4. While the condition number is CN $= \sqrt{\lambda_{max}/\lambda_{min}} = 20.6$; where λ_{max} is the largest value of the eigenvalues. The variance inflation factors (VIFs) are 38.50, 254.42, 46.87, and 282.51. Therefore, there is multicollinearity among the explanatory variables because the VIF values are greater than 10. The estimated variance of the error is $\widehat{\sigma}^2 = 5.8$ which shows high noise in the data. Table 14 displays the estimated coefficients of the model and the estimated MSE for each estimator. Note that the values of MSE of OLS, RR, Liu, MRT, KL, MOPL, and MTPL estimators that are presented in Table 14 are calculated based on equations (2.2), (2.6), (2.10), (2.14), (2.18), (2.22), and (2.26), respectively.

Table 14. Estimated coefficients and estimated MSE of Portland Cement Data

Estimator	x_1	x_2	x_3	x_4	k	d	MSE
OLS	2.19305	1.15333	0.75851	0.48632	——	——	0.063785
RR	2.19098	1.15380	0.75702	0.48663	0.149	——	0.063632
Liu	2.17931	1.15650	0.74864	0.48839	——	0.000	0.063029
KL	2.14734	1.16389	0.72567	0.49322	1.674	——	0.063636
MRT	2.15975	1.16101	0.73461	0.49134	1.234	0.990	0.063007
MOPL	2.17522	1.15745	0.74569	0.48901	——	0.298	0.062923
MTPL	2.16552	1.15969	0.73871	0.49048	0.707	0.298	0.062886

From Table 14, we can note the following:

1. The estimated regression coefficients of all estimators (OLS, RR, Liu, KL, MRT, MOPL, and MTPL) have the same signs; it means that the type of relationship between each x variable and y variable remains unchanged from the OLS estimation.
2. The MSE values of the RR, Liu, KL, MRT, MOPL, and MTPL estimators are lower than OLS. On the other hand, the MTPL estimator has the lowest MSE value.

Through this application, we can verify the above six theorems as follows:

1. The necessary condition of theorem 1 is satisfied for all $j = 1, \ldots, 4$, so $\widehat{\sigma}^2 \sum_{j=1}^{p} \left(A_j^2 - B_j^2\right) = 1208020 > (k_{\text{MTPL}} + d_{\text{MTPL}} + 1)^2 \sum_{j=1}^{p} \lambda_j \alpha_j^2 = 486415.1$ then $\text{MSE}\left(\widehat{\gamma}\right) > \text{MSE}\left(\widehat{\gamma}_{k,d_0}\right)$. Then the MTPL estimator outperforms the OLS estimator.

