UNITED ARAB REPUBLIC

THE INSTITUTE OF NATIONAL PLANNING

Memo. No. 825

Applications of Dynamic Programming on Simple Economic Problems

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22/2/1968.

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Introduction:

This note is a supplement to Memo. No. 783. It shows how to use the dynamic programming technique to solve two economic problems: the resourse allocation problem and the problem of finding the optimal economic plan. Both problems are stated in general terms, but solutions are given for simplified cases only.

This note together with Memo. No. 783, are the basis for a training course on "Dynamic Programming and its Application" given at the Institute of National Planning.

Resource Allocation

One of the central problems in economic theory is that of resource allocation: How to use the available resources in efficient ways in order to maximize either a firms profit or some welfare function of an economy.

Suppose that in a certain economy there are M different resources and we have an amount of Z_i from each $(i = 1 \ldots M)$. If these resources can be used in N different activities, and if $g_n (z_{1n} \ldots z_{Mn})$ is the return we get from using the amounts $z_{1n} \ldots z_{Mn}$ of resources in activity n, then the resource allocation problem is that of determining the value of Z_{in} $(i = 1 \ldots M)$, with $\sum_{n=1}^{N} z_{in} \leq Z_i$, $i = 1 \ldots M$, which maximize the total return : $\sum_{n=1}^{N} g_n (z_{1n} \ldots z_{Mn})$.

Consider now the simpler problem of allocating the amount Z of one resource to N different activities. Using the amount z_n in activity n yields the return $g_n(z_n)$. What are the values $z_1 \ldots z_N$, with $\sum_{n=1}^{N} z_n \leq Z$, which maximize the total return $\sum_{n=1}^{N} g_n (Z_n)$?

In stead of considering this as one N-dimentional problem, we can devide it into N simple one-dimention problems by using the dynamic programming set-up.

We will say that the economy is "in state s" if the available amount of resources is s. So, the set of states is:

$$S = \left\{ o \leq s = Z \right\}.$$

The decision- maker will make N successive decisions. These are the amounts $z_n (n = 1 \ldots N)$ of resource that will be allocated to activity n. So, the set of possible actions is:

$$A = \left\{ o \leq z_n \leq s;\ n = 1 \ldots N \right\}$$

[The immediate return from each allocation is $g_n (z_n)$ the problem is determiristicand $\beta = 1$].

Let $f_n(s, z_n)$ denote the total return if s is the available amount of resource, z_n is the amount allocated to activity n, and an optimal policy is followed in the remaining (n-1) activities, i.e., the remaining quantity $s - z_n$ is used to obtain a maximum return from the remaining (n-1) activities. Also, let $f_n(s) = \max_{o \leq z_n \leq s} f_n (s, z_n)$

$$\therefore\ f_n(s, z_n) = g_n(z_n) + f_{n-1}(s-z_n),$$

$$f_n(s) = \max_{o \leq z_n \leq s} \left\{ g_n(z_n) + f_{n-1}(s-z_n) \right\}, \text{ and}$$

$$f_1(s) = \max_{0 \leq z_1 \leq s} g_1(z_1).$$

$$\left[f_n(o) = o, \; f_o(s) = o\right].$$

Now, since s and z_n are continous veriables, and $f_n(\cdot)$, $g_n(\cdot)$ may be continous functions, then the direct enumeration of the values of $f_n\ (s, z_n)$ is imposible. Also as long as nothing is assumed about the analytic structure of $g_n(\cdot)$, then calculus cannot be used to locate the maximum of $f_n(s, z_n)$. So, in order to solve this problem $f_n(s, z_n)$ should be calculated for some chosen values of z_n and the maximum may be located by using some type of interpolation scheme. [For a detailed discussion see [2] page 16.]

A main advantage of using the dynamic programming approach is that its solution, expressed by the two functions $f_N(Z)$ and $z_N(Z)$ is obtained as a function of the available amount of resource, z, and the number of activities, N. So, it is easy to see the influence of these two parameters on the solution. [This is known by the sensitivity analysis.] In addition, this analysis does not impose any analytic restrictions on the functions involved- although it would be simplified if such restrictions are imposed. Also, it is applicable whether the functions are defined over discrete or continous sets.

