UNITED ARAB REPUBLIC

THE INSTITUTE OF NATIONAL PLANNING

التخطيط القومى

Memo. No. 804

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By

Prof. A. A. Anis
A. S. T. El-Naggar

October 1967

CAIRO
3. MOHAMED MAZHAR - ZAMALEK

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By

Prof. A. A. Anis
Ain Shams University

And

A. S. T. El-Naggar
Institute of National Planning

(To appear in the Journal of the Institute of Mathematics and its Applications, London)

I. Introduction:

In some cases, two streams of water A and B meet in a joint path C , but technical difficulties prohibit building a dam on C.

If a dam of capacity k is built on A, before meeting B, then naturally the classical Moran's model has to be modified.

In this paper, the problem of stationary distribution P_i , of the dam-storage (in case of discrete inputs) is discussed, and it is shown that the quantities.

$$\frac{P_r}{P_o} \quad , \quad r = o,1, \ldots , k-1$$

are independent of k, and that the P_r's (r=1, ... , k-1) are generated by the function V(z) given by

$$V(z) = \frac{P_o p_o q_o \quad (1-z)}{(q_1' z+q_o)G(z)-z}$$

where p_i is the input distribution of the stram A, q_i is the input distribution of the stream B, G(z) is the p.g.f. of the input A, and $q_1' = q_1 + q_2 + \ldots$

It is clear from V(z) that this problem is equivilent to the single stream case, provided the input is regarded as the convolution of two independent variates, one is given by the distribution p_i and the other is a Binomial distribution, which takes only the values 0,1 with probabilities q_o and $1-q_o$.

It is clear that if B does not exist, then $q_o = 1$, $q_1' = o$ and

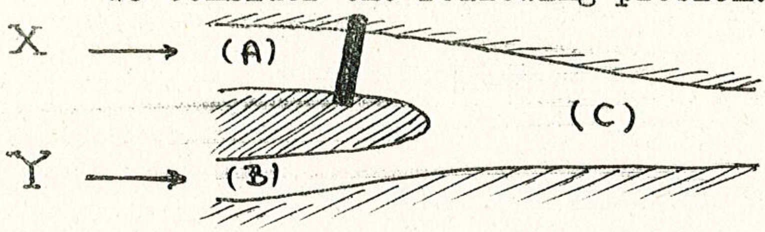
$$V(z) = \frac{P_o\, p_o\, (1-z)}{G(z) - z}$$

a result due to Moran (See [2]).

The probability of emptiness of the dam before filling completely is also obtained.

II. Two-streams Storage Problem:

We consider the following problem:



A and B are two streams which join to form a stream C. A dam is placed above the junction, on stream A, forming a finite reservoir, of capacity k. The object is to prevent the flow below the confluence from falling below a fixed value $M(<k)$ by releasing water (when available) from the reservoir to make up any deficit in C.

The technique is to use a discrete model, in which the following cycle of operations occurs:

(a) at time $t = n$, the quantity of water stored in the reservoir is Z_n

(b) during the time interval $(n,n+1)$, the inflow in A is X_n, with

$$P(X_n=r) = p_r\ ,\ P(X_n \geq r) = p_r'$$

and the flows in B is Y_n, with

$$P(Y_n=r)=q_r \quad , \quad p(Y_n \geqslant r) = q'_r,$$

X_n and Y_n being supposed mutually independent, non-seasonal, and without any serial correlation structure;

(c) just before time n+1 an instantaneous release R_n is allowed from the reservoir, in accordance with the above policy;

(d) at time t=n+1, the storage is Z_{n+1}.

Thus the successive values of storage during the cycle are:

i) Z_n

ii) Z_n + acceptable inflow = min $(X_n + Z_n , k) = c$, say

iii) $Z_{n+1} = c - R_n$,

where the phrase "acceptable inflow" refers of course to the possibility of overflow from the reservoir when the supply exceeds the spare storage; such spilled water is regarded as lost.

The novel aspect of this model lies in the fact that the draft R_n is controlled by the flow Y_n in the undammed stream.

We describe the situation for the case M=1, this being the one we deal with in detail.

The working equation for $\{Z_n\}$ in this case is

$$Z_{n+1} = \min (X_n+Z_n,k) - \begin{cases} 0 \ \text{ if } \ Y_n \geqslant 1 \\ \\ \min(1,X_n+Z_n) \ \text{ if } Y_n=0 \end{cases} \qquad (2.1)$$

The alternative cases that can arise can perhaps be most easily visualised in terms of a flow diagram (1) where current storage at any stage is marked by underlining.