2. The necessary condition of theorem 2 is satisfied for all $j = 1, \ldots, 4$, so $\widehat{\sigma}^2 \sum_{j=1}^{p} \left(\lambda_j^2 A_j^2 - B_j^2 (\lambda_j + \mathrm{k})^2\right) = 1.355\mathrm{e} + 15 > \sum_{j=1}^{p} \lambda_j \alpha_j^2 \left((\lambda_j + \mathrm{k})^2 (\mathrm{k} + d_{\mathrm{MTPL}} + 1)^2 - k^2 A_j^2\right) = 8.406\mathrm{e} + 14$, then $\mathrm{MSE}\left(\widehat{\gamma}_k\right) > \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right)$. Then the MTPL estimator outperforms the RR estimator.
3. The necessary condition of theorem 3 is satisfied for all $j = 1, \ldots, 4$, so $\widehat{\sigma}^2 \sum_{j=1}^{p} \left((\lambda_j + d_{\mathrm{Liu}})^2 - B_j^2\right) = 605399.5 > \left((k_{\mathrm{MTPL}} + d_{\mathrm{MTPL}} + 1)^2 - (d_{\mathrm{Liu}} - 1)^2\right) \sum_{j=1}^{p} \lambda_j \alpha_j^2 = 365379.7$, then $\mathrm{MSE}\left(\widehat{\gamma}_d\right) > \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right)$. Then the MTPL estimator outperforms the Liu estimator.
4. The necessary condition of theorem 4 is satisfied for all $j = 1, \ldots, 4$, so $\widehat{\sigma}^2 \sum_{j=1}^{p} \left(A_j^2 (\lambda_j - \mathrm{k})^2 - B_j^2 (\lambda_j + \mathrm{k})^2\right) = 6.171e + 14 > \sum_{j=1}^{p} \lambda_j \alpha_j^2 \left((\lambda_j + \mathrm{k})^2 (\mathrm{k} + d_{\mathrm{MTPL}} + 1)^2 - 4k^2 A_j^2\right) = 4.826e + 14$, then $\mathrm{MSE}\left(\widehat{\gamma}_{KL}\right) > \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right)$. Then the MTPL estimator outperforms the KL estimator.
5. The necessary condition of theorem 5 is satisfied for all $j = 1, \ldots, 4$, so $\widehat{\sigma}^2 \sum_{j=1}^{p} \left(\lambda_j^2 A_j^2 - B_j^2 (\lambda_j + \mathrm{k}\,(1 + d_{\mathrm{MRT}})\,)^2\right) = 6.243e + 14 > \sum_{j=1}^{p} \lambda_j \alpha_j^2 \left((\lambda_j + \mathrm{k}\,(1 + d_{\mathrm{MRT}})\,)^2 (\mathrm{k} + d_{\mathrm{MTPL}} + 1)^2 - \mathrm{k}^2 A_j^2 (1 + d_{\mathrm{MRT}})^2\right) = 4.872e + 14$, then $\mathrm{MSE}\left(\widehat{\gamma}_{MRT}\right) > \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right)$. Then the MTPL estimator outperforms the MRT estimator.
6. The necessary condition of theorem 6 is satisfied for all $j = 1, \ldots, 4$, so $\widehat{\sigma}^2 \sum_{j=1}^{p} \left((\lambda_j - d)^2 - B_j^2\right) = 425914.9 > k_{\mathrm{MTPL}}\left(k_{\mathrm{MTPL}} + 2\,(\mathrm{d} + 1)\right) \sum_{j=1}^{p} \lambda_j \alpha_j^2 = 282539.6$, then $\mathrm{MSE}\left(\widehat{\gamma}_{d_0}\right) > \mathrm{MSE}\left(\widehat{\gamma}_{k,d_0}\right)$. Then the MTPL estimator outperforms the MOPL estimator.