The following example is a simplified problem given to show the type of computations needed.

Example:

A firm has N different plants and each plant follows a certain production process.

Assume that this firm wants to invest a fixed amount of capital, Z, in the different plants in order to increase its total production. Assume also that having a new machine in plant n costs w_n and increases the production by v_n.

How should the firm devide the available capital, Z, among various plants in order to maximize the total production?

If we define x_n to be the number of new machines in plant n, then the firm's problem is to find the integers $x_1 \ldots x_N$ which maximize the total production $\sum_{n=1}^{N} v_n x_n$, subject to the constraint $\sum_{n=1}^{N} w_n x_n \leqslant Z$.

The dynamic programming formulation of this example is:

$$S = \left\{ o \leqslant s = z;\ s \text{ is the available capital} \right\}.$$

$$A = \left\{ x_n(s) : o \leqslant w_n x_n \leqslant s_n,\ n = 1 \ldots N \right\}.$$

The immediate return function of taking action x_n is $v_n x_n$

Let $f_n(s\ ,\ x_n)$ and $f_n(s)$ be defined as before. Then:

$$f_n(s_n, x_n) = v_n\ x_n + f_{n-1}\ (s_n - w_n\ x_n)$$

$$f_n(s_n) = \max_{x_n,\ o \leq x_n w_n \leq s_n} \left\{ v_n\ x_n + f_{n-1}\ (s_n - w_n\ x_n) \right\}$$

$$\therefore\quad f_1(s_1) = \max_{x_1,\ o \leq w_1 x_1 \leq s_1} \left\{ v_1\ x_1 \right\}$$

$$= v_1\ x_1^*(s)$$

$$\left[f_n(o) = o,\ n = 1 \ldots N\ ;\ f_o(s) = o \right]$$

So, using the recurrence relation, we can compute $f_2(s_2)$ and $x_2^*(s_2)$, and so on for all $n = 1 \ldots N$.

A Numerical Illustration:

Let:

$N = 3$, $Z = 50$, and

Plant, n.	Cost, w_n.	Production, v_n.
1	20	72
2	28	104
3	24	96

For n = 1:

	$f_1(s_1,x_1)$				
s_1 \ x_1	o	1	2	$f_1(s_1)$	$x_1^*(s_1)$
$0 \leqslant s_1 \leqslant 19$	o			o	o
$20 \leqslant s_1 \leqslant 39$	o	72		72	1
$40 \leqslant s_1 \leqslant 50$	o	72	144	144	2

For n = 2:

	$f_2(s_2,x_2)$			
s_1 \ x_2	o	1	$f_2(s_2)$	$x_2^*(s_2)$
$0 \leqslant s_2 \leqslant 19$	o+o		o	o
$20 \leqslant s_2 \leqslant 27$	o+72		72	o
$28 \leqslant s_2 \leqslant 39$	o+72	104+o	104	1
$40 \leqslant s_2 \leqslant 47$	o+144	104+o	144	o
$48 \leqslant s_2 \leqslant 50$	o+144	104+72	176	1

For n = 3:

	$f_3(s_3,x_3)$				
s_3 \ x_3	o	1	2	$f_3(s_3)$	$x_3^*(s_3)$
$o \leq s_3 \leq 19$	o+o			o	o
$20 \leq s_3 \leq 23$	o+72			72	o
$24 \leq s_3 \leq 27$	o+72	96+o		96	1
$28 \leq s_3 \leq 39$	o+104	96+o		104	o
$40 \leq s_3 \leq 43$	o+144	96+o		144	o
$44 \leq s_3 \leq 47$	o+144	96+72		168	1
$48 \leq s_3 \leq 50$	o+176	96+72	192+o	192	2

Since $s_3 = Z = 50$ $\therefore x_2^* = 2$

$\therefore s_2 = 50 - 2x24 = 2$ $\therefore x_2^* = o$

$\therefore s_1 = 2 - o = 2$ $\therefore x_1^* = o$

The optimal policy is to invest in plant 3 only by adding 2 machines to it. If this policy is used the total production will be 192.

