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Matrix Inversion
(Elimination Methods)

By

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CAIRO

"Opinions Expressed and Positions Taken by Authors are Entirely Their Own and do not Necessarily Reflect the Views of the Institute of National Planning".

PREFACE

Perhaps no field of science has changed so rapidly in the past few years as has the field of applied mathematics. The most important factor in this change has been the development of the high-speed digital computer. Indeed, it is not too much to claim that the development of the computer has revolutionized some parts of mathematics.

This memo. attempts to do two things in relation to the situation just descirbed - to give a short notes about Matrix Algebra and to give usable computer methods for solving systems of linear equations.

We aim that this would help as a benchmark for different researchers who need to use the computer in that field of algebra.

In the future, we hope to be able to extend our presentation to more detailed mathematical branches. In which a case, each will be presented in a separate memo'.s.

Finally we would like to thank Dr. Youssef Nasr El Din, for his cooperation which led to this memo., to Mrs. Ellen and Mrs. Sawsan for the great care and the many tedious hours of typing this memo.

AFAF - YEHIA

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CONTENTS

I. MATRIX OPERATIONS

I.1. Matrix Algebra:

Matrices provide a useful method for systematising both the theoretical and the practical aspects of certain computing procedures, particularly in connection with automatic computers.

Arectangular matrix A which is arranged in m rows and n columns is said to be of order m by n or m x n.

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The (i,j) the element a_{ij} represents the element in the ith row and the jth column of the matrix

Definitions:

1) A matrix with only one row is called a row matrix (or a row vector) and a matrix with only one column is called a column matrix (or a column vector).

2) The sealar product of a row matrix and a column matrix is meaningful if and only if the row and column have the same number of terms, and then it consists of a single number defined as in the following example,

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n .$$

3) A matrix with the same number of rows and columns is called a square matrix.

4) A diagonal matrix $D = d_{ij}$ is a square matrix with all the d_{ij} zero except d_{11}, d_{22}, ..., d_{nn} which are the diagonal elements. A diagonal matrix all of whose diagonal elements are unity is called a unit matrix and is denoted by I.

5) The transpose matrix A' of a matrix A is defined by the property that if A is an m x n matrix whose (i,j) th element is a_{ij} then A' is an n x m matrix whose (i,j) th element is a_{ji}.

i.e. $\left[a_{ij}\right]' = \left[a_{ji}\right]$

6) A symetrical matrix is a square matrix with

$$a_{ij} = a_{ji} \quad , \quad \text{i.e.} \quad A' = A$$

7) Given a matrix A of order n x n, there exists a matrix z such that

$$A\,Z = I$$

then the matrix z is called the inverse of A and it is denoted by A^{-1}. A necessary condition for the existence of the inverse of A is that det $A \neq 0$.

8) If $\det A = 0$ the matrix A is said to be singular and in this case the inverse does not exist, but if $\det A \neq 0$ the matrix is said to be non singular.

To multiply a matrix by a scalar, say k, each term of the matrix is multiplied by k:

$$kA = [ka_{ij}]$$

Consider operations involving two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$. Equality, addition and subtraction are meaningful terms if and only if the matrices have the same number of rows and the same number of columns. If this is true then:

$$A = B \text{ if and only if } a_{ij} = b_{ij} \text{ for all } i,j.$$

The sum or difference of two matrices with equal numbers of rows and columns is the matrix such that any element is the sum or difference of the corresponding elements in A and B. This is defined by the equation

$$A \pm B = [a_{ij} \pm b_{ij}]$$

Two matrices can be multiplied together if and only if the number of columns in the first is equal to the number of rows in the second. Then the element in the i th row and the jth column of the product is the scalar product of the ith row of the 1st matrix with the jth column of the second.

If A is m x n, B is n x p, and

$$AB = C$$

or

$$[a_{ij}] [b_{jk}] = [c_{ik}]$$

the elements of the product matrix C, is

$$C_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

and C is of order m x p. As an example,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} b_{11} + a_{22} b_{21} & a_{21} b_{12} + a_{22} b_{22} \\ a_{31} b_{11} + a_{32} b_{21} & a_{31} b_{12} + a_{32} b_{22} \end{bmatrix}$$

In the product AB the matrix A is said to premultiply B, and B is said to postmultiply A.

I.2. The Inverse Matrix

Let A be an n x n matrix consisting of elements a_{ij}. we will adopt the following usual notations.

$|A|$ or det A = determinant of A,

A_{ij} = cofactor of the element a_{ij},

A^{-1} = inverse of A.

For D = det A $\neq$ 0, we have A^{-1} defined as

$$A^{-1} = \begin{bmatrix} \frac{A_{11}}{D} & \frac{A_{21}}{D} & \cdots\cdots & \frac{A_{n1}}{D} \\ \frac{A_{21}}{D} & \frac{A_{22}}{D} & \cdots\cdots & \vdots \\ \frac{A_{1n}}{D} & \cdots & \cdots\cdots & \frac{A_{nn}}{D} \end{bmatrix} \qquad (I.2.1)$$

Evaluating A^{-1} from its definition is completely unsatisfactory from a computational point of view. However, the composition of inverse elements is useful in theory.

If D=0 the matrix A is said to be singular. For every nonsingular matrix, A, a unique inverse A^{-1} exists with the following properties:

(1) $AA^{-1} = A^{-1}A = I$ (the identity matrix),

(2) $\det A^{-1} = 1/\det A$,

(3) $(AB)^{-1} = B^{-1} A^{-1}$,

(4) $(A^{-1})^{-1} = A$,

(5) if two rows (columns) of A are transposed to form B, then transposing the corresponding columns (rows) of A^{-1} will yield B^{-1}.

Properties (1) and (4) are sometimes used inchecking the accuracy of a computed inverse.

I.3. Simultaneous Linear Equations

The system of equations

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1, \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdots \quad \cdot \quad \cdot \quad &\cdots \\
a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n &= b_n,
\end{aligned} \qquad \text{(I.3.1)}
$$

Can be represented in matrix form as

$$AX = B$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

The A matrix is called the coefficient matrix, the X vector is called the vector of unknowns, and the B vector is frequently referred to as the right-hand sides.

If A is nonsingular, then A^{-1} exists and we have

$$A^{-1} AX = A^{-1}B \quad , \quad IX = A^{-1}B \quad , \quad X = A^{-1}B.$$

This says that if we have A^{-1}, we can multiply this times B to obtain the solution to the system of equations.

I.4. Orthogonality and Orthonormality

Two vectors are said to be ORTHOGONAL if their product equals zero. Any two of the unit vectors are orthogonal, such as e_1 and e_2, since

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

A system of vectors all of which are orthogonal to each other is called an ORTHOGONAL SYSTEM.

The vecotors may be reduced to unit length, or NORMALIZED, by dividing by the respective values of the length of these vectors.

The system is then said to be ORTHONORMAL. It is convenient, to define an ORTHOGONAL MATRIX as a matrix consisting of a set of orthonormal vectors, which may be placed either column-wise or row - wise.

I.5. The Solution of Simultaneous Linear Equations

In the algebra of real numbers the inverse of any non zero number is defined, such that $a^{-1} \cdot a = a\, a^{-1} = 1$. The question then arises as to whether there is a similar behavior with arrays or whether it is possible to find an inverse of A such that:

$$A^{-1} A = A A^{-1} = I$$

where I is the identity, or unit, matrix. On the assumption for the moment that an inverse may exist, then a system of simultaneous equations might be solved as follows:

$$AX = C$$

Premultiplying both sides by the inverse of the coefficient matrix, A^{-1}, gives

$$A^{-1} A X = A^{-1} C$$

Since $A^{-1} A = I$, and since the scalar matrix I can be replaced by the scalar constant 1,

$$X = A^{-1} C$$

Thus, the result of the multiplication of the inverse of A and the column vector C is the column vector of the X values.

Let us now consider the problem of finding the inverse of a matrix. We may immediately limit ourselves to square matrices, since, for both $A^{-1}A$ and AA^{-1} multiplications to be possible, the column and row dimensions must be equal.

The problem is now one of solving M systems of M simultaneous equations for the unknowns.

Many methods may be used for that purpose.

II - ELIMINATION METHODS

II.1. Gauss Elimination

To illustrate the method, we should first consider the case of three equations in three unknowns:

$$a_{11}\, x_1 + a_{12}\, x_2 + a_{13}\, x_3 = b_1 \qquad \text{(II.1.1)}$$

$$a_{21}\, x_1 + a_{22}\, x_2 + a_{23}\, x_3 = b_2 \qquad \text{(II.1.2)}$$

$$a_{31}\, x_1 + a_{32}\, x_2 + a_{33}\, x_3 = b_3 \qquad \text{(II.1.3)}$$

At least one of a_{11}, a_{21} and a_{31} is not zero, otherwise only two unknowns would appear in the three equations. If a_{11} is zero, we reorder the equations so that the coefficients of x_1 in the first equation is not zero. Interchanging two rows in the system of equations, of course, leaves the system essentially unchanged.

Next define a multiplier

$$m_2 = \frac{a_{21}}{a_{11}}$$

We multiply the first equation (II.1.1) by m_2 and subtract from the second equation (II.1.2) the result is

$$(a_{21} - m_2\, a_{11})x_1 + (a_{22} - m_2\, a_{12})x_2 + (a_{23} - m_2\, a_{13})x_3 = b_2 - m_2 b_1$$

..... (II.1.4.)

But

$$a_{21} - m_2\, a_{11} = a_{21} - \frac{a_{21}}{a_{11}}\, a_{11} = 0$$

If we now define

$$a'_{22} = a_{22} - m_2\, a_{12}$$

$$a'_{23} = a_{23} - m_2\, a_{13}$$

$$b'_2 = b_2 - m_2\, b_1$$

then (II.1.4) becomes

$$a'_{22}\, x_2 + a'_{23}\, x_3 = b'_2 \qquad \text{(II.1.5)}$$

We replace the second of the original equation (II.1.2) by (II.1.5). Similarly, we define a multiplier for the third equation:

$$m_3 = \frac{a_{31}}{a_{11}}$$

We multiply the first equation by this multiplier and subtract from the third. Again the coefficient of x_1 vanishes and the result is

$$a'_{32}\, x_2 + a'_{33}\, x_3 = b'_3 \qquad \text{(II.1.6)}$$

where

$$a'_{32} = a_{32} - m_3\, a_{12}$$

$$a'_{33} = a_{33} - m_3\, a_{13}$$

$$b'_3 = b_3 - m_3\, b_1$$

If we now use (II.1.6) to replace (II.1.3) the resulting three equations in three unknowns are:

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1 \qquad \text{(II.1.1)}$$

$$a'_{22} x_2 + a'_{23} x_3 = b'_2 \qquad \text{(II.1.5)}$$

$$a'_{32} x_2 + a'_{33} x_3 = b'_3 \qquad \text{(II.1.6)}$$

If we can solve the last two equations for x_2 and x_3, the results can be substituted in the first to get x_1.

We can now proceed to eliminate x_2 from one of the last two equations. Again, if $a'_{22} = 0$, we interchange the last two equations. (if it should happen that $a'_{22} = 0$ and $a'_{32} = 0$, the equations are singular and have either no solutions or an infinite number of solutions).

We define a new multiplier m'_3:

$$m'_3 = \frac{a'_{32}}{a'_{22}}$$

We multiply (II.1.5) by m'_3 and subtract from (II.1.6). The result is

$$(a'_{32} - m'_3 a'_{22})x_2 + (a'_{33} - m'_3 a'_{23})x_3 = b'_3 - m'_3 b'_2$$

Again

$$a'_{32} - m'_3 a'_{22} = 0$$

and letting

$$a''_{33} = a'_{33} - m'_3 a'_{23}$$

$$b''_{33} = b'_3 - m'_3 b'_2$$

we get

$$a''_{33} x_3 = b''_3 \qquad \text{(II.1.7)}$$

It is now a straight forward process to solve (II.1.7) for x_3, to substitute that result in (II.1.5) to get x_1. This process, called BACK SUBSTITUTION, is given by

$$x_3 = \frac{b''_3}{a''_{33}}$$

$$x_2 = \frac{(b'_2 - a'_{23}\, x_3)}{a'_{22}}$$

$$x_1 = \frac{(b_1 - a_{12}\, x_2 - a_{13}\, x_3)}{a_{11}}$$

We have therefore found an exact solution in a finite number of arithmetic operations. In this case there were no round off errors.

