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On the Maintance of a system of machines by one repairman

by

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On the maintance of a system of machines by one repairman

Consider a system of n machines of the same type. Each of the machines "at randam instants" may require the attention of a repairman. The machines break down independantly of each other, the life time of any machine is a randam variable and has the expanential function distribution with parameter λ the time of repair of any broken down machine is a randam variable having the destribution function $G(x)$. All the machines are operating in parallel, and the whole system will be out of operation as soon as the number of broken machines becomes greater than m where $m < n$.

We shall try to find the probability that the whole system operates a time less or equal to t, and the expected time during which the system can operate.

1. We first define the following functions

   1. $\varphi_i(x,z)$ - the probability that i machines are out of operation, repairman spends time equals z in repairing one of them and the whole system can operate for time x before getting idle.

   2. $P(x)$ - the probability that the system can operate for time x.

      then $\varphi(x) = 1 - p(x)$ is the distbution function of the time of operation of the system.

      It is dear that $p(x) = \varphi_0(x,o)$

      We can easily see that

$$p(x) = e^{-n\lambda x} + \int_0^x n\lambda\, e^{-n\lambda\tau}\varphi_{i}(x-\tau, Q)\, d\tau \quad \ldots (1)$$

and

$$\varphi_i(x,z) = \frac{1-G(x+z)}{1-G(z)} \sum_{k=o}^{m-i} C_{ik}(x) + $$

$$+ \frac{1}{1-G(z)} \int_o^x \sum_{k=o}^{m-i} C_{ik}(\tau)\, \varphi_{k+i-1}(x-\tau, o)\, dG(x+\tau) \quad \ldots (2)$$

where

$$C_{ik}(t) = \binom{n-i}{k} e^{(n-i-k)\lambda t} (1-e^{-\lambda t})^k \quad \ldots (3)$$

To obviate these relations we explain the process, in words, as follows:

If the system starts functioning without any broken machine then it will centinue its functioning for time x till one of the following cases ocuers.

1. no machine goes out of operation during time x.
2. At time τ, from the begining one machine breeks and the repairman starts to repair it, but the whole system continues to operate for time $x-\tau$, this expains relation(1).
   If in the system at a given instance there are i machines idle and the repairman stants repairing one of them at time z before this moment, them the probability that it will continue to operate a time x after this given moment $\varphi_i(x,z)$ is equal to the probability of occurance of one of the following independant cases.

1. Repain which started a time x ago is not finished at the moment x, after the given instant, but not more than m-i machines could go out of operations

2. the repair which started a time x ago is finished at a time τ after the given moment, during this time not more than m-i machines go out of operation and the system will coutinue to operate for time $x-\tau$.

$C_{ik}(x)$-is the probability that from the n-i operating machines k go out of operation. during a time x

To find $\varphi(x)$, put $z = o$ in equation (2).
Using laplace transform relations (1) and (2) take the form:

$$\varphi(s) = \frac{n}{n+s}\left(1 - s\,\tilde{\varphi}_1(s)\right) \qquad (3)$$

$$\tilde{\varphi}_i(s) = g_i(s) + \sum_{k=o}^{m-i} f_k(s)\,\tilde{\varphi}_{k+i-1}(s) \qquad (4)$$

where

$$\varphi(s) = \int_o^\infty e^{-sx}\, d\Phi(x)$$

$$\tilde{\varphi}_i(s) = \int_o^\infty e^{-sx}\, \varphi_i(x,o)\, dx$$

$$g_i(s) = \int_o^\infty \left[1 - G(x)\right] \sum_{k=o}^{m-i} C_{ik}(x)\, dx$$

$$f_k(s) = \int_o^\infty e^{-sx}\, C_{ik}(x)\, d\,G(x).$$

we see that $\tilde{\varphi}_o(s) = \dfrac{1 - \varphi(s)}{s}$

put $i = 1$ in (4) we get

$$\tilde{\varphi}_1(s) = g_1(s) + f_0(s)\ \tilde{\varphi}_0(s) + \sum_{k=1}^{m-i} f_k(s)\, \tilde{\varphi}_k(s)$$

$$g_1(s) = \int_0^\infty [1 - G(x)] \sum_{k=0}^{m-1} \binom{n-1}{k} e^{-(n-k-1)\lambda x} .$$

$$\cdot (1 - e^{-\lambda x})^k e^{-sx} dx$$

$$= \int_0^\infty [1 - G(x)]\, e^{-((n-1)\lambda + s)} dx +$$

$$+ \tilde{g}_1(s)$$

$$= \frac{1}{(n-1)\lambda + s} [1 - g((n-1)\lambda + s)] + \tilde{g}_1(s)$$

where
$$\tilde{g}_1(s) = \int_0^\infty [1 - G(x)] \sum_{k=1}^{m-1} \binom{n-1}{k} e^{-(n-k-1)}\ (1 - e^{-\lambda x})^k dx$$

and
$$f_0(s) = \int_0 e^{-(n-1)\lambda x}\ e^{-sx}\ dG(x)$$

$$= g((n-1)\lambda + s)$$

substituting in (3) we get.