The routes through the diagram may be labelled from the left, (1), (2), ..., (5). It will be seen that the outcome $Z_{n+1} = k$ is achieved by route (1) only and that

$$P(Z_{n+1}=k) = P(Y_n \geqslant 1,\ X_n+Z_n \geqslant k)$$

$$= P(Y_n \geqslant 1)\ P(X_n+Z_n \geqslant k)$$

Taking for granted the existence of a limiting distribution for $\{Z_n\}$, as $n \longrightarrow \infty$, and denoting this by

$$\lim_{n \to \infty} P(Z_n=r) = P_r,\quad r=0,1,\ \ldots,\ k \qquad (2.2)$$

we have

$$P_k = q_1' \sum_{r=0}^{k} P_r\ p_{k-r}' \qquad (2.3)$$

The case $Z_{n+1} = k-1$ must also be considered separately. Here we have

$$P(Z_{n+1} = k-1) = P(Y_n=0\ ,\ X_n+Z_n \geqslant k) \quad \ldots \quad \text{route (2)}$$

$$+\ P(Y_n \geqslant 1,\ Y_n+Z_n = k-1) \ \ldots \ \text{route (3)}$$

whence

$$P_{k-1} = q_0 \sum_{r=0}^{k} P_r\ p_{k-r}' + q_1' \sum_{r=0}^{k-1} P_r\ p_{k-1-r}, \qquad (2.4)$$

$(q_1' = 1-q_0)$.

For $1 \leqslant r \leqslant k-2$ we have

$$P(Z_{n+1} = r) = P(Y_n \geqslant 1,\ X_n+Z_n=r) \quad \ldots \text{ route } (3)$$

$$+ P(Y_n=o,\ X_n+Z_n = r+1) \ldots \text{ route } (4)$$

whence

$$P_r = q_1' \sum_{s=o}^{r} P_s\, p_{r-s} + q_o \sum_{s=o}^{r+1} P_s\, p_{r+1-s}, \qquad (2.5)$$

$r = 1,2, \ldots , k-2$

Finally for $r = o$ we note that $Z_{n+1} = o$ may be achieved by routes (3), (4) and (5), whence

$$P_o = q_1' P_o\, p_o + q_o\, (P_o\, p_1 + P_1\, p_o + P_o\, p_o). \qquad (2.6)$$

Equations (2.3) , ... , (2.6) may be written in the form:

$$\left\{\begin{array}{l} P_o = q_1' P_o\, p_o + q_o(P_o p_1 + P_1 p_o) + q_o p_o P_o \\ P_1 = q_1' (P_o p_1 + P_1 p_o) + q_o(P_o p_2 + P_1 p_1 + P_2 p_o) \\ \vdots \\ P_{k-2} = q_1'(P_o p_{k-2} + \ldots + P_{k-2} p_o) + q_o(P_o p_{k-1} + \ldots + P_{k-1} p_o) \end{array}\right. \qquad (2.7)$$

$$\left\{\begin{array}{l} P_{k-1} = q_1'(P_o p_{k-1} + \ldots + P_{k-1} p_o) + q_o(P_o p_k' + \ldots + P_k p_o') \\ P_k = q_1'(P_o p_k' + \ldots + P_k p_o') \end{array}\right. \qquad (2.8)$$

The last two equations (2.8) have a different pattern from the rest (2.7). But in the equations (2.7) (apart from one addition term in the first) the pattern is clear, the coefficients of q_1' and q_o in each equation having a "convolution" form. As usual in such formulations, there is a redundancy of equations, since if we add all the equations in (2.7) and (2.8) we obtain the identity

$$\sum_o^k P_i = P_o p_o + (P_o p_1 + P_1 p_o) + \ldots + (P_o p_{k-1} + \ldots + P_{k-1} p_o) +$$

$$+ (P_o p_k' + \ldots + P_k p_o')$$

$$= \sum_{r \geqslant o} P(X_n + Z_n = r) = P(X_n + Z_n \geqslant o) = 1$$

We now attempt to extract information about the P_i's from (2.7) and (2.8). The structure of (2.7) and (2.8) is such that P_1 is determined in term of P_o , P_2 in terms of P_1 , ... , and P_{k-1} in terms of P_{k-2} , and these relations all have the same pattern, with coefficients that do not involve k, whence the theorem:

THEOREM: The ratios $\frac{P_r}{P_o}$, r=o, 1, ... , k-1

are independent of k.