We may now generalize the procedure to the case of n simultaneous linear equations in n unknowns.

Let the n unknowns be $x_1, x_2, \ldots, x_n$, and let the equations be

$$\begin{array}{ccccccc}
a_{11}x_1 + a_{12}x_2 + & \cdots & + a_{1i}x_i + & \cdots & + a_{1n}x_n & = & b_1 \\
a_{21}x_1 + a_{22}x_2 + & \cdots & + a_{2i}x_i + & \cdots & + a_{2n}x_n & = & b_2 \\
\cdots\ \ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots \\
a_{i1}x_1 + a_{i2}x_2 + & \cdots & + a_{ii}x_i + & \cdots & + a_{in}x_n & = & b_i \\
\cdots\ \ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots \\
a_{n1}x_1 + a_{n2}x_2 + & \cdots & + a_{ni}x_i + & \cdots & + a_{nn}x_n & = & b_n
\end{array} \qquad (II.1.8)$$

We assume that the equations have been so ordered that $a_{11} \neq 0$. Define (n-1) multipliers:

$$m_i = \frac{a_{i1}}{a_{11}} \; ; \quad i = 2,3, \ldots , n$$

and subtract m_i times the first equation from the i-th equation. If we define

$$a'_{ij} = a_{ij} - m_i \, a_{1j} \quad , \qquad i=2,3,\ldots,n$$

$$b'_i = b_i - m_i \, b_1 \quad , \qquad j=1, \ldots ,n$$

it is easy to see that

$$a'_{i1} = 0 \quad , \qquad i=2, \ldots ,n$$

The transformed equations are

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots\cdots & + & a_{1n}x_n & = & b_1 \\
0 & + & a'_{22}x_2 & + & \cdots\cdots & + & a'_{2n}x_n & = & b'_2 \\
\cdots & & \cdots & & \cdots\cdots & & \cdots & & \cdots \\
0 & + & a'_{i2}x_2 & + & \cdots\cdots & + & a'_{in}x_n & = & b'_i \\
\cdots & & \cdots & & \cdots\cdots & & \cdots & & \cdots \\
0 & + & a'_{n2}x_2 & + & \cdots\cdots & + & a'_{nn}x_n & = & b'_n
\end{array}$$

We continue in this way. At the K-th stage we eliminate x_k by defining multipliers

$$m_i^{(k-1)} = \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} \; , \quad i=k+1,\ldots,n \qquad \text{(II.1.9)}$$

where $a_{kk}^{(k-1)} \neq 0$ then

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - m_i^{(k-1)} a_{kj}^{(k-1)} \qquad (II.1.10)$$

$$b_i^{(k)} = b_i^{(k-1)} - m_i^{(k-1)} b_k^{(k-1)} \qquad (II.1.11)$$

for $i = k+1, \ldots, n$ and for $j = k, \ldots, n$.

The final triangular set of equations is given by

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + \cdots\cdots + a'_{2n}x_n &= b'_2 \\
\cdots \quad \cdots\cdots \quad \cdots \quad &\cdots \qquad (II.1.12) \\
a_{jj}^{(j-1)}x_j + \cdots + a_{jn}^{(j-1)}x_n &= b_j^{(j-1)} \\
\cdots\cdots \quad \cdots \quad &\cdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}$$

The round off errors in the values of the unknowns can be substantially reduced by a judicious choice of rows to interchange. The back substitution can be described as follows

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_{n-1} = \frac{b_{n-1}^{(n-2)} - a_{n-1,n}^{(n-2)} x_n)}{a_{n-1,n-1}^{(n-2)}}$$

$$\ldots \quad \ldots\ldots\ldots\ldots$$

$$x_j = (b_j^{(j-1)} - a_{jn}^{(j-1)} x_n \ldots a_{j,j+1}^{(j-1)} x_{j+1})/a_{jj}^{(j-1)} \qquad \ldots \text{(II.1.13)}$$

for $j = n-2, \ldots, 1$

Example:

As an example of the use of Gaussian elimination, let us solve the equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 10 \\ 2x_1 + x_2 + 3x_3 &= 21 \\ x_1 + 3x_2 + 2x_3 &= 17 \end{aligned} \qquad \text{(II.1.14)}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 10 \\ 2 & 1 & 3 & 21 \\ 1 & 3 & 2 & 17 \end{bmatrix} \begin{matrix} \equiv R_1 \\ \equiv R_2 \\ \equiv R_3 \end{matrix}$$

The multipliers

$$m_2 = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2$$

$$m_3 = \frac{a_{31}}{a_{11}} = \frac{1}{1} = 1$$

Elimination of coefficient of x_1:

$$\begin{bmatrix} 1 & 1 & 1 & 10 \\ 2 & 1 & 3 & 21 \\ 1 & 3 & 2 & 17 \end{bmatrix} \begin{matrix} R_1/a_{11} \\ R_2 - m_2 R'_1 \\ R_3 - m_2 R'_1 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 10 \\ 0 & -1 & 1 & 1 \\ 0 & 2 & 1 & 7 \end{bmatrix}$$

The multiplier

$$m'_3 = \frac{a'_{32}}{a'_{22}} = \frac{2}{-1} = -2$$

Elimination of coefficients of x_2

$$\begin{bmatrix} 1 & 1 & 1 & 10 \\ 0 & -1 & 1 & 1 \\ 0 & 2 & 1 & 7 \end{bmatrix} \begin{matrix} \\ R_2/-1 \\ R_3 - m'_3 R'_2 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 10 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

The forward elimination is now complete, and the equations corresponding to the matrix form are

$$x_1 + x_2 + x_3 = 10$$

$$x_2 - x_3 = -1$$

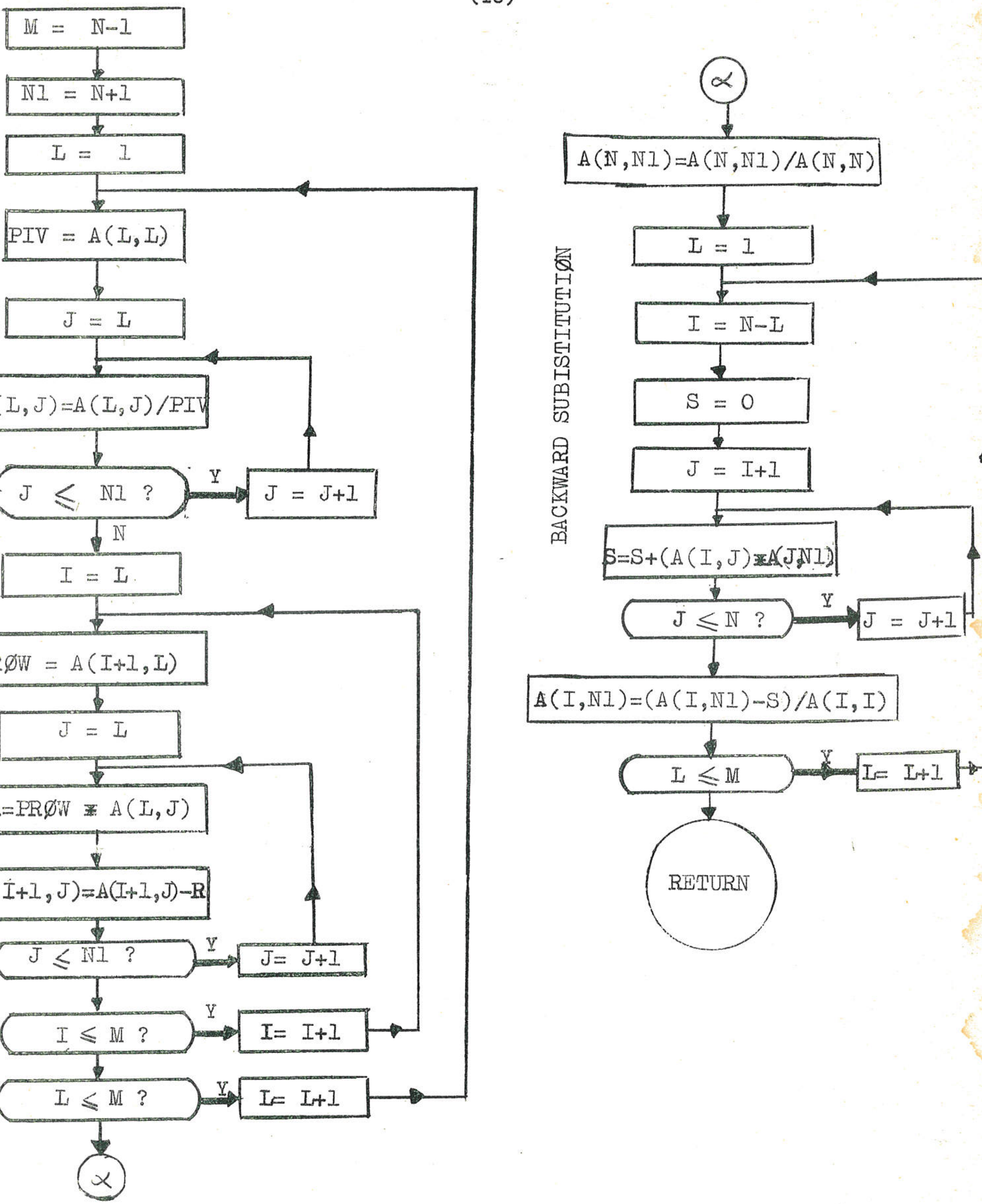
$$3x_3 = 9$$

Backward substitution may now be used to solve for the x_i's in reverse order. Hence,

$$x_3 = 3$$

$$x_2 = -1 + x_3 = 2$$

$$x_1 = 10 - x_2 - x_3 = 5$$


M = N-1
N1 = N+1
L = 1
PIV = A(L,L)
J = L
(L,J)=A(L,J)/PIV
J ≤ N1 ?
Y
J = J+1
N
I = L
RØW = A(I+1,L)
J = L
=PRØW * A(L,J)
I+1,J)=A(I+1,J)-R
J ≤ N1 ?
Y
J= J+1
I ≤ M ?
Y
I= I+1
L ≤ M ?
Y
L= L+1
α
BACKWARD SUBISTITUTIØN
α
A(N,N1)=A(N,N1)/A(N,N)
L = 1
I = N-L
S = 0
J = I+1
S=S+(A(I,J)*A(J,N1)
J ≤ N ?
Y
J = J+1
A(I,N1)=(A(I,N1)-S)/A(I,I)
L ≤ M
Y
L= L+1
RETURN

```
C     A PROG. TO READ A SQUARE MATRIX AND GET ITS SOLUTION.
C     N ... IS THE MAX. NO OF ITS ROWS OR COLUMNS.
C     M ... IS THE NO. OF THE CONSTANT VECTORS.
      DIMENSION A(30,31)
    1 READ 2,N,M
    2 FORMAT(2I2)
      N1=N+1
      DO 3 I=1,N
    3 READ 4,(A(I,J),J=1,N1)
    4 FORMAT(10F8.3)
      CALL GAUSEL(A,N)
      IF(SENSE SWITCH 1)5,7
    5 PRINT 6,(A(I,N1),I=1,N)
    6 FORMAT(5(F14.8,2X))
    7 PUNCH 6,(A(I,N1),I=1,N)
      GO TO 1
      END

      SUBROUTINE GAUSEL (A,N)
      DIMENSION A(30,31)
C     FORWARD ELIMINATION.
      M=N-1
      N1=N+1
    1 DO 5 L=1,M
    2 PIV=A(L,L)
      DO 3 J=L,N1
    3 A(L,J)=A(L,J)/PIV
      DO 4 I=L,M
      PROW=A(I+1,L)
      DO 4J=L,N1
      R=PROW*A(L,J)
    4 A(I+1,J)=A(I+1,J)-R
    5 CONTINUE
```

```
C     BACKWARD SUBISTITUTION.
      A(N,N1)=A(N,N1)/A(N,N)
      DO 8 L=1,M
      I=N-L
      S=0.
      KK=I+1
      DO 7 J=KK,N
    7 S=S+A(I,J)*A(J,N1)
    8 A(I,N1)=(A(I,N1)-S)/A(I,I)
      RETURN
      END
```

```
GAUSEL
READ SUBPROGRAMS NAMED ABOVE
LOAD SUBROUTINES

ENTER DATA

    5.00000000        2.00000000        3.00000000
```

II.2. Jordan Elimination

If the coefficients matrix is reduced to the identity by row operations on the rectangualr systems matrix, the solutions to the system are obtained in the last reduction. No back solutions are required.