It can easily be shown that if Z = 40 then the optimal policy will be: $x_3^* = o$ ($\therefore s_2 = 40$), $x_2^* = o$ ($\therefore s_1 = 40$), $x_1^* = 2$, and the total production will be 144 if this policy is followed.

If N=2, then the optimal policy will be:

$x_2^*=1$ ($\therefore s_1=50-28=22$), $x_1^*=1$, and the total production under this policy will be 176.

This shows that it is easy to find out the effect of any changes in the parameters of the problem (Z and N) on the optimal solution.

Economic Planning

Economic planning is a decision-making process in which the planner, describing the economy by a mathematical model, and using the available knowledge about the prevailing economic situation, tries to choose the economic program (i.e. the sequence of economic decisions) which will maximize a predefined welfare function over a certain number of periods.

Suppose the planning horizon is T periods long and that we use a mathematical model where there are two types of commodities only: produced commodities and primary resources. In our mathematical model, the produced commodities are assumed to follow the generalized Cobb-Douglas production function, while the primary resourses are made available exogenously, i.e., do not depend upon the program adopted.

Let: $z_i(t) \geqslant 0$ denote the available stock of commodity i at the begining of period t,

$c_i(t) \geqslant 0$ denote the quantity of commodity i devoted to consumption in period t,

$x_{ij}(t) \geqslant 0$ denote the quantity of commodity i used as input into the production of commodity j in period t,

$$\left[\text{thus: } z_i(t) \geq c_i(t) + \sum_{j=1}^{M} x_{ij}(t) \text{ , all } i\right],$$

$y_j(t)$ denote the quantity of commodity j produced at the end of period t as a result of using inputs $x_{ij}(t)$, i = 1 ... M in the production process during period t,

[thus, using the generalized cobb-Douglas function we get

$$y_j = e^{\beta_j} \prod_{i=1}^{M} x_{ij}^{\alpha_{ij}}(t) \Big],$$

$q_i(t)$ denote the quantity of commodity i that is exogenously available at the begining of period t,

$$\left[\text{thus : } z_i(t-1) = y_i(t) + q_i(t-1) \text{ , all } i\right].$$

Note: The convention of numbering the periods in a backward order is used through out the paper.

$u_t(\cdot)$ denote the one period welfare function,

$U(\cdot)$ denote the welfare function over the planning horizon.

We are going to use consumption as the criterion for comparing alternative programs. Consequently, the one-period welfare is a function of that period's consumption, i.e.

$u_t(\cdot) = u_t(c_1(t) \ldots c_M(t))$. This function is assumed to be of the linear logarithmic form:

$$u_t(c_1(t) \ldots c_M(t)) = \sum_{i=1}^{M} w_i \log c_i(t),$$

which means that the marginal welfare is decreasing. We also assume that the welfare function over the whole horizon is the sum of the discounted one-period welfares, i.e.

$$U(\cdot) = \sum_{t=1}^{T} \delta^t \; u_t \,(c_1(t) \ldots c_M(t)),$$

where $o \leq \delta \leq 1$ is the "social time discount factor".

So, given the initial stocks : $[\ _i(T),\ i=1 \ldots M]$ and the exogenous quantities: $[q_i(t),\ t=1 \ldots T,\ i=1 \ldots M]$, the planner has to decide the quantities $[c_i(t)$ and $x_{ij}(t)$, $t=1 \ldots T$; i and $j = 1 \ldots M]$ and consequently the quantities: $[z_i(t)$ and $y_i(t)$, all i, j , $t]$ which maximize the function $U(\cdot)$ and satisfy the constraints:

$$z_i(t) \leq c_i(t) + \sum_{j=1}^{M} x_{ij}(t) \qquad \text{all } i,t \text{ ,}$$

$$z_i(t+1) = y_i(t) + q_i(t-1) \qquad \text{all } i,t \text{ ,}$$

$$y_i = e^{\beta_i} \prod_{i=1}^{M} x_{ij}^{\alpha_{ij}}(t) \qquad \text{all } j,t \text{ .}$$

