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CONFIDENCE INTERVALES FOR ECONOMIC QUANTITIES DERIVED FROM IMPERICAL PRODUCTION FUNCTIONS

by

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(A) "Introduction"

The production function is a concept in physical and biological science. However, it was largely developed and, until recently, used mainly by economists. Estimating production functions provides basic scientific knowledge. Society is likely to invest in production function estimates as long as economic criteria can be employed in putting them to use in decisions which benefit producers and / or consumers. For this reason several quantities, of great interest, are usually derived from production functions. These quantities provide one of the two sets of information needed for choice and decision - making in the field of economic analysis. The other set of information needed is price data or other quantities which serve as economic criteria. The first set of physical quantities derived from emperical production functions is under focus in this study, while the second set is beyond the scope of this paper. The estimation of the marginal physical productivity, the isoquants, and the isoclines and their use in the opecification of optimum levels and combinations of resource inputs as well as the determination of the optimum production plans has been quite common in ogricultural economics research. However, measures of reliability for these quantities are not generally investigated. Since conelusions and recommendations are stated on the bases of these derived quantities, it is of great interest to have avialable some measures of reliability for these estimates.

It is the purpose of this paper to compute confidence regions for the marginal physical products, the isoquants (product countors), and theisoclines derived from a given type of production functions. However, the procedure is quiete general and with little modification, can be applied to almost every type of the production functions.

(B) Analytical Procedure

Fuller (3) proposed a methodological framework for the computations of the confidence intervals for the quantities derived from the estimated production functions. This proposal is used as a basic tool for the analysis to follow. He indicates that the derived quantities can in general be expressed as functions of the estimatedcoefficients of the production function. Thus the statistical properties of these quantities are determined by the statistical properties of the estimated production function coefficients.

For the determination of the variances of our quantities (i.e., variances of the marginal physical products, isoquants, and isoclines) the following formulation are used. Given that "β_1 , β_2 , ... , β_m" are "m" variables distributed with variance-covariance matrix estimated unbiasedly by "G" Where β's are the production function's coefficients and, $G = (X'X)^{-1} s^2$, and where $(X'X)^{-1}$ is the familier inverse of the sum of squares and cross product matrix, and s^2 is the sum of squares residuals devided by the degrees of freedom. And given that H_1 , H_2, ... , H_n are "n" variables defined by given functions of the β's (i.e., the H's are the quantities derived from the emperical production functions). That is the H's are functions of β's or : $H_1 = f_1(\beta_1, \beta_2, \ldots, \beta_m)$; $H_2 = f_2(\beta_1, \beta_2, \ldots, \beta_m)$, ... , $H_n = f_n(\beta_1, \beta_2, \ldots, \beta_m)$. Then the variance-covariance matrix of the quantities derived from the production function, (which we derive at in this analysis) or in other words the variance-covariance matrix of the H's is estimated by:

$$R\,G\,R' \quad \ldots\ldots \quad (1)$$

where "R" is the "n" by "m" matrix;

$$R = \begin{bmatrix} \frac{\partial H_1}{\partial \beta_1} & \frac{\partial H_1}{\partial \beta_2} & \cdots & \frac{\partial H_1}{\partial \beta_m} \\ \frac{\partial H_2}{\partial \beta_1} & \frac{\partial H_2}{\partial \beta_2} & \cdots & \frac{\partial H_2}{\partial \beta_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial H_n}{\partial \beta_1} & \frac{\partial H_n}{\partial \beta_2} & & \frac{\partial H_n}{\partial \beta_m} \end{bmatrix} \quad \ldots\ldots \quad (2)$$

If the H's or the quantities derived from the production function are linear distribution of the "β's", then all the elements of "R" are constants, and if all elements of "R" are constants, expression (1) provides unbiased estimates of the variance convariance matrix of the H's. For more complicated functions the estimates of equation (1) are only asympototically unbiased.

(C) A Land-Fertilizer production

Function as a case of study :

In this study a land-fertilizer production function where the output (corn production) is measured against the change in both factors of production namely land and fertilizer is under focus. This function is homogeneous of the first degree, that is constant returnes to scale are assumed to exist. The estimated land-fertilizer production function under consideration in this study is of the form:

$$Y = aA + b_1F - b_2F^2 A^{-1} \quad \ldots\ldots \quad (3)$$

where: Y = corn yield in the U. A. R.