To illustrate this procedure, we will solve the system:

$$\begin{aligned} 4x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 + x_2 + 3x_3 &= 2 \\ 2x_1 \qquad + x_3 &= 4 \end{aligned} \qquad \text{(II.2.1)}$$

and at the same time obtain the inverse of the coefficients matrix. The rectangular systems matrix is given by:

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 3 & 1 & 3 & 2 \\ 2 & 0 & 1 & 4 \end{bmatrix} \qquad \text{(II.2.2)}$$

We augment this matrix by the identity matrix, thus obtaining the matrix:

$$\begin{bmatrix} 4 & 2 & 1 & 3 & 1 & 0 & 0 \\ 3 & 1 & 3 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 4 & 0 & 0 & 1 \end{bmatrix}$$

The row operations described below are now performed to reduce the coefficients matrix to identity.

First reduction

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{9}{4} & -\frac{1}{4} & -\frac{3}{4} & 1 & 0 \\ 0 & -1 & \frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{array}{l} (1)\ a'_{1j}=a_{1j}/a_{11} \\ (2)\ a'_{2j}=a_{2j}-a_{21}a'_{1j} \\ (3)\ a'_{3j}=a_{3j}-a_{31}a'_{1j} \end{array} \Bigg\} 1 < j \leqslant 5.$$

Second reduction

$$\begin{bmatrix} 1 & 0 & \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & -\frac{9}{2} & \frac{1}{2} & \frac{3}{2} & -2 & 0 \\ 0 & 0 & -4 & 3 & 1 & -2 & 1 \end{bmatrix} \begin{array}{l} (2)\ a'_{1j}=a_{1j}-a_{12}a'_{2j} \\ (1)\ a'_{2j}=a_{2j}/a_{22} \\ (3)\ a'_{3j}=a_{3j}-a_{32}a'_{2j} \end{array} \Bigg\} 2 < j \leqslant 6.$$

Third reduction

$$\begin{bmatrix} 1 & 0 & 0 & \frac{19}{8} & \frac{1}{8} & -\frac{1}{4} & \frac{5}{8} \\ 0 & 1 & 0 & -\frac{23}{8} & \frac{3}{8} & \frac{1}{4} & -\frac{9}{8} \\ 0 & 0 & 1 & -\frac{3}{4} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \begin{array}{l} (2)\ a'_{1j}=a_{1j}-a_{13}a'_{3j} \\ (3)\ a'_{2j}=a_{2j}-a_{23}a'_{3j} \\ (1)\ a'_{3j}=a_{3j}/a_{33} \end{array} \Bigg\} 3 < j \leqslant 7$$

Numbers in parenthesis denote the ordering of operations within a reduction stage.

In a computer solution to the problem, the identity matrix need not occupy storage locations in the memory. We can set up avery orderly procedure that avoids the computation of predetermined elements and does not utilize computer storage for the initial identity matrix or the unit vectors formed in the reduction process. We will consider two methods of storage allocation that are frequently used to accomplish this task.

Method 1.

For the r-th reduction, we have

$$a'_{ij} = a_{ij} - \frac{a_{ir}\ a_{rj}}{a_{rr}} , \quad i \neq r , \quad j \neq r ,$$

$$a'_{ir} = - \frac{a_{ir}}{a_{rr}} , \quad i \neq r ,$$

$$a'_{rj} = \frac{a_{rj}}{a_{rr}} , \quad j \neq r ,$$

$$a'_{rr} = \frac{1}{a_{rr}} ,$$

Method 2.

For every reduction, we have

$$a'_{i-1,j-1} = a_{ij} - \frac{a_{i1}\ a_{1j}}{a_{11}} , \quad i=2,3,\ldots,n, \quad j=2,3,\ldots,n+m;$$

$$a'_{i-1,n+m} = - \frac{a_{i1}}{a_{11}} , \quad i=2,3,\ldots,n;$$

$$a'_{n,j-1} = \frac{a_{1j}}{a_{11}} , \quad j=2,3,\ldots,n+m;$$

$$a'_{n,n+m} = \frac{1}{a_{11}} .$$

Method 1 requires exactly nx(n+m) storage locations for array elements, However in method 2,a work row is utilized for temporary storage of the pivot row. As new rows are computed, they are stored one row above their prior location in the nx(n+m) array. The work row finally repalces the n-th row at the end of each reduction. In this manner, the first row in the nx(n+m) array will always be the pivot row for next reduction, the pivotal element being the first element in the first row.

Using as an example the system matrix (II.2.2), we shall observe that the solution vectors appear in different columns of the final array, dependent on the method used.

Initial array

$$\begin{bmatrix} 4 & 2 & 1 & 3 & 1 & 0 & 0 \\ 3 & 1 & 3 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 4 & 0 & 0 & 1 \end{bmatrix}$$

Method 1

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 3 & 1 & 3 & 2 \\ 2 & 0 & 1 & 4 \end{bmatrix}$$

Method 2

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 3 & 1 & 3 & 2 \\ 2 & 0 & 1 & 4 \end{bmatrix}$$

First reduction

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{9}{4} & -\frac{1}{4} & -\frac{3}{4} & 1 & 0 \\ 0 & -1 & \frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{1}{2} & \frac{9}{4} & -\frac{1}{4} \\ -\frac{1}{2} & -1 & \frac{1}{2} & \frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{4} & \frac{9}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -1 & \frac{1}{2} & \frac{5}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

Second reduction

$$\begin{bmatrix} 1 & 0 & \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & -\frac{9}{2} & \frac{1}{2} & \frac{3}{2} & -2 & 0 \\ 0 & 0 & -4 & 3 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 & \frac{5}{2} & \frac{1}{2} \\ \frac{3}{2} & -2 & -\frac{9}{2} & \frac{1}{2} \\ 1 & -2 & -4 & 3 \end{bmatrix} \begin{bmatrix} -4 & 3 & 1 & -2 \\ \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} & 1 \\ -\frac{9}{2} & \frac{1}{2} & \frac{3}{2} & -2 \end{bmatrix}$$

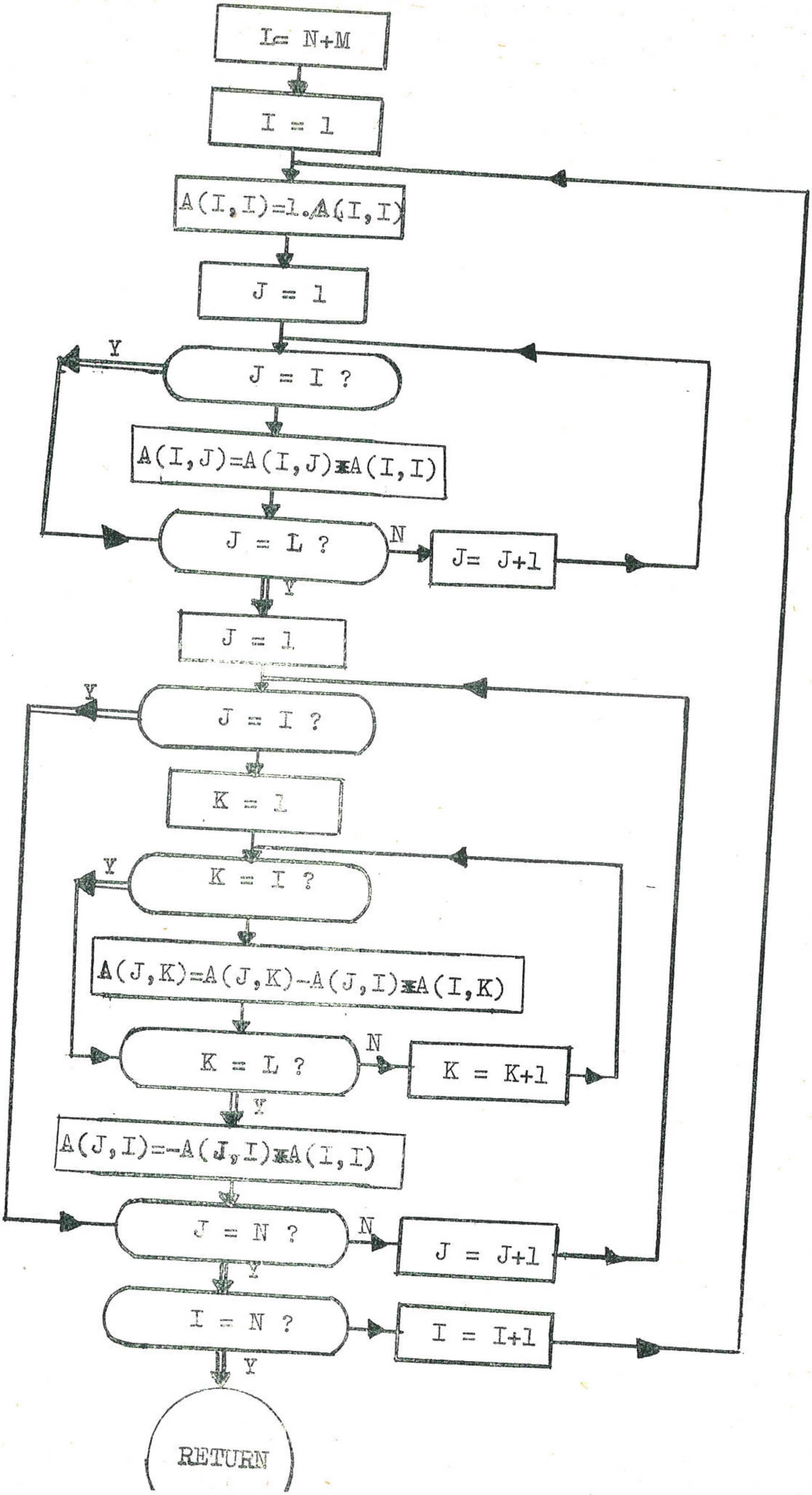
$$\begin{bmatrix} -\frac{9}{2} & \frac{1}{2} & \frac{3}{2} & -2 \end{bmatrix}$$

Third reduction

$$\left[\begin{array}{ccc|c|ccc} 1 & 0 & 0 & \frac{19}{8} & \frac{1}{8} & -\frac{1}{4} & \frac{5}{8} \\ 0 & 1 & 0 & -\frac{23}{8} & \frac{3}{8} & \frac{1}{4} & -\frac{9}{8} \\ 0 & 0 & 1 & -\frac{3}{4} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{array}\right] \left[\begin{array}{ccc|c} \frac{1}{8} & -\frac{1}{4} & \frac{5}{8} & \frac{19}{8} \\ \frac{3}{8} & \frac{1}{4} & -\frac{9}{8} & -\frac{23}{8} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & -\frac{3}{4} \end{array}\right] \left[\begin{array}{c|ccc} \frac{19}{8} & \frac{1}{8} & -\frac{1}{4} & \frac{5}{8} \\ -\frac{23}{8} & \frac{3}{8} & \frac{1}{4} & -\frac{9}{8} \\ -\frac{3}{4} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{array}\right]$$

$$\begin{bmatrix} -\frac{3}{4} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

In general, the accuracy of the reduction computations depends considerably upon the pivotal elements used in each reduction stage. A pivotal element of zero, for example, at any stage will make it impossible to continue the process. True zero pivotal elements are unlikely to be formed beyond the first few reductions, even in the case of a singular coefficients matrix, due to truncation or rounding. Although a relative zero pivotal element may not stop the process, it will yield inaccurate results.