The dynamic programming set-up for this problem is:

The economy is in state $(z_1(t) \ldots z_M(t)$ in period t if the available stocks of commodities at the begining of this period are $(z_1(t) \ldots z_M(t))$.

$$\therefore\ S = \left\{ (z_i(t)\ ,\ \text{all } i) : z_i(T) \text{ given; } z_i(t) = y_i(t+1) + q_i(t),\ t = 1 \ldots T-1\quad i = 1 \ldots M \right\}.$$

At the begining of each period the planner decides how much of the available stocks should be devoted to consumption in this period.

$$\therefore\ A = \left\{ (c_i(t)\ ,\ x_{ij}(t),\ \text{all } i,j) : \sum_{j=1}^{M} x_{ij}(t) + c_i(t) \leq z_i(t) \quad \text{all } i \quad \text{all } t \right\}.$$

The immediate return function in period t is

$$u_t\ (c_1(t) \ldots c_M(t)) = \sum_{i=1}^{M} w_i \log c_i(t)\ .$$

The problem is assumed to be deterministic, and the discount factor is δ.

The essential idea of the dynamic programming approach is to determine the maximum welfare acheviable at the begining of any period as a function of the current stocks of commodities and the number of periods remaining in the program.

In what follows we will discuss the solution of the problem in two simple cases: first the case of an economy with only one produced commodity; then the case with two commodities, one produced and the other a primary resource.

The Case of One Commodity:

If there is only one produced commodity in the economy and no primary resources, then the planning problem becomes:

Given $z(T)$ and T, find the sequence $c(t)$, $t = 1 \ldots T$ which maximizes the function:

$$U(\cdot) = \sum_{t=1}^{T} \delta^{t} \log c(t) ,$$

and which satisfies the constraints:

$$z(t) = c(t) + x(t) \quad ; \; c(t) , x(t) \geq 0 \quad ,$$

$$z(t) = y \; (t+1) \; ,$$

$$y(t) = e^{\beta} \; x(t).$$

[Take $\alpha = 1$, which means that we have constant returns to scale.]

The dynamic programming set-up is:

$$S = \{ z(t) ; \; z(t) = y(t+1) \text{ for } t = 1 \ldots T-1 \text{ and } z(T) \text{ given} \}$$

$$A = \{ c(t) ; \; o \leq c(t) \leq z(t) \; , \; t = 1 \ldots T \}$$

The immediate return function is: $u_t(c(t)) = \log c(t)$.

Define $f_t(z(t) \; , \; c(t))$ to be the total discounted welfares in period t, if action $c(t)$ is taken in period t and an optimal policy is followed in the remaining periods and,

$$f_t(z(t)) = \max_{o \leq c(t) \leq z(t)} \left\{ f_t \; (z(t), \; c(t) \;) \right\}.$$

$$\therefore \quad f_t(z(t), \; c(t)) = \log c(t) + \delta \, f_{t-1} \; (z(t-1)).$$

But $z(t-1) = y(t)$

$$= e^{\beta} \; x(t)$$

$$= e^{\beta} \left[z(t)-c(t) \right]$$

$$\therefore \quad f_t(z(t), c(t)) = \log c(t) + \delta \, f_{t-1} \left\{ e^{\beta} \left[z(t)-c(t) \right] \right\},$$

$$f_t(z(t)) = \max_{o \leq c(t) \leq z(t)} \left\{ \log c(t) + \delta \, f_{t-1}(e^{\beta}[z(t)-c(t)]) \right\}.$$