$$\varphi(s) = \frac{n(n-1)\lambda^2 \left[1 - g((n-1)\lambda + s)\right] - (n-1)\lambda s \left[(n-1)\lambda + s\right]}{s + n\lambda(1 - g((n-1)\lambda + s)}$$

$$\cdot \frac{\tilde{g}_1(s) + \sum_{k=1}^{m-1} f_k(s)\, \tilde{\varphi}_k(s)}{(S + (n-1) \quad)}$$

or

$$\varphi(s) = \frac{(n\lambda)(n-1)\lambda\left[1 - g((n-1)\lambda + s)\right]}{((n-1)\lambda + s)\left[s + n\lambda(1 - g(n-1)\lambda + s)\right]} -$$

$$- \frac{(n-1)\lambda s\left[\tilde{g}_1(s) + \sum_{k=1}^{m-1} f_k(s)\, \tilde{\varphi}_k(s)\right]}{s + n\lambda\left[1 - g((n-1)\lambda + s)\right]}$$

To obtain the expected length of the operating period of the system we find the derivative of relation (5) at s = o.

Let μ be the expected duration of the operating period of the system.

then

$$\mu = \frac{(n-1)\lambda + (n\lambda)\left[1 - g(n-1)\lambda\ )\right]}{(n-1)n\lambda^2\left[1 - g(n-1)\lambda + s\ )\right]} +$$

$$+ \frac{(n-1)\lambda\left[\tilde{g}_1(o) + \sum_{k=1}^{m-1} f_k(o)\, \tilde{\varphi}_k(o)\right]}{\left[n\lambda\,(1 - g((n-1)\lambda\ )\right]} \qquad (6)$$

where $\tilde{g}_1(o) = \sum_{k=1}^{m-1} \binom{n-1}{k} \int_0^\infty \left[1 - G(x)\right] e^{(n-1-k)\ x} (1 - e^{\lambda x})^k\, dx$

$$f_k(o) = \int_0^\infty q_k(x)\ d\,G(x)$$

$$\tilde{\varphi}_k(o) = \int_o^\infty \varphi_k\,(x,o)\,dx = \mu_k$$

where μ_k is the expected duration of the operating period of the system if it started with k broken machines.

μ_k could be found by solving the following system of equations

from relations (3) and (4)

$$\mu = \frac{1}{n\lambda} + \mu_1$$

$$\mu_i = g_i(o) + \sum_{k=o}^{m-i} f_{ik}(o)\ \mu_{k+i-1} \qquad i=1,2,\ldots,m \qquad \ldots (8)$$

we notice that $\mu = \mu_o$

Solving this set of algebraic equations we can find u_k, for all $k = o,1,2,\ldots,m$.

The first term in the R.H.S. of relation (6) is the same as the one in the relation of expected duration of the operating period, which was obtained in refrence, (1) but with slight modification. In other words this first term is the expected duration of the operating period of a system containing (n-1) main machines and one stand-by relief, for peak operation, "hot reserve which can go out of operation in the same manner of other operating machines.

Then the second term in the R.H.S. of relation (6) is the expected value of the mean duration of the period of operation of the whole system due to (m-1) other leaded stand by machines.

II. How to compute the optimal size of reserve:

We can use relation (6) to find the optimal size of reserve in our case:

Let C_1 be the cost of one machine and C_2 - the profet per unite oparating time of the system. them for a system consisting of $n + r - m$ machines from which $(n - m)$ main oparating ones and r reserve,

the optimal size of r (where $r > m$) will satisfy the following inequality

$$C_2 \mu' - C_2 \mu - (r-m) \; C_1 > 0$$

μ is given by relation (6) and μ' is the mean oparating time of the system consisting of $n - m$ main machines and r hot stand-by ones.

μ' could be found, if we put $n - m + r$ in relation (6) instead of n.

III. We now prove a theorem which is the same as the one proved in the case of non-loaded reserve. (1).

theorem

If the distribution function of repair G(x) depends on a parameter ν such that

$$\int_0^\infty \left[1 - e^{-(n-1)\lambda x}\right] d \; G_\nu(x) = \alpha_\nu \longrightarrow 0 \text{ as } \nu \to \infty \quad (8)$$

them the distribution function of the operating time of the whole system tends to the exponential one.