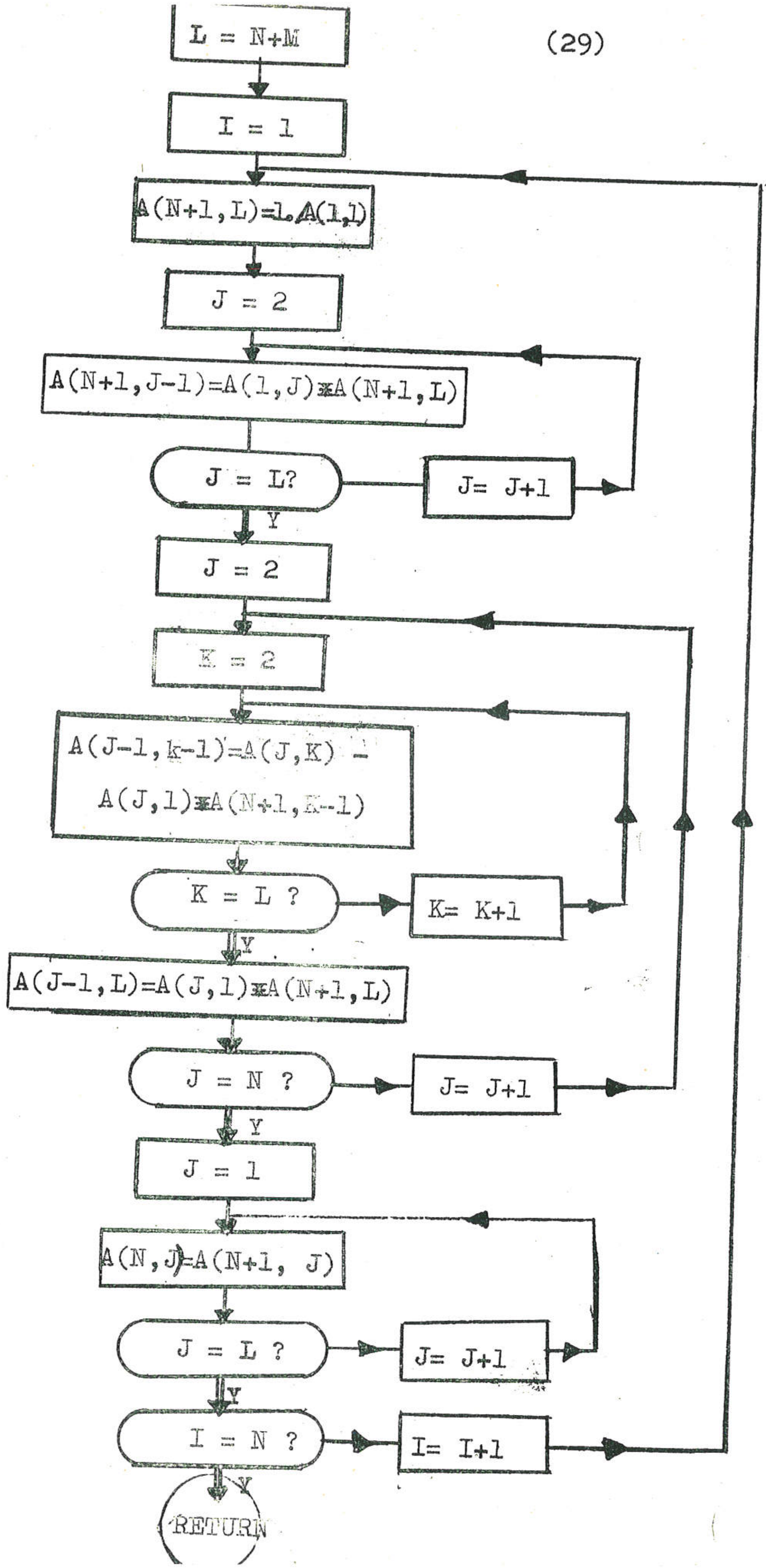
L= N+M
I = 1
A(I,I)=1./A(I,I)
J = 1
J = I ?
Y
A(I,J)=A(I,J)*A(I,I)
J = L ?
N
Y
J= J+1
J = 1
J = I ?
Y
K = 1
K = I ?
Y
A(J,K)=A(J,K)-A(J,I)*A(I,K)
K = L ?
N
Y
K = K+1
A(J,I)=-A(J,I)*A(I,I)
J = N ?
N
Y
J = J+1
I = N ?
Y
I = I+1
RETURN

```
  SUBROUTINE GORD 1(A,N,M)
  DIMENSION A(15,30)
  L=N+M
  DO 7 I=1,N
  A(I,I)=1.0/A(I,I)
  DO 3 J=1,L
  IF(J-I)2,3,2
2 A(I,J)=A(I,J)*A(I,I)
3 CONTINUE
  DO 7 J=1,N
  IF(J-I)4,7,4
4 DO 6 K=1,L
  IF(K-I)5,6,5
5 A(J,K)=A(J,K)-A(J,I)*A(I,K)
6 CONTINUE
  A(J,I)=-A(J,I)*A(I,I)
7 CONTINUE
  RETURN
  END
```

```
ENTER SOURCE PROGRAM, PRESS START
        DIMENSION A(15,30)
C       N IS THE NO OF ROWS OF THE ORIGINAL MATRIX.
C       M IS THE NO OF THE SOLUTION VECTORS.
      1 READ 2,M,N
      2 FORMAT(2I2)
        L=N+M
        DO 3 I=1,N
      3 READ 4,(A(I,J),J=1,L)
      4 FORMAT(10F8.3)
        CALL GORD 1(A,N,M)
        DO 5I=1,N
      5 PRINT 6,(A(I,J),J=1,L)
      6 FORMAT(5(F14.8,2X))
        GO TO 1
        END

        GORD 1
        READ SUBPROGRAMS NAMED ABOVE
        LOAD SUBROUTINES
        ENTER DATA

          .12500000   -.25000000     .62500000    2.37500000
          .37500000    .25000000   -1.12500000   -2.87500000
         -.25000000    .50000000   - .25000000   - .75000000
```


L = N+M
I = 1
A(N+1,L)=1.A(1,1)
J = 2
A(N+1,J-1)=A(1,J)*A(N+1,L)
J = L?
J= J+1
Y
J = 2
K = 2
A(J-1,k-1)=A(J,K) -
A(J,1)*A(N+1,K-1)
K = L ?
K= K+1
Y
A(J-1,L)=A(J,1)*A(N+1,L)
J = N ?
J= J+1
Y
J = 1
A(N,J)=A(N+1, J)
J = L ?
J= J+1
Y
I = N ?
I= I+1
Y
RETURN

```
      DIMENSION A(15,30)
C     N IS THE NO. OF ROWS OF THE ORIGINAL MATRIX.
C     M IS THE NO. OF THE SOLUTION VECTORS.
    1 READ 2,M,N
    2 FORMAT(2I2)
      L=N+M
      DO 3 I=1,N
    3 READ 4,(A(I,J),J=1,L)
    4 FORMAT(10F8.3)
      CALL GORD 2(A,N,M)
      DO 5 I=1,N
    5 PRINT 6,(A(I,J),J=1,L)
    6 FORMAT(5(F14.8,2X)
      GO TO 1
      END
```

```
      SUBROUTINE GORD 2(A,N,M)
      DIMENSION A(15,30)
      L=N+M
      DO 1 I=1,N
      A(N+1,L)=1.0/A(1,1)
      DO 2 J=2,L
    2 A(N+1,J-1)=A(1,J)*A(N+1,L)
      DO 3 J=2,N
      DO 4 K=2,L
    4 A(J-1,K-1)=A(J,K)-A(J,1)*A(N+1,K-1)
    3 A(J-1,L)=-A(J,1)*A(N+1,L)
      DO 1 J=1,L
    1 A(N,J)=A(N+1,J)
      RETURN
      END
```

GORD 2

READ SUBPROGRAMS NAMED ABOVE

LOAD SUBROUTINES

ENTER DATA

2.37500000	.12500000	-.2500000	.62500000
-2.87500000	.37500000	.2500000	-1.12500000
-.75000000	-.25000000	.5000000	-.25000000

II.3. Crout Reduction Method

Consider the set of n linear equations in n unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \qquad \text{(II.3.1)} \\
\dots \quad \dots \quad \dots \quad \dots & \quad \dots \\
a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n
\end{aligned}
$$

The process of eliminating one unknown at a time from the set of equations is perhaps the simplest approach to their solution and at the same time one of the shortest methods known. Some case must be exercised in the order of elimination of the x's, especially if they are of different magnitudes, it is advisable to begin with the smallest one, proceeding in order of increasing magnitude,

The first equation of(II.3.1)can be multiplied through by the reciprocal of a_{11} and written

$$x_1 = a_{11}^{-1} b_1 - \sum_{i=2}^{n} (a_{11}^{-1} a_{1i})x_i \qquad \text{(II.3.2)}$$

In the other equations, which can be written in the form,

$$a_{j1}x_1 + \sum_{i=2}^{n} a_{ji}x_i = b_j \qquad j=2,3,\dots,n$$

x_1 can be eliminated by the use of equation (II.3.2) and one obtains the (n-1) equations in (n-1) unknowns $x_2, x_3, \dots, x_n$

$$\sum_{i=2}^{n} (a_{ji}-a_{j1}\, a_{11}^{-1}\, a_{1i})x_i = b_j - a_{j1}\, a_{11}^{-1}\, b_1 \qquad \ldots \qquad \text{(II.3.3)}$$

where $j = 2,3, \ldots , n$

We can next proceed to eliminate x_2. Writing equations (II.3.3) as,

$$\sum_{i=2}^{n} a'_{ji}\, x_i = b'_j \qquad j=2,3,\ldots,n \qquad \text{(II.3.4)}$$

The process can be repeated. Thus from the first equation

$$x_2 =(a'_{22})^{-1}b'_2 - \sum_{i=3}^{n} (a'_{22})^{-1}\, a'_{2i}x_i \qquad \text{(II.3.5)}$$

and substituting this expression for x_2 in the remaining equation, one has corresponding to equations (II.3.3), the equations

$$\sum_{i=3}^{n} \left[a'_{ji} - a'_{j2}(a'_{22})^{-1}\, a'_{2i}\right] x_i=b'_j-a'_{j2}(a'_{22})^{-1}b'_2 \qquad \ldots\ldots \qquad \text{(II.3.6)}$$

where now $j = 3,4, \ldots , n$

By repeated elimination we arrive at a single equation in the unknown x_n, which can be solved by a single division. Having x_n, we can substitute to find x_{n-1}; and having x_{n-1}, we can substitute in the appropriate equation to find x_{n-2}, etc, and finally having x_n, x_{n-1}, ... , x_3, we can substitute in equation (II.3.5) to find x_2 and in equation (II.3.2) to find x_1.