For $t = 1$:

$$f_1(z(1)) = \max_{o \leq c(1) \leq z(1)} \left\{ \log c(1) \right\}$$

$$= \log z(1).$$

$$\therefore \quad c^{*}(1) = z(1) \text{ and } x^{*}(1) = o.$$

For $t = 2$:

$$f_2(z \; (2) \; , \; c(2) \;) = \log c(2) + \delta \, f_1 \left(e^{\beta} [z \; (2) - c(2)] \right)$$

$$= \log c(2) + \delta \left(\beta + \log [z(2)-c(2)] \right)$$

$$\because \quad \frac{\partial}{\partial c(2)} \; f_2(z(2), \; c(2)) = \frac{1}{c(2)} - \frac{\delta}{(2)-c(2)} \; , \text{ and}$$

$$\frac{\partial^2}{\partial c^2(2)} f_2(z(2), c(2)) < o.$$

$\therefore$ $f_2(z(2), c(2))$ reachs its maximum when its first derivative equals zero, i.e. at:

$$c^*(2) = \left(\frac{1}{1+\delta}\right) z(2) .$$

$$x^*(2) = \left(\frac{\delta}{1+\delta}\right) z(2)$$

$$\therefore f_2(z(2)) = \log\left(\frac{1}{1+\delta}\right) z(2) + \delta\left(\beta + \log\left(\frac{\delta}{1+\delta}\right) z(2)\right)$$

$$= (1+\delta) \log z(2) + \delta \log \delta - (1+\delta) \log(1+\delta) + \delta\beta .$$

For $t = T$:

$$f_T(z(T)) = \left(\sum_{t=o}^{T-1} \delta^t\right) \log z(T) + K_T , \text{ where}$$

$$K_T = \left(\sum_{t=1}^{T-1} \delta^t\right) \log \left(\sum_{t=1}^{T-1} \delta^t\right) - \left(\sum_{t=o}^{T-1} \delta^t\right) \log\left(\sum_{t=o}^{T-1} \delta^t\right) + \beta \sum_{t=1}^{T-1} \delta^t + \delta K_{T-1}$$

$$K_1 = o .$$

<u>Proof</u>:

[By induction on T] :

The given result is true-as shown above - for T = 1 and T = 2. Suppose it holds for all t including T-1, then:

$$f_{T-1}(z(T-1)) = \left(\sum_{t=o}^{T-2} \delta^{t}\right) \log z(T-1) + K_{T-1}$$

$$\therefore f_T(z(t)) = \max_{o \leq c(t) \leq z(T)} \left\{ \log c(T) + \delta f_{T-1}\left(e^{\beta}\left[z(T)-c(T)\right]\right)\right\}$$

$$\therefore f_T(z(T)) = \max_{o \leq c(t) \leq z(T)} \left\{ \log c(T) + \delta\left(\sum_{t=o}^{T-2} \delta^{t}\right)\left(\beta + \log\left[z(T)-c(T)\right]\right) + \delta K_{T-1}\right\},$$

Equating the first derivative with zero, (notice that the second derivative is negative), we get:

$$c^{*}(T) = \left(\frac{1}{\sum_{t=o}^{T-1} \delta^{t}}\right) z(T) \quad , \quad x^{*}(T) = \left(\frac{\sum_{t=1}^{T-1} \delta^{t}}{\sum_{t=o}^{T-1} \delta^{t}}\right) z(T)$$

$$\therefore f_T(z(T)) = \left(\sum_{t=o}^{T-1} \delta^{t}\right) \log z(T) + K_T ,$$

with K_T as defined before.

This completes the induction proof.

The properties of the solution:(*)

1. $$f_T(z(T)) = \max_{o \leq c(t) \leq z(t)} \left[\log c(T) + \left(\sum_{t=1}^{T-1} \delta^{t}\right) \log z(T-1)\right] + \delta K_{T-1}.$$

(*)All these results are stated in Radnen [3], pages 90-94.

Notice that K_{T-1} does not depend on c(T), and that maximizing the quantity $\left[\log c(T) + \left(\sum_{t=1}^{T-1} \delta^{t}\right) \log z(T-1)\right]$ is formally the same as maximizing $f_2($ (2), c(2) except that is replaced by$\left(\sum_{t=1}^{T-1} \delta^{t}\right)$.

This shows that: "The problem of determining any single step of a program with arbitrary finite horizon can be transformed into an "equivalent" problem of determining the "second"step of a program with two periods".