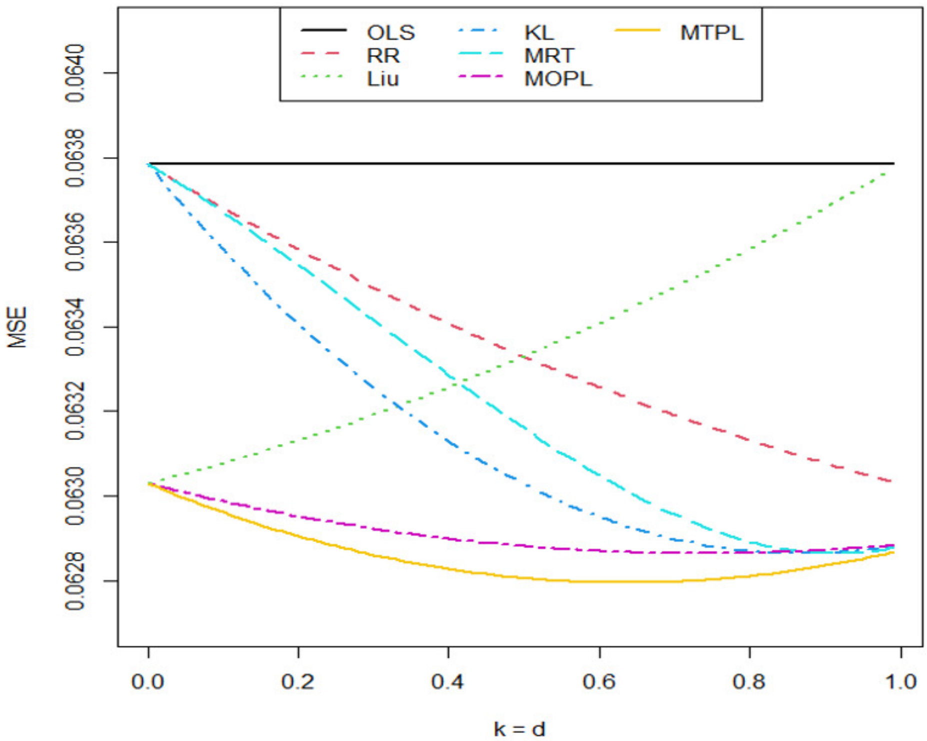

Figure 5. The MSE values of each estimator at different values of k and d (Portland cement data).

This real data is used to study the efficiency of the proposed MTPL estimator and the other estimators at different values of k and d. Figure 5 displays the MSE values for the OLS, RR, Liu, KL, MRT, MOPL, and MTPL estimators at the interval of k, d from zero to one ($0 \leq k = d \leq 1$). Figure 5 shows that the proposed MTPL estimator has the lowest MSE values, especially when $0 < k = d < 0.8$. This means that the proposed MTPL estimator is more efficient than the OLS, RR, Liu, KL, MRT, and MOPL estimators even if $k = d$.

6.2. Application II: Company Efficiency Data

The second application is related to finance data. The study uses annual data for 100 Egyptian companies listed on the stock exchange in 2021. Secondary data that is used is extracted from Thomson Reuters DataStream (https://www.library.ucsb.edu/node/7813). The response variable (y) represents the company efficiency (asset turnover ratio). After reviewing the literature [43, 44, 45], various factors (explanatory variables) affecting the efficiency of the company were identified as follows: tangibility (x_1), profitability (x_2), working capital (x_3), productivity (x_4), liquidity (x_5), leverage (x_6), and volatility (x_7) . The considered model of this data is

$$y_i = \gamma_1 x_{i1} + \gamma_2 x_{i2} + \gamma_3 x_{i3} + \gamma_4 x_{i4} + \gamma_5 x_{i5} + \gamma_6 x_{i6} + \gamma_7 x_{i7} + \varepsilon_i;\ i = 1, \ldots, 100. \tag{6.2}$$

The eigenvalues of $Z = X'X$ are 660.40, 6.11, 5.42, 5.31, 4.41, 3.67, and 2.81, then CN is equal to 15.33. The values of the VIF for the explanatory variables are 10.96, 9.74, 12.10, 8.86, 10.66, 9.40, and 54.43, respectively. Therefore, multicollinearity exists among the explanatory variables. Table 15 displays the estimated coefficients and the estimated MSE for each estimator.

Table 15. Estimated coefficients and estimated MSE of Company Efficiency Data

Variable	**OLS**	**RR**	**Liu**	**KL**	**MRT**	**MOPL**	**MTPL**
x_1	1.04671	1.04612	1.04238	1.04632	1.04612	1.04238	1.04082
x_2	0.79353	0.80416	0.82637	0.80420	0.80416	0.82637	0.83819
x_3	1.22368	1.20963	1.18009	1.20960	1.20963	1.18009	1.16441
x_4	1.07320	1.06713	1.05407	1.06714	1.06713	1.05407	1.04718
x_5	0.71142	0.72577	0.75597	0.72580	0.72577	0.75597	0.77200
x_6	0.91637	0.91954	0.92672	0.91951	0.91954	0.92672	0.93044
x_7	1.28482	1.27540	1.25639	1.27530	1.27540	1.25639	1.24617
k	——	0.281	——	0.137	0.270	——	0.360
d	——	——	0.000	——	0.041	0.000	0.000
MSE	0.41282	0.36131	0.27286	0.36086	0.36131	0.27286	0.23387