Crout modified the elimination method of Gauss. The procedure is based on the following variation. Consider the set of n linear equation (II.3.1) and the first equation (II.3.2) can be written in the form

$$x_1 = b'_1 - \sum_{i=2}^{n} a'_{1i}\, x_i \qquad \text{(II.3.7)}$$

where b'_1 and a'_{1i} are calculated from the relation

$$b'_1 = \frac{b_1}{a_{11}} \quad \text{and} \quad a'_{1i} = \frac{a_{1i}}{a_{11}} \qquad i \geqslant 2 \qquad \text{(II.3.8)}$$

Letting

$$a'_{j1} = a_{j1}$$

$$a'_{j2} = a_{j2} - a'_{j1}\, a'_{12} \qquad j \geqslant 2 \qquad \text{(II.3.9)}$$

then equations (II.3.3) can be written

$$a'_{j2}x_2 + \sum_{i=3}^{n} (a_{ji} - a'_{j1}\, a'_{1i})x_i = b_i - a'_{j1}\, b'_1$$

$$\ldots \qquad \text{(II.3.10)}$$

where j = 2,3, ... , n. The first equation of this set can be solved for x_2, yielding the equation

$$x_2 = b'_2 - \sum_{i=3}^{n} a'_{2i}\, x_i \qquad \text{(II.3.11)}$$

where

$$b'_2 = (b_2 - a'_{21}\, b'_1)\ \frac{1}{a'_{22}}$$

$$a_{2i} = (a_{2i} - a'_{21}\, a'_{1i})\ \frac{1}{a'_{22}} \qquad i \geqslant 2 \qquad \text{(II.3.12)}$$

Using this expression for x_2 to eliminate it from the other equations of equations (II.3.10) then

$$a'_{j3}x_3 + \sum_{i=4}^{n} (a_{ji} - a'_{j1}a'_{1i} - a'_{j2}a'_{2i})x_i$$

$$= b_j - a'_{j1}b'_1 - a'_{j2}b'_2 \qquad \text{(II.3.13)}$$

where $j = 3,4, \ldots, n$ and

$$a'_{j3} = a_{j3} - a'_{j1}\, a'_{13} - a'_{j2}\, a'_{23} \qquad j \geqslant 3 \text{(II.3.14)}$$

Again the first equation of (II.3.13) can be solved for x_3, yielding the e quation,

$$x_3 = b'_3 - \sum_{i=4}^{n} a'_{3i}\, x_i \qquad \text{(II.3.15)}$$

where

$$\left.\begin{aligned} b'_3 &= (b_3 - a'_{31}b'_1 - a'_{32}b'_2)\frac{1}{a'_{33}} \\ a'_{3i} &= (a_{3i} - a'_{3i}a'_{1i} - a'_{32}a'_{2i})\frac{1}{a'_{33}} \end{aligned}\right\} i \geqslant 4 \qquad \text{(II.3.16)}$$

Therefore, $x_j = b'_j - \sum_{i=j+1}^{n} a'_{ji}\, x_i \qquad j=1,2,\ldots,n$ (II.3.17)

which have as an augmented matrix the triangular matrix

$$\begin{bmatrix} 1 & a'_{12} & a'_{13} & a'_{14} & \cdots & a'_{1n} & b'_1 \\ 0 & 1 & a'_{23} & a'_{24} & \cdots & a'_{2n} & b'_2 \\ 0 & 0 & 1 & a'_{34} & \cdots & a'_{3n} & b'_3 \\ 0 & 0 & 0 & 1 & \cdots & a'_{4n} & b'_4 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 & b'_n \end{bmatrix} \qquad \text{(II.3.18)}$$

The method of determining these constants is indicated by equations (II.3.8), (II.3.9), (II.3.12), (II.3.14), and (II.3.16) and is summarized by the equations

$$a'_{ji} = a_{ji} - \sum_{k=1}^{i-1} a'_{jk}\, a'_{ki} \qquad i \leqslant j$$

$$a'_{ji} = \frac{1}{a'_{jj}} \left(a_{ji} - \sum_{k=1}^{j-1} a'_{jk}\, a'_{ki}\right) \qquad i > j \qquad \text{(II.3.19)}$$

$$b'_{j} = \frac{1}{a'_{jj}} \left(b_{j} - \sum_{k=1}^{j-1} a'_{jk}\, b'_{k}\right)$$

All these primed constants can be thought of as belonging to the matrix

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & \cdots & a'_{1n} & b'_{1} \\ a'_{21} & a'_{22} & a'_{23} & \cdots & a'_{2n} & b'_{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a'_{n1} & a'_{n2} & a'_{n3} & \cdots & a'_{nn} & b'_{n} \end{bmatrix} \qquad \text{(II.3.20)}$$

Which is termed the auxiliary matrix.

Then having the elements of the auxiliary matrix (II.3.20) and the elements of the triangular matrix (II.3.18) one can solve the equations (II.3.17) in reverse order of x_n, x_{n-1}, $\cdots$, x_1. Then the solutions can be represented by the equations,

$$x_n = b'_n$$

$$x_{n-1} = b'_{n-1} - a'_{n-1,n}\, x_n$$

$$x_{n-2} = b'_{n-2} - a'_{n-2,n-1}\, x_{n-1} - a'_{n-2,n} x_n$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$x_j = b'_j - \sum_{i=j+1}^{n} a'_{ji}\, x_i \qquad \text{(II.3.21)}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$x_1 = b'_1 - \sum_{i=2}^{n} a'_{2i} x_i$$

Crout gives the following working rules for obtaining the auxiliary matrix (II.3.20) from the given augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & b_n \end{bmatrix} \qquad \text{(II.3.22)}$$

1) The various elements are determined in the following order; elements of the first column; then elements of the first row the right of the first column, elements of the second column below the first row; then elements of the second row to the right of the second column; and so on untill all elements are determined.

2) The first column is identical with the first column of the given matrix. Each element of the first row except the first is obtained by dividing the corresponding element of the given matrix by that first element.

3) Each element on or below the principal diagonal is equal to the corresponding element of the given matrix minus the sum of those products of elements in its row and corresponding elements in its column (in the auxiliary matrix) which involve only previously computed elements.

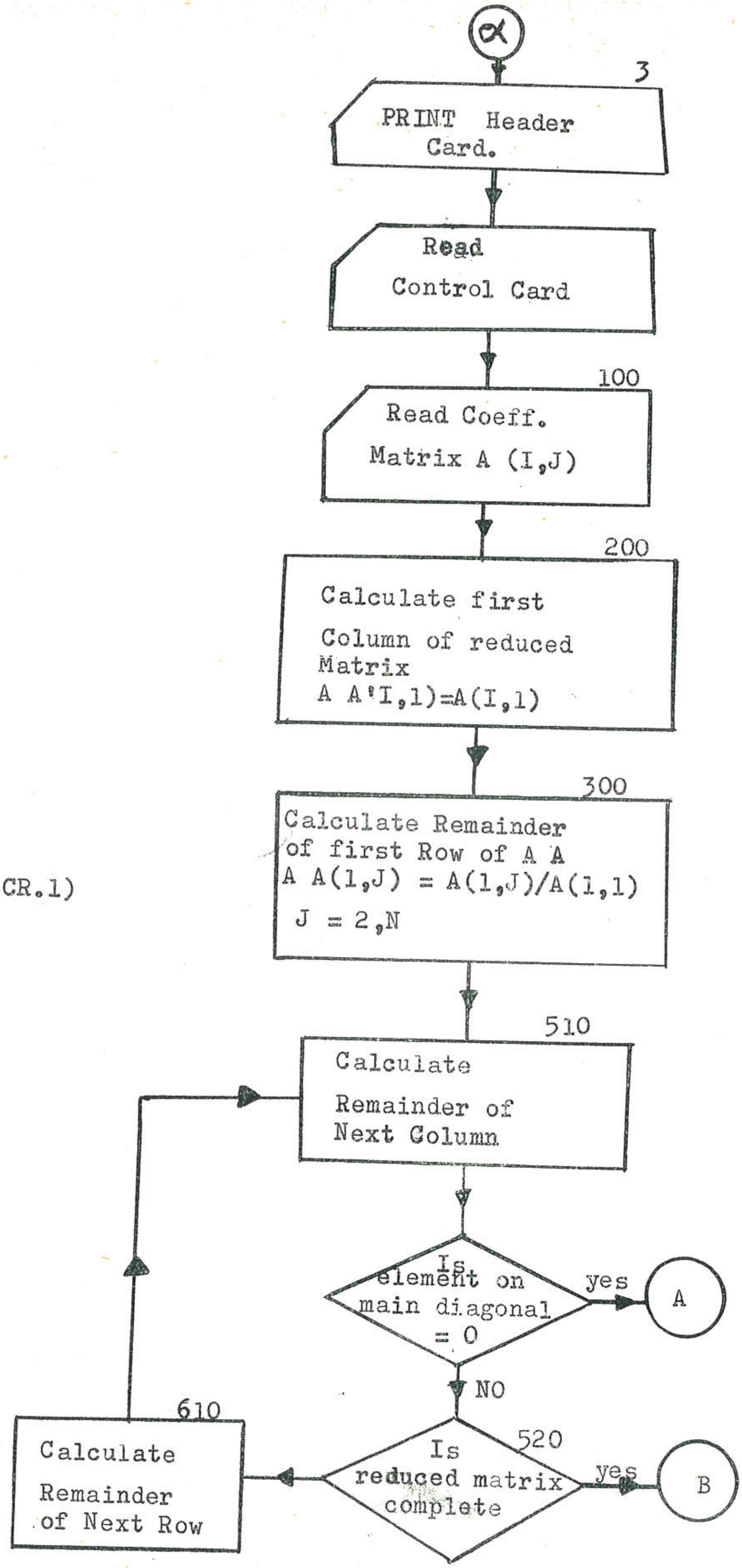
4) Each element to the right of the principal diagonal is given by a calculation which differs from rule (3) only in that there is a final division by its diagonal element (in the auxiliary matrix).

Crout gives the following working rules for obtaining the one-column final matrix from the auxiliary matrix:

1) The elements are determined in the following order: last, next to the last, second from the last, third from the last, and so forth.

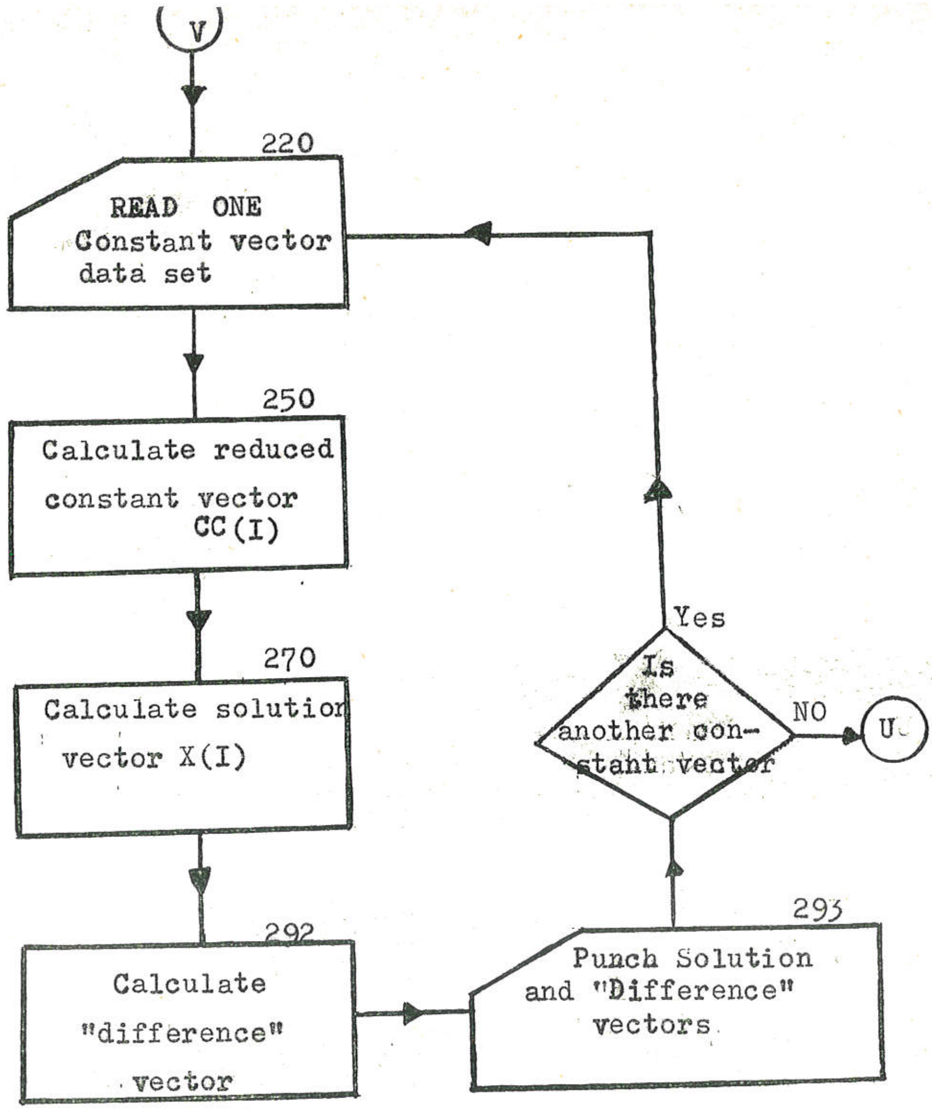
2) The last element is equal to the corresponding element in the last column of the auxiliary matrix.