A = Land.

F = Nitrogen fertilizer.

a, b_1, and b_2 = the production function coefficients

In the analysis to follow, the analytical procedure for deriving confidence regions for : a) the marginal physical products, b) isoquants , and C) the isoclines, are specified. Then solving numerically for these findings, as a working example, follows.

a) Marginal physical productivity:

The marginal physical productivity of land and fertilizer are:

$$MPP_A = \frac{\partial Y}{\partial A} = a + b_2 F^2 A^{-2} \quad \ldots \quad (4)$$

$$MPP_F = \frac{\partial Y}{\partial F} = b_1 - 2b_2F A^{-1} \quad \ldots \quad (5)$$

It is clear that thise marginal physical quantities are linear functions of the coefficients of the original production function. Thus the estimate R G R' will give an unbiased estimate of the variances and covariances of these quantities.

The "R" matrix of the marginal physical products are given by:

$$R = \begin{bmatrix} 1 & 0 & F^2 A^{-2} \\ 0 & 1 & -2F A^{-1} \end{bmatrix} \qquad \ldots \qquad (6)$$

and : $G = (X'X)^{-1} s^2$

Hence the estimated variance-covariance matrix of the marginal physical products of land and fertilizer is:

$$\overset{R}{\begin{bmatrix} 1 & 0 & F^2 A^{-2} \\ 0 & 1 & -2F A^{-1} \end{bmatrix}} \overset{G}{\begin{bmatrix} X'X^{-1} \end{bmatrix}} \overset{R'}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ F^2 A^{-2} & -2F A^{-1} \end{bmatrix}} s^2$$

$$= \begin{bmatrix} \sigma_A^2 & \alpha \\ \alpha & \sigma_F^2 \end{bmatrix} \qquad \ldots \qquad (7)$$

Where σ_A^2 is the estimated variance of the marginal physical productivity of land, and σ_F^2 is the estimated variance of the marginal physical productivity of fertilizer, and α is the covariance between the two marginal productivities. Equations (8) and (9) give the 95% confidence intervales for the marginal physical productivity of land and fertilizer respectively.

$$P\left[\hat{MPP}_A - t_{.05}\sqrt{\sigma_A^2} \leq MPP_A \leq \hat{MPP}_A + t_{.05}\sqrt{\sigma_A^2}\right] = 95\% \quad \ldots. (8)$$

$$P\left[\hat{MPP}_F - t_{.05}\sqrt{\sigma_F^2} \leq MPP_F \leq \hat{MPP}_F + t_{.05}\sqrt{\sigma_F^2}\right] = 95\% \quad \ldots. (9)$$

Where ; $\hat{MPP}$ is the calculated marginal physical productivity, σ_A^2 and σ_F^2 are variances of MPP_A and MPP_F respectively, and t is the tabuler t at .05 probability level and the appropriate degrees of freedom.

b) Isoquants

The $\bar{Y}$ - isoquant consists of those points for which total product equals $\bar{Y}$. Isoquants are oftenderived from the emperically estimated production function by equating the estimated Y in terms of X's (inputs) with the fixed number $\bar{Y}$ and solve for one x as a function of another.

For the establishment of the confidence region for an isoquant, from our land-fertilizer production function, the following steps are carried out:

a) We start by finding the confidence in tervales for a given $\bar{Y}$, at a given $\bar{F}$ and $\bar{A}$. This be doen by the same method as

$$R = \left[\frac{\partial Y}{\partial a} \quad \frac{\partial Y}{\partial b_1} \quad \frac{\partial Y}{\partial b_2}\right] \quad (10)$$

$$R = \left[A \quad F \quad -F^2 A^{-1}\right] \quad (11)$$

and $G = (X'X)^{-1} \ S^2$ (12)

and the variance of "Y" is determined by RGR'. Therefore the limits on $\bar{Y}$ can be stated as: $\bar{Y}_1 < \bar{Y} < \bar{Y}_2$

b) By the use of the land-fertilizer production function, the limits on Y can be translated into land equivelance. That is knowing $\bar{Y} - \bar{Y}_1 = \bar{Y}_0$ and using : $\bar{Y}_0 = a\bar{A}_0 + b_1F - b_2F^2 \bar{A}_0^{-1}$, therefore at the given F, then, A_0 can be solved for. Hence confidence intervales for A as $(\bar{A}_1 < \bar{A} < \bar{A}_2)$ where: $\bar{A}_0 = \bar{A} - \bar{A}_1$ or $\bar{A}_0 = \bar{A}_2 - \bar{A}$ can be determined. Repeating the same procedure at different points gives a confidence interval around the isoquant function.