As in the first application, the results of Table 15 show that the estimated coefficients of OLS, RR, Liu, KL, MRT, MOPL, and MTPL have the same signs. Furthermore, the MSE values of the biased estimators (RR, Liu, KL, MRT, MOPL, and MTPL) are lower than the OLS estimator. Whereas, the MTPL estimator has the lowest MSE value. The above six theorems have been verified as follows:

1. For theorem 1, the required condition is fulfilled for all $j = 1, \ldots, 7$, so $\widehat{\sigma}^2 \sum_{j=1}^{p} \left(A_j^2 - B_j^2\right) = 68367.13 >$ $(k_{\text{MTPL}}+d_{\text{MTPL}}+1)^2 \sum_{j=1}^{p} \lambda_{\text{j}} \alpha_{\text{j}}^2 = 29263.82$. Then the MTPL estimator outperforms the OLS estimator.
2. For theorem 2, the required condition is fulfilled for all $j = 1, \ldots, 7$, so $\widehat{\sigma}^2 \sum_{j=1}^{p} \left(\lambda_j^2 A_j^2 - B_j^2 (\lambda_{\text{j}}+\text{k})^2\right) = 1.7\text{E}+10 >$ $\sum_{j=1}^{p} \lambda_{\text{j}} \alpha_{\text{j}}^2 \left((\lambda_{\text{j}}+\text{k})^2 (\text{k}+d_{\text{MTPL}}+1)^2 - k^2 A_j^2\right) = 1.1\text{E}+10$. Then the MTPL estimator outperforms the RR estimator.
3. For theorem 3, the required condition is fulfilled for all $j = 1, \ldots, 7$, so $\widehat{\sigma}^2 \sum_{j=1}^{p} \left((\lambda_{\text{j}}+d_{\text{Liu}})^2 - B_j^2\right) = 54499.49 >$ $\left((k_{\text{MTPL}}+d_{\text{MTPL}}+1)^2 - (d_{\text{Liu}}-1)^2\right) \sum_{j=1}^{p} \lambda_{\text{j}} \alpha_{\text{j}}^2 = 28083.84$. Then the MTPL estimator outperforms the Liu estimator.
4. For theorem 4, the required condition is fulfilled for all $j = 1, \ldots, 7$, so $\widehat{\sigma}^2 \sum_{j=1}^{p} \left(A_j^2 (\lambda_{\text{j}}-\text{k})^2 - B_j^2 (\lambda_{\text{j}}+\text{k})^2\right) =$ $5.88\text{E}+09 > \sum_{j=1}^{p} \lambda_{\text{j}} \alpha_{\text{j}}^2 \left((\lambda_{\text{j}}+\text{k})^2 (\text{k}+d_{\text{MTPL}}+1)^2 - 4k^2 A_j^2\right) = 4.71\text{E}+09$. Then the MTPL estimator outperforms the KL estimator.
5. For theorem 5, the required condition is fulfilled for all $j = 1, \ldots, 7$, so $\widehat{\sigma}^2 \sum_{j=1}^{p} \left(\lambda_j^2 A_j^2 - B_j^2 (\lambda_{\text{j}}+\text{k}\,(1+d_{\text{MRT}}))^2\right) =$ $1.16E+10 > \sum_{j=1}^{p} \lambda_{\text{j}} \alpha_{\text{j}}^2 \left((\lambda_{\text{j}}+\text{k}\,(1+d_{\text{MRT}}))^2 (\text{k}+d_{\text{MTPL}}+1)^2 - \text{k}^2 A_j^2 (1+d_{\text{MRT}})^2\right) = 8.27E+09$. Then the MTPL estimator outperforms the MRT estimator.