3) Each element is equal to the corresponding element of the last column of the auxiliary matrix minus the sum of those products of elements in its row in the auxiliary matrix and corresponding elements in the final matrix which involve only previously computed elements.

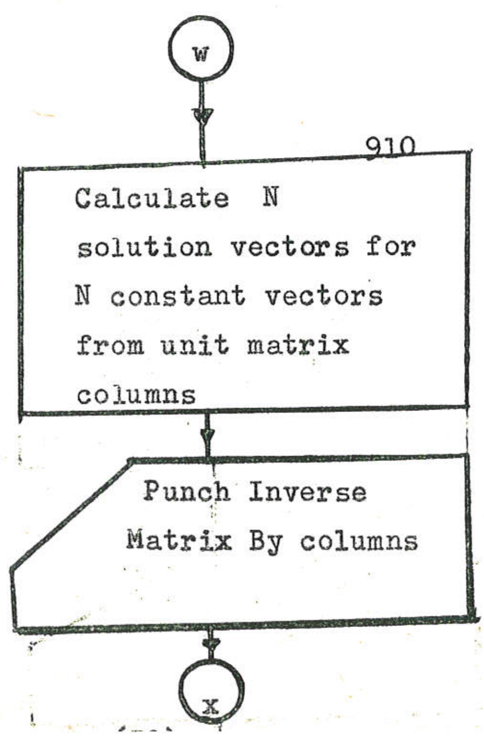


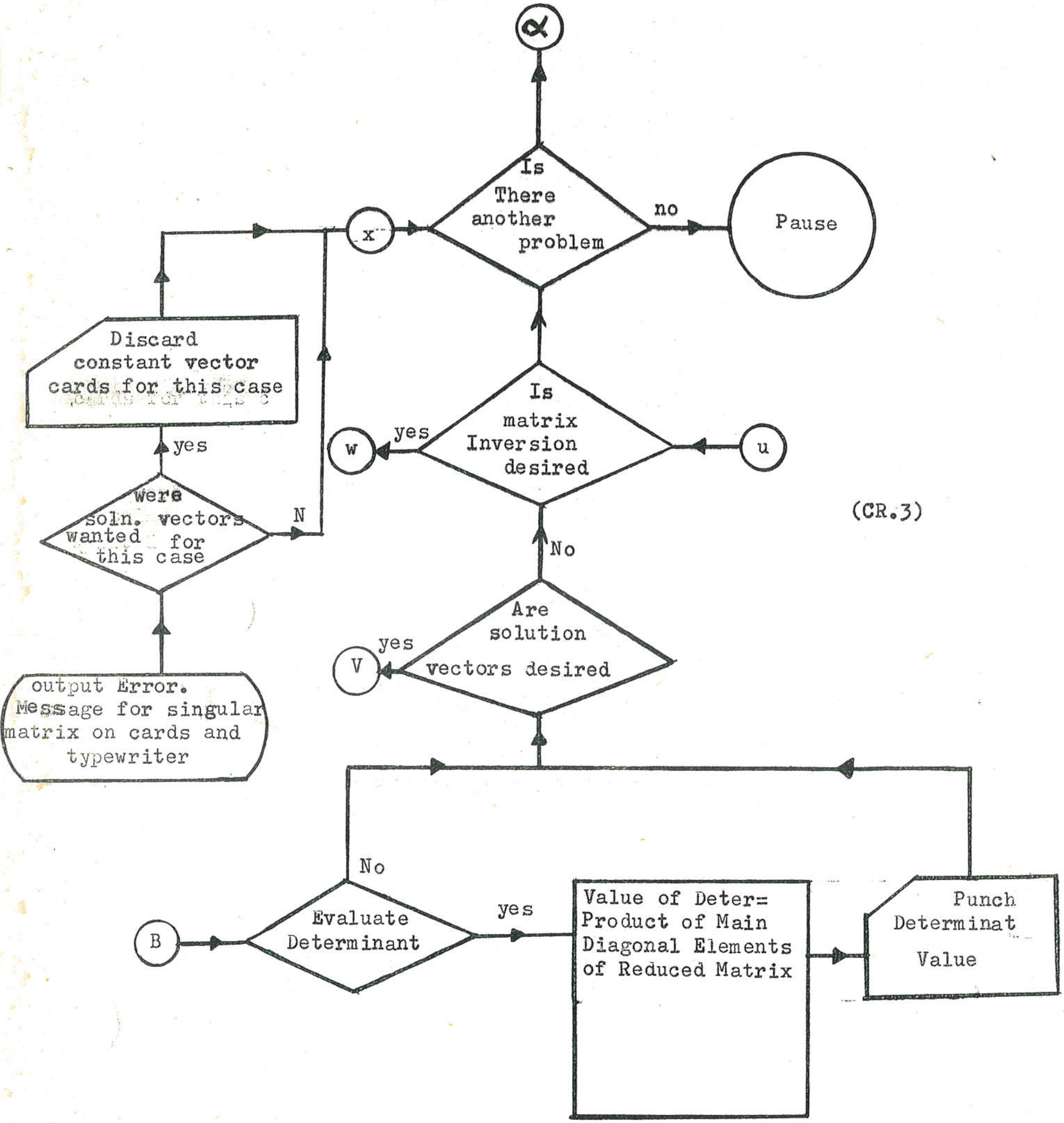
α
3
PRINT Header Card.
Read
Control Card
100
Read Coeff.
Matrix A (I,J)
200
Calculate first
Column of reduced
Matrix
A A'I,1)=A(I,1)
300
Calculate Remainder
of first Row of A A
A A(1,J) = A(1,J)/A(1,1)
J = 2,N
510
Calculate
Remainder of
Next Column
Is
element on
main diagonal
= 0
yes
A
NO
520
Is
reduced matrix
complete
yes
B
610
Calculate
Remainder
of Next Row

(CR.1)


V
220
READ ONE Constant vector data set
250
Calculate reduced constant vector CC(I)
270
Calculate solution vector X(I)
292
Calculate "difference" vector
293
Punch Solution and "Difference" vectors
Is there another constant vector
Yes
NO
U

(CR.2)


W
910
Calculate N solution vectors for N constant vectors from unit matrix columns
Punch Inverse Matrix By columns
X



α
Is There another problem
no
Pause
x
Discard constant vector cards for this case
yes
Were soln. vectors wanted for this case
N
output Error. Message for singular matrix on cards and typewriter
Is matrix Inversion desired
yes
w
u
No
Are solution vectors desired
yes
V
B
Evaluate Determinant
No
yes
Value of Deter= Product of Main Diagonal Elements of Reduced Matrix
Punch Determinat Value

(CR.3)

Variables used in the Crout reduction program:

Input

All input is from punched cards and shall consist of the following

Card No.	Data	CC	Remarks
1	---	1-80	Header Card
2	N	1-5	I5-Order of matrix
	MTRX	6-10	I5- > 0, invert matrix, ≤ 0 do not invert matrix.
	LSIN	11-15	I5- > 0, solution (S) to simult-aueous system desired. ≤ 0, no solution desired
	KNO	16-20	I5-number of constant vectors in input
3	JVAL	21-25	I5- > 0 evaluate determinant of coefficient matrix, ≤ 0, do not evaluate determinat.
	INXT	26-30	I5- > 0, read data for next problem, ≤ 0, do not read.
	A(I,J)	1-80	10F8.3-values of coefficients of input matrix arranged by.rows up to 10 elements per card. (elements must all be from the same row on any given card)
4	C(I)	1-80	10F8.3-values of constant vector elements up to 10 elements per card.(Elements must all be from same constant vector on any given card).

Output

All output is on punched cards except for the singular input matrix error message which will be typed as well as punched. The output deck will consist of appropriate header cards, and any or all of the following data as called for an input card 1:

1) The value of the determinant of the coefficient matrix complete with an identifying label.

2) The solution vector (S) together with the "difference" vector (S) which is the difference between the input constant vector and a calculated constant vector.- These will appear with identifying header cards and labels.