This result enables us to form a useful generating function relating to the P_i. Consider the infinite sequence P_o, P_1 , ... , defined by an associated system (2.7') which could be obtained from (2.7) by allowing $k \to \infty$. In the light of the above theorem, it is clear that (normalization apart) the values of P_o, P_1, ... , P_{k-1} that satisfy (2.7) will coincide with the first k P's in (2.7') . Now let

$$V(z) = \sum_{r=o} P_r \; z^r$$

denote the generating function of the P's defined by (2.7').
The "convolution" structure of the coefficients of q_1 and of q_o in (2.7) makes it clear that

$$V(z) = q_1' \, V(z) \, G(z) + q_o \left\{ \frac{V(z) \; G(z) - P_o p_o}{z} + P_o p_o \right\}$$

where $G(z) = \sum_{r=o}^{\infty} p_r \, z^r$ is the generating function of the inflow distribution $\{p_r\}$. Then

$$V(z) = \frac{p_o \; q_o \; P_o \; (1-z)}{(q_1{}' z + q_o) G(z) - z} \qquad (2.9)$$

We shall first prove that V(z) can be expanded as a power series which is convergent for suitable values of $|z|$. Let us consider the case $\rho \leq q_o$, where ρ is the mean inputs of stream A.

$$(q_o + q_1{}'z)G(z) - z = q_o(1-z) - (q_o + q_1{}'z)\left[1 - G(z)\right]$$

$$= q_o(1-z)\left[1 - \frac{q_o + q_1{}'z}{q_o} \cdot \frac{1-G(z)}{1-z}\right]$$

then since $\frac{1-G(z)}{1-z} = \sum_{n=o} z^n \cdot \sum_{i=n+1}^{\infty} p_i$, we obtain for $|z| < 1$,

$$\left| \frac{q_o + q_1{}'z}{q_o} \cdot \frac{1-G(z)}{1-z} \right| < \frac{1}{q_o} \sum_{n=o}^{\infty} \sum_{i=n+1}^{\infty} p_i = \frac{1}{q_o} \sum_{i=1}^{\infty} i p_i = \frac{\rho}{q_o} \leq$$

so that $|(q_o + q_1{}'z) \; G(z) - z| \neq o$, and we have the power series

$$V(z) = p_o\ P_o \left[1 - \frac{q_1 + q_1{}'z}{q_o} \cdot \frac{1-G(z)}{1-z} \right]^{-1}$$

$$= P_o + P_1 z + P_2 z^2 + \ldots , \qquad (2.10)$$

convergent for $|z| < 1$.

Next suppose that $\rho > q_o$. If we regard $(q_o + q_1{}'z)\ G(z)$ as the p.g.f. of the convolution of two independent variates, one of them has a p.g.f. $G(z)$ and the other has a p.g.f. $(q_o + q_1{}'z)$ then

$$\frac{1}{(q_o + q_1{}'z)\ G(z) - z}$$

will have a power series expansion convergent for $|z| < \lambda$, $\lambda > o$ provided the mean of the convolution is greater than unity (Knopp [1], pp 182).

Now the mean of the convolution is easily seen to be

$$q_1' + \rho = 1 - q_o + \rho ,$$

i.e. we get as the condition of the expansion

$$1 - q_o + \rho > 1$$

i.e. $$\rho > q_o$$

which is satisfied in our case.

Thus whether $\rho \leq q_o$ or $\rho > q_o$, $V(z)$ has a power series expansion (2.10).

This generating function determines P_1, P_2 , ... , P_{k-1}, whilst P_k is determined by either of the two equations (2.8). Finally P_o must be determined by normalization.

III. We shall work out an example, specifying only the distribution of the inputs from stream A. If the inputs in A has a geometric distribution $P_i = ab^i$, $i=o,1, \dots \infty$ where $a+b=1$ and $G(z) = \frac{a}{1-bz}$ and $p'_r = b^r$, we get from (2.9)

$$V(z) = \frac{aq_oP_o\,(1-z)\,(1-bz)}{bz^2-(b+aq_o)z+aq_o}$$

$$= P_o\,\frac{(1-bz)}{(1-\rho z)}\ ,\ \text{where}\ \rho = \frac{b}{aq_o}$$

and on choosing z such that $|z|<\frac{1}{\rho}$, we obtain

$$V(z) = P_o\,(1-bz)\,(1-\rho z)^{-1}$$

whence

$$P_r = P_o\ \rho^{r-1}\,(\rho-b)\ ,\ r = 1,2,\dots,k-1$$

Now P_k is obtained by either of the two equations (2.8) and hence,

$$P_k\ \frac{q_o}{1-q_o} = P_o\left[\,b^k+(\rho-b)b^{k-1}+(\rho-b)\rho b^{k-2}+\dots+(\rho-b)\rho^{k-2}b\,\right]$$