6. For theorem 6, the required condition is fulfilled for all $j = 1, \ldots, 7$, so $\widehat{\sigma}^2 \sum_{j=1}^{p} \left((\lambda_{\mathrm{j}} - d)^2 - B_j^2\right) = 26974.21 > k_{\mathrm{MTPL}}\left(k_{\mathrm{MTPL}} + 2\,(\mathrm{d}+1)\right) \sum_{j=1}^{p} \lambda_{\mathrm{j}} \alpha_{\mathrm{j}}^2 = 14037.91$. Then the MTPL estimator outperforms the MOPL estimator.

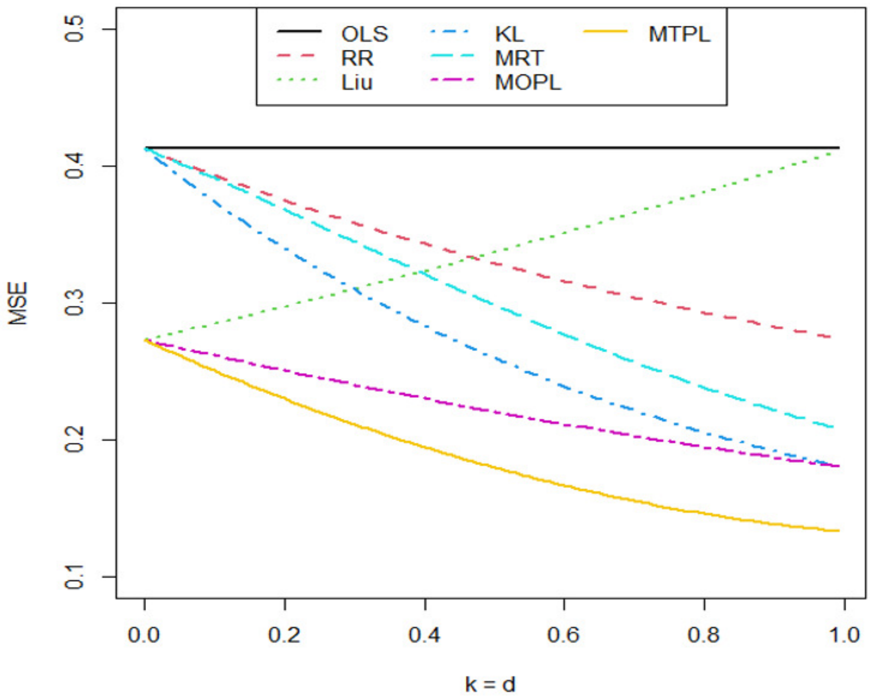

Figure 6. The MSE values of each estimator at different values of k and d (Company Efficiency data).

Again, the efficiency of the MTPL estimator is examined at different values of k and d based on the second real-life application (company efficiency data). Figure 6 displays the MSE values for the OLS, RR, Liu, KL, MRT, MOPL, and MTPL estimators at the interval of k, d from zero to one ($0 \leq k = d \leq 1$). Figure 6 confirms that the proposed MTPL estimator is more efficient than the OLS, RR, Liu, KL, MRT, and MOPL estimators in case of $0 < k = d < 1$.

7. Conclusion

This paper developed the modified two-parameter Liu (MTPL) estimator for the regression model to deal with the issue of multicollinearity. We demonstrated that our proposed estimator is more efficient than previous biased estimators proposed in the literature (ridge, Liu, KL, MRT, and modified one–parameter Liu estimators). The simulation study and empirical application were also carried out to examine the performance of the proposed estimator and compare it with the OLS, ridge, Liu, KL, MRT, and modified one–parameter Liu estimators. The results indicated that the proposed MTPL estimator outperforms these estimators, in terms of MSE reduction, especially when the explanatory variables are highly correlated (correlation coefficient more than or equal to 0.85) and when the sample size is less than or equal to 50. Therefore, we recommend that practitioners use the proposed estimator to obtain an efficient estimate of the regression parameters if the multicollinearity problem occurs in the model. In future work, we can propose new shrinkage parameters (k, d) for the proposed MTPL estimator, as well as provide a robust version of the MTPL estimator for the linear regression model and/or generalized linear model as in [46, 47, 48, 49, 50, 51, 52, 53].

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