3) The inverse of the coefficient matrix listed by columns in five column blocks. The column number will appear above the appropriate column; the row number will appear to the left of the appropriate row.

```
C       CROUT REDUCTION
C       MAY BE USED TO.
C       1. EVALUATE DETERMINANTS(MAXIMUM ORDE R=20)
C       2. SOLVE UP TO 20 SIMULTANEOUS LINEAR EQUATIONS.
C       3. FIND THE INVERSE OF A MATRIX
C       A(1,1), MAY NOT BE ZERO.
        DIMENSIONA(24,24),AA(20,20),X(20),C(20),CC(20),NQ(24)
        READ 3,
        PRINT 3,
   1000 READ1,N,MTRX,LSIN,KNO,JVAL,INXT
        DO100I=1,N
    100 READ2,(A(I,J),J=1,N)
        DO 200 I=1,N
    200 AA(I,1)=A(I,1)
        DO 300 J=2,N
    300 AA(1,J)=A(1,J)/A(1,1)
        DO 400 I=2,N
        DO 400 J=2,N
    400 AA(I,J)=0.
        J=2
    490 II=J
        DO 510 I=II,N
        LIM1=J-1
        DO 500 K=1,LIM1
    500 AA(I,J)=AA(I,J)+(AA(I,K)*AA(K,J))
    510 AA(I,J)=A(I,J)-AA(I,J)
        IF(AA(J,J))520,900,520
    520 IF(N-J)700,700,530
    530 I=J
        J=J+1
        JJ=J
        DO 610 J=JJ,N
```

```
      LIM 2 = I-1
      DO 600 K=1,LIM 2
600   AA(I,J)=AA(I,J)+(AA(I,K)*AA(K,J))
      AA(I,J)=A(I,J)-AA(I,J)
610   AA(I,J)=AA(I,J)/AA(I,I)
      J=I+1
      GO TO 490
700   IF(JVAL)800,800,720
720   VALUE=AA(1,1)
      DO 710 I=2,N
710   VALUE=VALUE*AA(I,I)
      PRINT 4,VALUE
800   IF(LSIN)730,730,220
220   READ2,(C(I),I=1,N)
      DO 240 I=1,N
      X(I)=0.
240   CC(I)=0.
      CC(1)=C(1)/AA(1,1)
      DO 250 I=2,N
      LIM6=I-1
      DO 260 K=1,LIM6
260   CC(I)=CC(I)+(AA(I,K)*CC(K))
      CC(I)=C(I)-CC(I)
250   CC(I)=CC(I)/AA(I,I)
      X(N)=CC(N)
      LIM7=N-1
      DO 270 I=1,LIM7
      II=N-I
      LIM8=II+1
      DO 280 K=LIM8,N
280   X(II)=X(II)+(AA(II,K)*X(K))
270   X(II)=CC(II)-X(II)
      DO 290 I=1,N
```

```
290   CC(I)=0.
      DO 291 I=1,N
      DO 291 J=1,N
291   CC(I)=CC(I)+(A(I,J)*X(J))
      DO 292 I=1,N
292   CC(I)=C(I)-CC(I)
      PRINT 8
      DO 293 I=1,N
293   NQ(I)=I
      DO 294 I=1,N
      PRINT 9,NQ(I),X(I),CC(I)
294   CONTINUE
      KNO=KNO-1
      IF(KNO)730,730,220
730   IF(MTRX)210,210,810
810   C(1)=1.
      CC(1)=1./AA(1,1)
      DO 820 I=2,N
      C(I)=0.
820   CC(I)=0.
      NI J=N+4
      DO 830 I=1,NIJ
      DO 830 J=1,NIJ
830   A(I,J)=0.
      J=0
840   J=J+1
      DO 860 I=2,N
      LIM 3=I-1
      DO 890 K=1,LIM3
890   CC(I)=CC(I)+(AA(I,K)*CC(K))
      CC(I)=C(I)-CC(I)
860   CC(I)=CC(I)/AA(I,I)
      A(N,J)=CC(N)
      LIM4=N-1
```

```
      DO 910 I=1,LIM4
      II=N-I
      LIM5=II+1
      DO 920 K=LIM5,N
920   A(II,J)=A(II,J)+(AA(II,K)*A(K,J))
910   A(II,J)=CC(II)-A(II,J)
      IF(N-J)110,110,120
120   C(J)=0.
      C(J+1)=1.
      DO 130 I=1,N
130   CC(I)=0.
      GO TO 840
110   PRINT 5
      DO 160 I=1,N
160   NQ(I)=I
      DO 170 J=1,N,5
      PRINT 6,NQ(J),NQ(J+1),NQ(J+2),NQ(J+3),NQ(J+4)
      DO 170 I=1,N
      PRINT 7,NQ(I),A(I,J),A(I,J+1),A(I,J+2),A(I,J+3),A(I,J+4)
170   CONTINUE
210   IF(INXT)50,50,1000
900   PRINT 10
50    IF(LSIN)210,210,901
901   READ 2,(C(I),I=1,N)
      KNO=KNO-1
      IF(KNO)210,210,901
1     FORMAT(6I5)
2     FORMAT(10F8.3)
3     FORMAT (80H
     X                                                   ))
4     FORMAT(//43HVALUE OF DETERMINANT OF COEFFICIENT MATRIX=E11.5)
5     FORMAT(//31HINVERSE OF COEFFICIENT MATRIX --)
```

```
6   FORMAT(//5(10X,I3))
7   FORMAT(I3,2X,E11.5,2X,E11.5,2X,E11.5,2X,E11.5,2X,E11.5)
8   FORMAT(//41H      SOLUTION VECTOR   ACTUAL C-CALC. C)
9   FORMAT(2HX(I3,4H)  E11.5,7X,E11.5)
10  FORMAT(/24H INPUT MATRIX IS SINGULAR)
    END
```

ENTER DATA

*** MODIFIED PROGRAM FOR CROUT REDUCTION 01/08/1969 ***

VALUE OF DETERMINANT OF COEFFICIENT MATRIX = .54000E+02

	SOLUTION VECTOR	ACTUAL C-CALC. C
X(1)	.10000E+01	.00000E-99
X(2)	-.10000E+01	.00000E-99
X(3)	.20000E+01	.00000E-99
X(4)	-.20000E+01	.00000E-99

INVERSE OF COEFFICIENT MATRIX--

	1	2	3	4	5
1	-.33333E-00	.55555E-00	-.14814E-00	.37037E-01	.00000E-99
2	.83333E-00	-.55555E-00	.14814E-00	-.37037E-01	.00000E-99
3	.66666E-00	-.44444E-00	-.14814E-00	.37037E-01	.00000E-99
4	.16666E-00	-.11111E-00	-.37037E-01	.25925E-00	.00000E-99

II.4 Choleski's Method

We consider only 3 x 3 matrices when describing the theory. Lower and upper traingular matrices L, U are defined as matrices with zero elements above and below the principle diagonal, respectively,

$$L = \begin{bmatrix} 11 & 0 & 0 \\ 21 & 22 & 0 \\ 31 & 32 & 33 \end{bmatrix}, \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

The set of linear simultaneous equations for the case n = 3,

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 &= b_1 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 &= b_2 \qquad \text{(II.4.1)} \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 &= b_3 \end{aligned}$$

Can be expressed in matrix notation as

$$Ax = b$$

where,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

We first of all show that A can be expressed as the product of two matrices in the form A = LU where L is a lower triangular matrix and U is an upper triangular matrix units along the principle diagonal (i.e. $U_{ii} = 1$), we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \ell_{11} & \ell_{11} u_{12} & \ell_{11} u_{13} \\ \ell_{21} & \ell_{21} u_{12} + \ell_{22} & \ell_{21} u_{13} + \ell_{22} u_{23} \\ \ell_{31} & \ell_{31} u_{12} + \ell_{32} & \ell_{31} u_{13} + \ell_{32} u_{23} + \ell_{33} \end{bmatrix}$$

On equating the individual elements in the first and last matrices we obtain equations which determine the elements of L and U. These equations are:

$$\left.\begin{aligned} L(I,J) &= A(I,J) - \sum_{N=1}^{J-1} L(I,N)*U(N,J) \\ U(I,J) &= (A(I,J) - \sum_{N=1}^{I-1} L(I,N)*U(N,J))/L(I,I) \end{aligned}\right\} \begin{aligned} &I=1,\ldots,k; \\ &J=1,\ldots,k \end{aligned} \qquad \text{(II.4.2)}$$

a) the first column of L

$$\ell_{11} = a_{11} \quad \ell_{21} = a_{21} \quad \ell_{31} = a_{31}$$

b) the first row of U

$$u_{12} = a_{12}/\ell_{11} \quad , \quad u_{13} = a_{13}/\ell_{11}$$

c) the second column of L

$$\ell_{22} = a_{22} - \ell_{21}u_{12} \quad , \quad \ell_{32} = a_{32} - \ell_{31}u_{12}.$$

d) the second row of u.

$$u_{23} = (a_{23} - \ell_{21}u_{13}) / \ell_{22}$$

e) the third column of L

$$\ell_{33} = a_{33} - \ell_{31}u_{13} - \ell_{32}u_{23}$$

Assuming that L,U are known; we can wirte

$Ax = b$ as.

$Lux = b$

we introduce y defined by

$y = ux$ (1)

$Ly = b$ (2)

written out in full, equation 1 & 2

$$\ell_{11}\ y_1 = b_1$$
$$\ell_{21}\ y_1 + \ell_{22}\ y_2 = b_2 \qquad \text{(II.4.3)}$$
$$\ell_{31}\ y_1 + \ell_{32}\ y_2 + \ell_{33}\ y_3 = b_3$$

$$x_1 + u_{12}\, x_2 + u_{13}\, x_3 = y_1$$
$$x_2 + u_{23}\, x_3 = y_2 \qquad \text{(II.4.4)}$$
$$x_3 = y_3$$

the values of y_1 , y_2 , y_3 can be computed from the set 3 as,

$$y_1 = b_1 / \ell_{11}$$
$$y_2 = (b_2 - \ell_{21}\ y_1) / \ell_{22}$$
$$y_3 = (b_3 - \ell_{31}\ y_1 - \ell_{32}\, y_2) / \ell_{33}$$

and then x_1 , x_2 , x_3 can be computed from the second set, 4/.

It is convenient to divide the calculation into three stages.

1) write down the original matrix with check sums:

a_{11}	a_{12}	a_{13}	b_1	(s_1)
a_{21}	a_{22}	a_{23}	b_2	(s_2)
a_{31}	a_{32}	a_{33}	b_3	(s_3)
(s_1)	(s_2)	(s_3)	(s_4)	$(s=s)$

2) Write down an auxiliary matrix

l_{11}	u_{12}	u_{13}	y_1
l_{21}	u_{22}	u_{23}	y_2
l_{31}	l_{32}	l_{33}	y_3

3) obtain the unknowns by back- substitution in the second set (4) and apply the final check:-

$$S_4' = s_1x_1 + s_2x_2 + s_3x_3 = s_4.$$

Numerical example:

$$x_1 + 4x_2 + x_3 = 1$$
$$- x_2 + 3x_3 = -4$$
$$3x_1 + x_2 + 6x_3 = -11$$

The original matrix

1	4	1	1	(7)
0	-1	3	-4	(-2)
3	1	6	-11	(-1)
(4)	(4)	(10)	(-14)	(4)