$$= P_o\left[\,b^k+(\rho-b)b^{k-1}\left(1+\frac{\rho}{b}+\frac{\rho^2}{b^2}+\dots+\frac{\rho^{k-2}}{b^{k-2}}\right)\right]$$

$$= P_o\,b\,\rho^{k-1}$$

$$\therefore\quad P_k = \left(\frac{1-q_o}{q_o}\right)P_o\ b\ \rho^{k-1}$$

$$= (a\rho-b)P_o\ \rho^{k-1}\qquad \text{where}\quad q_o = \frac{b}{a\rho}$$

To determine P_o we use the fact that $\sum_{r=0}^{k} P_r = 1$, therefore

$$1 = P_o \left[1 + (\rho - b) + \rho(\rho - b) + \rho^2(\rho - b) + \dots + \rho^{k-2}(\rho - b) + (a\rho - b)\rho^{k-1}\right]$$

$$= P_o \left[1 + (\rho - b)(1 + \rho + \rho^2 + \dots + \rho^{k-2}) + (a\rho - b)\rho^{k-1}\right]$$

$$= P_o \left[1 + (\rho - b)\left(\frac{1 - \rho^{k-1}}{1 - \rho}\right) + (a\rho - b)\rho^{k-1}\right]$$

$$1 - \rho = P_o \left[(1 - \rho) + (\rho - b)(1 - \rho^{k-1}) + (1 - \rho)(a\rho - b)\rho^{k-1}\right]$$

$$= P_o \, a(1 - \rho^{k+1})$$

$$P_o = \frac{1 - \rho}{a(1 - \rho^{k+1})}$$

and hence

$$P_r = \frac{\rho^{r-1}(1 - \rho)(\rho - b)}{a(1 - \rho^{k+1})}, \quad r = 1, 2, \dots, k-1$$

and

$$P_k = \frac{\rho^{k-1}(1 - \rho)(a\rho - b)}{a(1 - \rho^{k+1})}$$

The cases when the input distribution of stream A is Negative Binomial, Binomial or Poission, could be also worked out along similar lines.

IV. The Problem of Emptiness Before Filling:

For the same model we shall discuss the emptiness of the dam before filling completely. We define H_i to be the conditional probability that starting with an initial storage i, the dam becomes empty before it fills completely. We shall deal in detail with the case M=1. The alternative cases that can arise can also be shown in the flow-diagram(2) where current storage at any stage is marked by underlining.

It is clear that $H_o=1$ and $H_k=o$. The values of $X_n \geqslant k-1$ when $i = 1$ and $X_n \geqslant k-s$ when $i = s$ causes the dam to be filled completely or overflow and hence these cases are not included in the problem under discussion.

The value $Z_{n+1} = o$ in route (2) indicates that the dam will be empty at cycle n+1 if the initial storage Z_n is unity and the inflow of streams A and B during the cycle interval (n,n+1) are respectively $X_n = o$, $Y_n \geqslant 1$. Other values in route (2) and values in routes (1), (3) and (4) indicate that the reservoir still contains r units ($r > o$) with a probability H_r that the dam will be empty before filling completely.

The equations for the conditional probability H_i are defined for H_1 from routes (1) and (2), and for H_s ($1 < s < k$) from routes (3) and (4).

Here we have

$$H_1 = \sum_{r=o}^{k-2} P(Y_n \geqslant 1,\ X_n=r)H_{r+1} + \qquad \text{... route (1)}$$

$$+ \sum_{r=o}^{k-2} P(Y_n=o,\ X_n=r)\ H_r \qquad \text{... route (2)}$$

whence

$$H_1 = q_1' \sum_{r=o}^{k-2} p_r\ H_{r+1} + q_o \sum_{r=1}^{k-2} p_r\ H_r + q_o p_o \qquad (4.1)$$

For $1 < s < k-1$ we have

$$H_s = \sum_{r=o}^{k-s-1} P(Y_n \geqslant 1,\ X_n=r)\ H_{r+s} + \qquad \text{... route (3)}$$

$$+ \sum_{r=o}^{k-s-1} P(Y_n=o,\ X_n=r)\ H_{r+s-1} \qquad \text{... route (4)}$$

whence

$$H_s = q_1' \sum_{r=o}^{k-s-1} p_r \; H_{r+s} + q_o \sum_{r=o}^{k-s-1} p_r \, H_{r+s-1} \quad (4.2)$$

$$s = 2,3, \ldots , k-1$$

We shall consider the notation

$$w_i = H_{k-i} \quad (4.3)$$

then clearly $w_o = o$ and $w_k = 1$.