Condition (8) means that the probability that no machine can break down during the repair of the broken one

Proof

$$\frac{1 - g_v\,(s\alpha_v + (n-1)\lambda)}{1 - g_v\,((n-1)\lambda)} = 1 + \frac{g_v((n-1)\lambda) - g_v(s\alpha_v - (n-1)\lambda)}{1 - g_v\,((n-1)\lambda)}$$

$$g_v\,((n-1)\lambda) - g_v\,((n-1)\lambda + s\,\alpha_v)$$

$$= \int_0^\infty \left[1 - \bar{e}^{s\alpha_v x}\right] \bar{e}^{(n-1)\lambda x}\, d\,G_v(x) \leq$$

$$\leq \int_0^\infty x\,\alpha_v\, s\;\bar{e}^{(n-1)\lambda x}\, d\,G_v(x)$$

and

$$\int_0^\infty x\,\bar{e}^{(n-1)\lambda x} = \int_0^\varepsilon x\,\bar{e}^{(n-1)\lambda x}\, d\,G_v(x) +$$

$$+ \int_\varepsilon^\infty x\,\bar{e}^{(n-1)\lambda x}\, d\,G_v(x) \leq$$

$$\leq \varepsilon + \max_x\; x\,\bar{e}^{(n-1)\lambda x} \int_\varepsilon^\infty d\,G_v\,(x) \leq$$

$$\leq \varepsilon + \frac{\varepsilon}{\lambda e} = \varepsilon\,A$$

Where ε is any small + ve number

then $$0 \leq g_v\,((n-1)\lambda) - g_v\,((n-1)\lambda + s\,\alpha_v) \leq \varepsilon\,A$$

and as $\varepsilon \longrightarrow 0$

then $$\frac{1 - g_v\,((n-1)\lambda + s\,\alpha_v)}{1 - g_v\,((n-1)\lambda)} \longrightarrow 1 \qquad (10)$$

for every finite interval of s. Putting α_v's instead of s in (5) we have .

$$\varphi(s\alpha_v) = \frac{n\,(n-1)\lambda^2\left[1 - g_\nu((n-1)\lambda + \alpha_v\, s)\right]}{\left[n-1\lambda + s\alpha_v\right]\left[\alpha_\nu s + n\lambda\left(1-g_\nu((n-1)\lambda + s\alpha_\nu)\right)\right]}$$

$$- \frac{(n-1)\lambda\, s\,\alpha_\nu\, \tilde{g}_{1\nu}(s) + \sum_{n=1}^{m-1} f_n(s\alpha_v)\ \tilde{\varphi}_k(s\alpha_v)}{\left[s + n\lambda\,(\,1-g_\nu(n-1)\lambda + s\alpha_v)\right.}$$

But using (9) we can see that

$$\frac{\dfrac{n\,(n-1)\lambda^2\left[1 - g_\nu(n-1)\lambda + s\,\alpha_v\right)}{\alpha_v}}{((n-1)\lambda + s\alpha_v)\left[\dfrac{s\alpha_v}{\alpha_\nu} + n\lambda\,\dfrac{(1-g_\nu((n-1)\lambda + s\alpha_v)}{\alpha_\nu}\right]}$$

tends to $\dfrac{n\lambda}{n\lambda + s}$ (11)

as $\alpha_v \longrightarrow 0$

and

$$\tilde{g}_{1\nu}(s\alpha_v) = \int_0^\infty (1 - G_v\,(x)) \sum_{n-1}^{m-1} \binom{n-1}{k} .$$

$$. \, e^{-(n-1-k)\lambda x}\,(1-e^{-\lambda x})^k\, e^{-s\alpha_v x}\, d\,x \longrightarrow 0 \quad (12)$$

as $\alpha_v \longrightarrow 0$

that is because as $\alpha_v \longrightarrow 0$

$$\tilde{g}_{1\nu}(s\alpha_v) \longrightarrow \int_0^\infty (1-G_v(x)) \sum_{n=1}^{m-1} \binom{n-1}{k} e^{-(n-1-n)\lambda x} \cdot$$

$$\cdot (1-e^{-\lambda x})^k \; d\,x$$

and this is the probability that at least one machine will breaks down before the finishing of repair of the broken one, and from condition (8), this probability tends to zero as ν tends to ∞.

$$\sum_{k-1}^{m-1} f_{1k}(s\alpha_v) \; \tilde{\varphi}_k (s\alpha_v) \longrightarrow 0 \qquad (13)$$

because $\tilde{\varphi}_k(s\alpha_v)$ is finite for all k, and $\lim_{v \to 0} f_n (s_v) =$

$$\lim_{\&\to 0} f_x (s\&y) = \int_0^\infty C_{1k}(x) \; d\, G_v (x)$$

which is the probability that k machines will break down before the finishing of repair of the broken one and from condition (8) this tends to zero as $\gamma \longrightarrow \infty$.

from (11) and (12) we have

$$\frac{\dfrac{(n-1)\lambda \; s\alpha_v \; \tilde{g}_{1\nu}(s\alpha_v) + \sum_{n=1}^{m-1} f_k (s\alpha_v) \; \tilde{\varphi}_k (s\alpha_v)}{\alpha_\nu}}{\dfrac{s\alpha_v + n\lambda\,(1-g_\nu((n-1)\lambda + s\alpha_v)}{\alpha_v}} \longrightarrow 0$$

as $\quad \alpha_\nu \longrightarrow 0$ (14)

from (11) and (14) we finaly see that

$$\varphi(s\alpha_v) \longrightarrow \frac{n\lambda}{n\lambda+s} \qquad \text{as } \alpha_v \longrightarrow 0$$

which complete the proof of the theorem.

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