2. It is clear that the optimal consumption and "savings" in _e period t for a program with horizon T are given by:

$$c^{*}(t) = \left(\frac{1}{\sum_{\ell=0}^{t-2} \delta^{\ell}}\right) z(t), \text{ and } x^{*}(t) = \left(\frac{\sum_{\ell=1}^{t-2} \delta^{\ell}}{\sum_{\ell=0}^{t-2} \delta^{\ell}}\right) z(t).$$

Notice also that $\frac{c^{*}(t)}{z(t)}$ and $\frac{x^{*}(t)}{z(t)}$ depend on t but not on z(t).

3. In the infinite horizon case:

If $\delta < 1$ then, as $T \longrightarrow \infty$, we get, for a fixed t, :

$$c^{*}(t) = (1 - \delta)\, z(t), \quad x^{*}(t) = \delta z(t), \text{ and}$$

$$z(t) = (\delta e^{\beta})^{t}\, z(T),$$

$\left[\text{by solving the difference equation } z(t) = e^{\beta}\delta z(t+1)\right]$.

"Hence output, and therefore consumption, grows exponentially with growth factor δe^{β} (the growth rate is $\delta e^{\beta} - 1$)".

"But if $\delta = 1$, we cannot strictly speak of the optimal infinite program because ($\sum_{\ell=0}^{t-2} \delta^{\ell}$) and ($\sum_{\ell=0}^{t-1} \delta^{\ell}$) do not converge. Nevertheless we see that as $\delta \to 1$, for fixed t, consumption approaches o and " savings" approache z (t)".

The Case of Two Commodities:

In this case, assume that commodity 1 is produced from inputs of the two commodities, while commodity 2 is a primary resourse. Thus, the planning problem becomes:

Given $\left[T , z_i(T), i=1,2 \text{ and } q(t), t=1 \ldots T \right]$, find the squence $\left[c_i(t), i=1,2 ; t = 1 \ldots T \right]$ which maximizes the function:

$$U(\cdot) = \sum_{t=1}^{T} \delta^t \left[w_1 \log c_1(t) + w_2 \log c_2(t) \right],$$

and which satisfied the constraints:

$$z_i(t) = c_i(t) + x_i(t) \qquad i = 1,2 ; t=1 \ldots T$$

$$z_1(t) = y_1(t+1) \qquad t = 1 \ldots T-1$$

$$z_1(T) \text{ given}$$

$$z_2(t) = q(t) \text{ given for all } t = 1 \ldots T$$

$$y_1(t) = e^{\beta} x_1^{\alpha_1}(t) x_2^{\alpha_2}(t) .$$

We will take $\alpha_1 + \alpha_2 = 1$, i.e., constant returns to scale.

The dynamic programming set-up is:

$$S = \left\{ z_i(t), i=1,2 : z_1(t) = y_1(t+1) \ t=1 \ldots T-1; \quad z_2(t) = q(t) \ t=1 \ldots T; \ q(T) \text{ and } z_1(T) \text{ given} \right\}$$

$$A = \left\{ c_i(t), i=1,2 : 0 \leq c_i(t) \leq z_i(t) \quad t=1 \ldots T \right\}$$

The immediate return function is:

$$u_t(c_1(t),\ c_2(t)) = w_1 \log c_1(t) + w_2 \log c_2(t).$$

Define $f_t(z_i(t), c_i(t))$ to be the total discounted welfares in period t if action $c_i(t)$ is taken in this period and an optimal policy is followed in the remaining periods. Also define $f_t(z_i(t))$ by:

$$f_t(z_i(t)) = \max_{c_i(t)} \{ f_t(z_i(t),\ c_i(t))\}.$$

Note: Whenever the subscript i is used, it is meant to take values 1 and 2.

$$\therefore \quad f_t(z_i(t),\ c_i(t)) = w_1 \log c_1(t) + w_2 \log c_2(t) + \delta f_{t-1}(z_i(t-1)).$$

But $z_1(t-1) = y_1(t) = e^{\beta} x_1^{\alpha_1}(t)\, x_2^{\alpha_2}(t)$

$$= e^{\beta}\left[z_1(t) - c_1(t)\right]^{\alpha_1}\left[z_2(t) - c_2(t)\right]^{\alpha_2},$$