The auxiliary matrix

1	4	1	1
0	-1	-3	4
3	-11	-30	-1

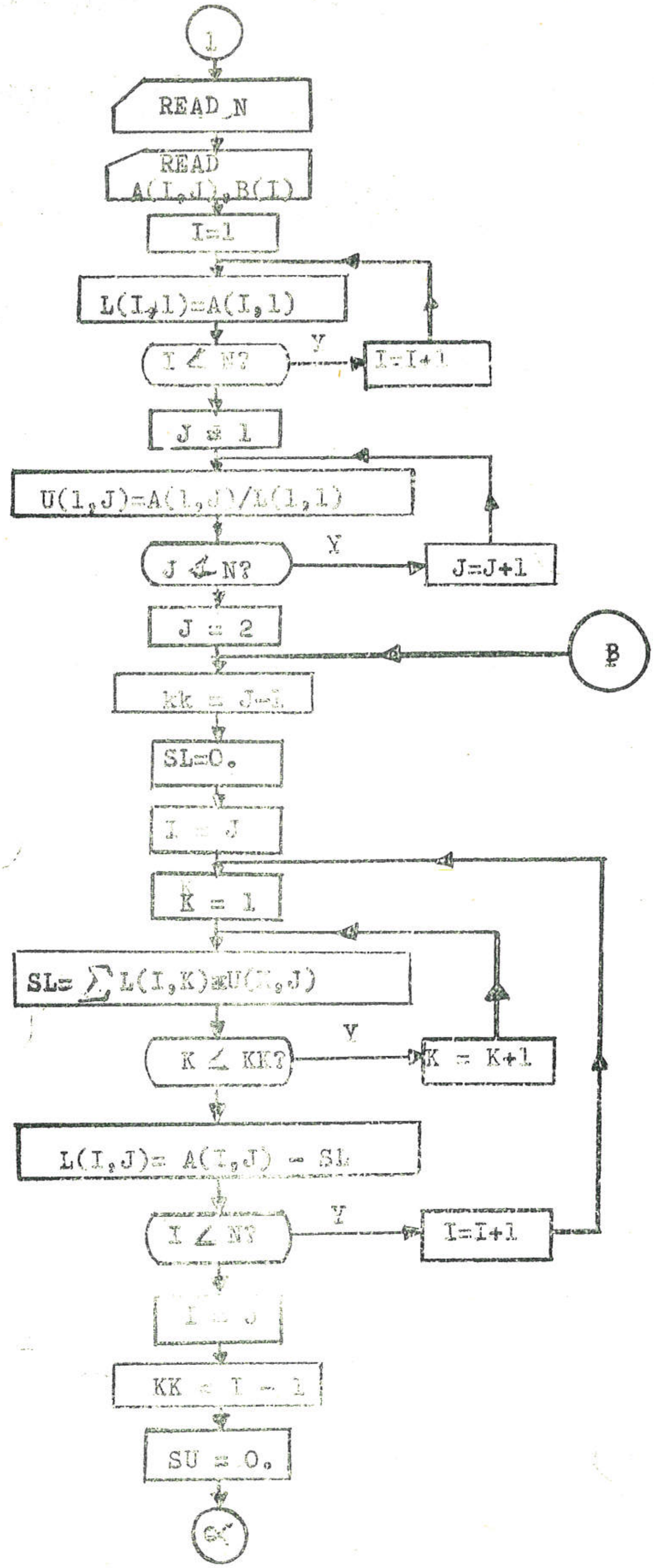
$y_1 = 1$, $y_2 = 4$, $y_3 = -1$

$x_3 = -1$ $\quad x_2 = 4 - 3 = 1$

$x_1 = 1 - 4 + 1 = -2$

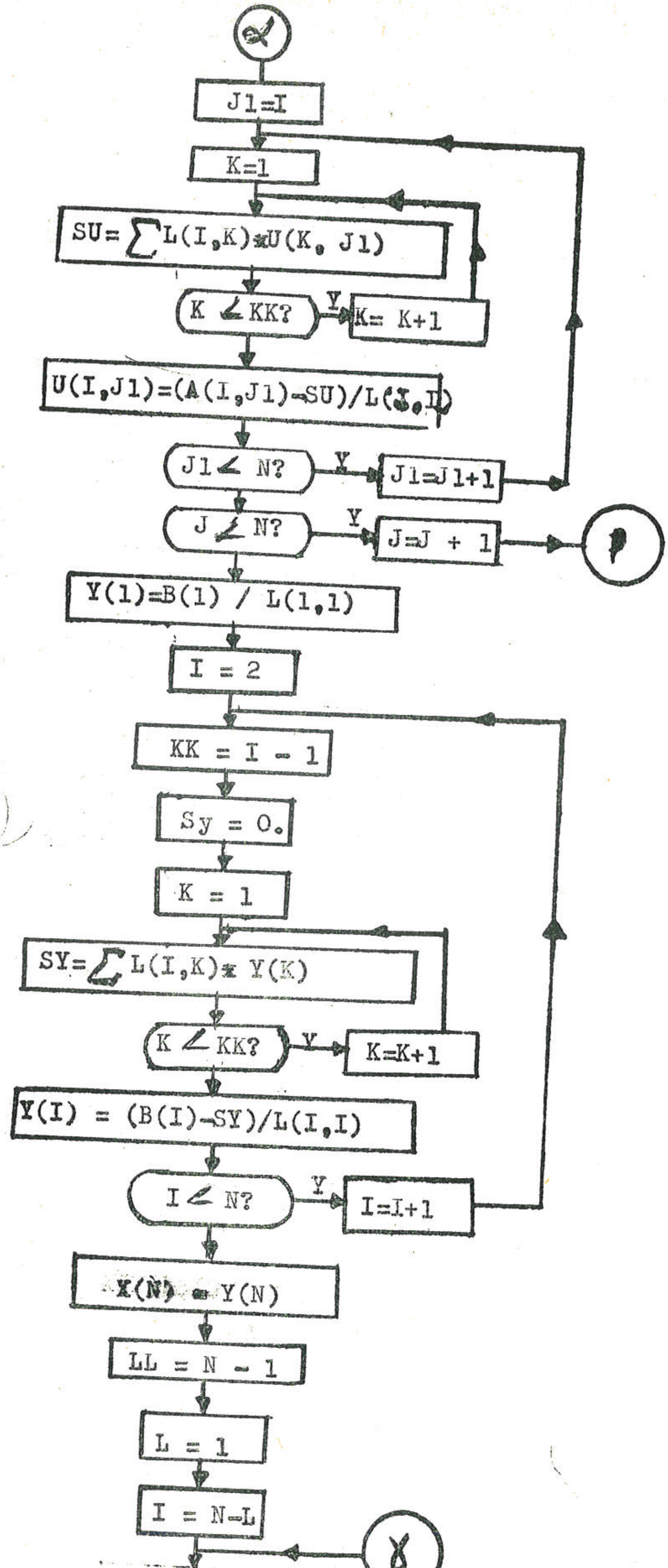
The final check

$S_4 = 4x - 2 + 4 \times 1 + 10x - 1 = -8 + 4 - 10 = -14 = S_4$


1
READ N
READ
A(I,J),B(I)
I=1
L(I,1)=A(I,1)
I ≤ N?
Y
I=I+1
J = 1
U(1,J)=A(1,J)/L(1,1)
J ≤ N?
Y
J=J+1
J = 2
B
kk = J-1
SL=O.
I = J
K = 1
SL= ∑ L(I,K)*U(K,J)
K ≤ KK?
Y
K = K+1
L(I,J)= A(I,J) - SL
I ≤ N?
Y
I=I+1
I = J
KK = I - 1
SU = O.
α

(CHOL.1)

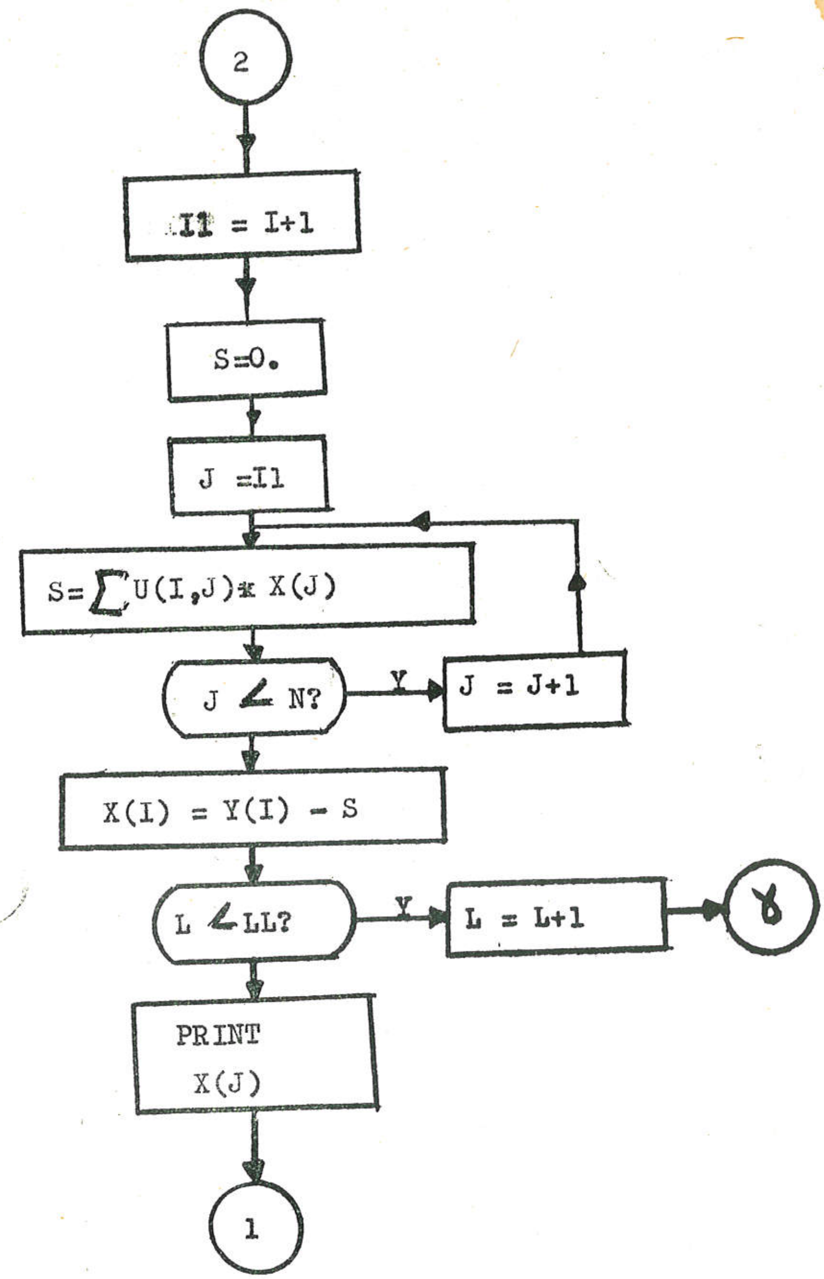
(CHOL.2)



J1=I
K=1
SU= ∑L(I,K)*U(K, J1)
K ∠ KK?
Y
K= K+1
U(I,J1)=(A(I,J1)-SU)/L(I,I)
J1∠ N?
Y
J1=J1+1
J ∠ N?
Y
J=J + 1
Y(1)=B(1) / L(1,1)
I = 2
KK = I - 1
Sy = 0.
K = 1
SY= ∑L(I,K)* Y(K)
K ∠ KK?
Y
K=K+1
Y(I) = (B(I)-SY)/L(I,I)
I ∠ N?
Y
I=I+1
X(N) = Y(N)
LL = N - 1
L = 1
I = N-L

2

(CHOL.3)



Variables used in the Program:-

N	:	The order of original matrix.
A(I,J)	:	The original matrix.
AL(I,J)	:	The lower triangular matrix.
U(I,J)	:	The upper triangular matrix.
B(I)	:	The constant column vector.
Y(I)	:	The computed column vector from set of equations (II.4.3)
X(I)	:	The solution of the set of equations (II.4.1)

```
C     CHOLESKI,S METHOD.
C     A COMPACT ELIMINATION METHOD.
      DIMENSION A(20,20),AL(20,20),U(20,20),B(20),Y(20),X(20)
    1 READ 2,N
    2 FORMAT (I2)
      DO 3 I=1,N
    3 READ 4,(A(I,J),J=1,N),B(I)
    4 FORMAT(10 F8.3)
C     CALCULATION OF THE FIRST COLUMN OF AL.
      DO 5 I=1,N
    5 AL(I,1)=A(I,1)
C     CALCULATION OF THE FIRST ROW OF U.
      DO 6 J=1,N
    6 U(1,J)=A(1,J)/AL(1,1)
      DO 10 J=2,N
      KK=J-1
      SL=0.
      DO 8 I=J,N
      DO 7 K=1,KK
    7 SL=SL+AL(I,K)*U(K,J)
    8 AL(I,J)=A(I,J)-SL
      I=J
      KK=I-1
      SU=0.
      DO 10 J1=I,N
      DO 9 K=1,KK
    9 SU=SU+AL(I,K)*U(K,J1)
   10 U(I,J1)=(A(I,J1)-SU)/AL(I,I)
      Y(1)=B(1)/AL(1,1)
      DO 12 I=2,N
      KK=I-1
      SY=0.
```

```
      DO 11K=1,KK
   11 SY=SY+AL(I,K)*Y(K)
   12 Y(I)=(B(I)-SY)/AL(I,I)
C     BACKWARD SUBISTITUION
      X(N)=Y(N)
      LL=N-1
      DO 14 L=1,LL
      I=N-L
      I1=I+1
      S=0.
      DO 13 J=I1,N
   13 S=S+U(I,J)*X(J)
   14 X(I)=Y(I)-S
C     SOLUTION
      PRINT 15,(X(J),J=1,N)
   15 FORMAT(5(F14.8,2X))
      GO TO 1
      END
```

END OF PASS 1

LOAD SUBROUTINES
ENTER DATA

-2.00000000   1.00000000   -1.00000000

REFERENCES

1. SOUTHWORTH - DELEEUW
   DIGITAL COMPUTATION & NUMERICAL METHODS.

2. BEN NOBLE
   Numerical Methods I.

3. WILLIAM J. HEMMERLE
   Statistical Computations on a Digital Computer.

4. Ralph H. Pennington
   Introductory Computer Methods And Numerical Analysis.

5. Hildebrand, F.B.
   Introduction to Numerical Analysis

6. 1620 GENERAL PRØGRAM LIBRARY.
   (5-0-021)