c) Isocline

The "K" isocline is defined as the locus of points for which the marginal rates of substitution of fertilizer for land (or : input (1) for input (2)) is k. This trace in our land-fertilizer production function is:

$$MPS_{\text{of F for A}} = \frac{\partial A}{\partial F} = -\frac{MPP_F}{MPP_A} \qquad \ldots (12)$$

$$= \frac{2b_2 F A^{-1} - b_1}{a + b_2F^2 A^{-2}} = K \qquad \ldots (13)$$

$$aK + b_2K F^2 A^{-2} = 2b_2 F A^{-1} - b_1$$

$$A = -b_2F \pm \left[b_2^2 F^2 - b_2F^2 K^2a + b_2F^2Kb_1\right]^{1/2} \left[aK + b_1\right]^{-1} \qquad \ldots (14)$$

To set the confidence intervale the variance of "A" can be determined through the RGR' procedure as follows:

$$R = \left[\frac{\partial A}{\partial a} \quad \frac{\partial A}{\partial b_1} \quad \frac{\partial A}{\partial b_2} \right] \quad \ldots \quad (15)$$

Solving for "R" quantities, we obtaine:

$$\frac{\partial A}{\partial a} = (-K\ A^2) \left[2(aAK + b_1 A - b_2 F) \right]^{-1} \quad \ldots \quad (16)$$

$$\frac{\partial A}{\partial b_1} = (-A^2) \left[2(aAK + b_1 A - b_2 F) \right]^{-1} \quad \ldots \quad (17)$$

$$\frac{\partial A}{\partial b_2} = (2AF - KF^2) \left[2(aAK + b_1 A - b_2 F) \right]^{-1} \quad \ldots \quad (18)$$

Thus the variance of A can be solved for. Hence, a confidence interval for A at $F = F_o$ and $A = A_o$ can be established as:

$$P \left\{ \hat{A}_o - t_\alpha \sqrt{\sigma^2_{A_o}} \leq A_o \leq A_o + t_\alpha \sqrt{\sigma^2_{A_o}} \right\} = 1 - \alpha \quad \ldots \quad (19)$$

Where α represents the probability level. Solving for a set of A's a confidence intervale is determined for the isocline function.

The application of the previous analysis to an estimated land-fertilizer production function (1) follows:

The function is

$$Y = 10.423\ A + 2.312\ F - .218\ F^2\ A^{-1} \quad \ldots \quad (20)$$

where : Y = corn yield in ardab (in U.A.R., Alex.)

A = Land in feddans.

F = Nitrogen fertilizer in units of 15 Kgs:

a) <u>The marginal productivities</u>:

The marginal physical products for fertilizer and land are:

$$MPP_F = 2.312 - .436\ F\ A^{-1} \quad \ldots \quad (21)$$

$$MPP_A = 10.423 + .218\ F^2\ A^{-2} \quad \ldots \quad (22)$$

and hence the "R" matrix of the marginal products is given by:

$$R = \begin{bmatrix} 0 & 1 & -2FA^{-1} \\ 1 & 0 & F^2A^{-2} \end{bmatrix} \quad \ldots \quad (23)$$

and since the estimated variance-convariance matrix of the estimated production function (G) is given by:

$$G=(X'X)^{-1}S^2= \begin{bmatrix} .76190 & -.46428 & .059523 \\ -.46428 & .46428 & -.071428 \\ .05952 & -.071428 & .011904 \end{bmatrix} 5.324 \quad \ldots \quad (24)$$

then the estimated variance-covariance matrix of MPP_F and MPP_A is:

$$\underset{(R)}{\begin{bmatrix} 0 & 1 & -2FA^{-1} \\ 1 & 0 & F^2A^{-2} \end{bmatrix}} \underset{(G)}{\begin{bmatrix} .76190 & -.46428 & .059523 \\ -.46428 & .46428 & -.071428 \\ .05952 & -.071428 & .011904 \end{bmatrix}} \underset{(R')}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -2FA^{-1} & F^2A^{-2} \end{bmatrix}} 5.324$$

$$= \begin{bmatrix} (\sigma^2_{MPP_F}) ; (\sigma_{MPP_F\ MPP_A}) \\ (\sigma_{MPP_F\ MPP_A}) ; (\sigma^2_{MPP_A}) \end{bmatrix} \quad (25)$$