$z_2(t-1) = q(t-1).$

$$\therefore \quad f_t(z_i(t),\ c_i(t)) = w_1 \log c_1(t) + w_2 \log c_2(t)$$

$$+ \delta f_{t-1}\left(e^{\beta}\left[z_1(t)-c_1(t)\right]^{\alpha_1}\left[z_2(1)-c_2(1)\right]^{\alpha_2};\ q(t-1)\right).$$

For t = 1:

$$f_1(z_i(1)) = \max_{0 \le c_i(1) \le z_i(1)} \left\{ w_1 \log c_1(1) + w_2 \log c_2(1)\right\}$$

$$= w_1 \log z_1(1) + w_2 \log z_2(1).$$

$$\therefore \quad c_i^*(1) = z_i(1) \quad \text{and} \quad x_i^*(1) = 0 \ .$$

For t = 2:

$$f_2(z_i(2), c_i(2)) = w_1 \log c_1(2) + w_2 \log c_2(2) + \delta\Big\{ w_1 \beta + w_1 \alpha_1 \log\big[z_1(2) - c_1(2)\big] + w_1 \alpha_2 \log\big[z_2(2) - c_2(2)\big] + w_2 \log q(1) \Big\} .$$

Setting the partial first derivatives of this function with respect to $c_1(2)$ and $c_2(2)$ equal to zero gives:

$$c_i^*(2) = \frac{1}{1+\delta\alpha_1} z_1(2) \quad \text{and} \quad c_2^*(2) = \frac{w_2}{w_2 + \delta w_1 \alpha_2} z_2(2).$$

Consequently:

$$x_1^*(2) = \frac{\delta\alpha_1}{1+\delta\alpha_1} z_1(2) \quad \text{and} \quad x_2^*(2) = \frac{\delta w_1 \alpha_2}{w_2 + \delta w_1 \alpha_2} z_2(2).$$

The given function reaches its maximum at the point $c_i^*(2)$ (the second derivatives are negative), and its maximum value is given by:

$$f_2(z_i(2)) = w_1(1+\delta\alpha_1)\log z_1(2) + (w_2 + \delta w_1 \alpha_2)\log z_2(2) + K_2\big[q(1)\big]$$

where $K_2[q(2)]$ is independent of $z_i(2)$ and is a function of $q(1)$, β, δ, α_i, and w_i.

For t = T:

It can be proved by induction on T (in a way similar to that used in the one commodity case) that:

$$f_T(z_i(T)) = \left[w_1 \sum_{t=0}^{T-2} (\delta\alpha_1)^t\right] \log z(T) + \left[w_2 + \delta\alpha_2\, w_1 \sum_{t=0}^{T-3} (\delta\alpha_1)^t\right]$$

$$\cdot \log z_2(T) + K_T\left[q(1)\ \ldots\ q(T-1)\right],$$

where K_T does not depend on $z_i(T)$, and $K_1 = 0$.

Also, the optimal values of $c_i(T)$ and $x_i(T)$ are given by:

$$c_1^*(T) = \frac{1}{\sum_0^{T-1} (\delta\alpha_1)^t}\, z_1(T), \quad x_1^*(T) = \frac{\sum_0^{T-2} (\delta\alpha_1)^{t-1}}{\sum_0^{T-2} (\delta\alpha_1)^t}\, z_1(T)$$

$$c_2^*(T) = \frac{w_2}{w_2 + \delta\alpha_2 w_1 \sum_0^{T-3} (\delta\alpha_1)^t}\, z_2(T), \quad x_2^*(T) = \left(\frac{\delta\alpha_2 w_1 \sum_0^{T-3} (\delta\alpha_1)^t}{w_2 + \delta\alpha_2 w_1 \sum_0^{T-3} (\delta\alpha_1)^t}\right) z_2(T)$$

It also can be proved that, for any t_0, the optimal values of $c_i(t_0)$ and $x_i(t_0)$ are given by the previous formulaes after replacing T by t_0.

Properties of the solution:

1. The problem of determining the t^{th} step of a program with arbitrary finite horizon can be transformed into an "equivalent" problem of determining the second step of a program with two periods.