Now to investigate the marginal products at the points: F = 1 and A = 1, we substitute in equation (25) to obtain the variances at these levels of the inputs. And given the assumption of normal errors, "t" statistic may be used to construct the confdence intervales. Thus the 95% confidence intervale for the marginal product of nitrogen at the point F = 1 (15 Kg.) and A = 1 (Feddan), is:

$$P(-3.15680 \leq MPP_F \leq + 6.92480) = 95\% \quad \ldots \quad (26)$$

The similar confidence intervale for the marginal product of land (A) at the same point F=1, A=1 is:

$$P(5.28207 \leq MPP_A \leq 15.99993) = 95\% \quad \ldots \quad (27)$$

b. Isoquants

The procedure for the determination of the confidence itervales around the isoquant function has been explained before. The method was used and the estimated 70% confidence intervles for an isoquant is plotted in figure (1), however, the procedure followed can be summarized as: a) by applying the RGR' technique the variance of Y was determined as:

$$\sigma_Y^2 = \begin{bmatrix} A & F & -F^2A^{-1} \end{bmatrix} \begin{bmatrix} .76190 & -.46428 & .059523 \\ -.46428 & .46428 & -.071428 \\ .05952 & -.071428 & .011904 \end{bmatrix} \begin{bmatrix} A \\ F \\ -F^2A^{-1} \end{bmatrix} (5.324) \quad \ldots \quad (28)$$

$$\sigma_Y^2 = \left[.76190A^2 - .92856FA + .345234F^2 + 1.42956\, F^3 A^{-1} + .011904F^4A^{-2} \right] (5.324) \quad \ldots \quad (30)$$

Equation (30) shows the variance of the predicted Y as a function of the values of F and A, b) we asign values for F and A and solve for the equivelant σ^2_Y at these specfied values. c) knowing this variance intervles around Y can be established, d) given the land-fertilizer production function, and knowing Y and F, we can solve for (A). These A's are plotted in the land-fertilizer isoquant diagram indicating the specified cconfidence intervale.

c) <u>Isocline</u>

The procedure for the determination of confidence intervle around an isocline function is a direct one as specified before. Sine "R" is a function of K, a, b_1, b_2, F and A and we know all of these constants but F and A, then we asign values for F and A and solve for the corresponding "R" and hence "RGR" and therefore σ^2_A at the asigned "F" and "A". Repeating this procedure a set of the variances of "A" at a corresponding values of F and A can be established. And assuming the normality criterion, a (1 - α) confidence intervale around the isocline curve is set up. The confidence limits around an isocline which is an expansion path, in the same time; is plotted in the diagram.

In the diagram too, the intersection area between the confidence limits around the expansion path, and the confidence intervles around the Y isoquant (where Y indicates optimum output level) gives a confidence region (with probability level less than either of the two areas) around the point of optimum resource combination.

LITERATURE CITED

1. Bishay, F.K. Marginal rates of substitution between land, labor, and fertilizer in relation to the optimum planning of resource combination. Ph.D. thesis. Ames, Iowa, Library, Iowa State University of Science and Technology. 1965.

2. Doll, J. P., Jebe, E. H., and Mansons, R. Computations of variance estimates for marginal physical products and marginal rates of substitutions. Journal of Form Economics 42: 596 - 607 . 1960

3. Fuller, W. Estimating the relaibilities of quantities derived from emperical production functions. Journal of Form Economics 44: 82 - 99. 1962.

4. Ostle, B. Statistics in research , basic concepts and techniques for research workers. 2nd. Ames, Iowa, Iowa State University Press. 1963.

5. Snedecor, G. W. Statistical methods. 4th ed. Ames, Iowa, Iowa State University Press. 1962.

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