Rewriting equations (4.1) and (4.2) using the above new variable w_i in (4.3) we have

$$w_{k-1} = q_1' \sum_{r=o}^{k-2} p_r \, w_{k-r-1} + q_o \sum_{r=1}^{k-2} p_r \, w_{k-r} + q_o p_o$$

and

$$w_{k-s} = q_1' \sum_{r=o}^{k-s-1} p_r \, w_{k-r-s} + q_o \sum_{r=o}^{k-s-1} p_r \; w_{k-r-s+1}$$

$$\text{for } s = 2,3, \ldots , k-1$$

Or, in general we have

$$w_1 = q_1' \, p_o \, w_1 + q_o \, p_o \, w_2$$

and

$$w_i = q_1' \, p_{i-1} \, w_1 + \sum_{r=1}^{i-1} (q_o \, p_{i-r} + q_1' \, p_{i-r-1}) w_{r+1} + q_o p_o w_{i+1}$$

$$\text{for } i=2,3, \ldots , k-1$$

This may be written in detail,

$$\left\{\begin{array}{lll} w_1 & = & q_1' p_o w_1 + q_o p_o w_2 \\ w_2 & = & q_1' p_1 w_1 + q_o p_o w_3 + (q_o p_1 + q_1' p_o) w_2 \\ w_3 & = & q_1' p_2 w_1 + q_o p_o w_4 + (q_o p_2 + q_1' p_1) w_2 + (q_o p_1 + q_1' p_o) w_3 \\ w_4 & = & q_1' p_3 w_1 + q_o p_o w_5 + (q_o p_3 + q_1' p_2) w_2 + \dots + (q_o p_1 + q_1' p_o) w_4 \\ \vdots & & \\ w_{k-1} & = & q_1' p_{k-2} w_1 + q_o p_o w_k + (q_o p_{k-2} + q_1' p_{k-3}) w_2 + \dots + (q_o p_1 + q_1' p_o) w_{k-} \end{array}\right. \quad (4.4)$$

The structure of (4.4) is such that w_2 is determined in terms of w_1, w_3 in terms of w_2 , ... , w_{k-1} in terms of w_{k-2}, and these relations all have the same pattern with coefficients that do not involve k and whence the therem:

THEOREM: The ratios $\frac{w_i}{w_1}$, $i = 1,2,\dots,k-1$

are independent of k.

Now Consider the infinite sequence w_1, w_2, ... defined by an associated system (4.4$'$) which could be obtained from (4.4) by allowing $k \longrightarrow \infty$

It is clear that the values of w_1 , w_2, ... , w_{k-1} that satisfy (4.4) are the same for extended system (4.4$'$).

Define the generating function of w_i's to be

$$W(z) = w_1 + w_2 z + w_3 z^2 + \dots = \sum_{r=o}^{\infty} w_{r+1} \; z^r$$

On multiplying w_i in the extended system (4.4') by Z^{i-1} for $i=1,2,\ldots$, we get after some manipulation,

$$zW(z) = W(z)\ (q_o+q_1'z)\ G(z) - q_o w_1\ G(z)$$

and hence

$$W(z) = \frac{q_o\ w_1\ G(z)}{(q_o+q_1'z)G(z)-z} \qquad (4.5)$$

As in section II, we can prove that W(z) can be expanded as power series convergent for suitable values of $|z|$ -

The conditional probability w_i $(i=2,3,\ldots,k-1)$ is the coefficient of z^{i-1} in W(z). To obtain w_1 we have to use the fact that $w_k = 1$

As an example consider the case where stream A has geometric input with $G(z) = \frac{a}{1-bz}$ and $a+b = 1$. Thus, on using (4.5), we have ,

$$W(z) = \frac{w_1}{(1-\rho z)(1-z)}, \text{ where } \rho = \frac{b}{aq_o}$$

$$= \frac{w_1}{(1-\rho)}\left[\frac{1}{1-z} - \frac{\rho}{1-\rho z}\right]$$

whence

$$w_i = \frac{w_1}{1-\rho}(1-\rho^i) \quad i=2,\ldots,k-1$$

and since $w_k = 1$, when w_1 has the value

$$w_1 = \frac{1-\rho}{1-\rho^k}$$

Thus

$$w_i = \frac{1 - \rho^i}{1 - \rho^k} \qquad i = 2, \ldots , k-1$$

and using (4.3), therefore

$$H_i = \frac{1 - \rho^{k-i}}{1 - \rho^k} , \qquad \rho \neq 1$$

$$= 1 - \frac{i}{k} , \qquad \rho = 1$$

Acknowledgment

We are deeply grateful to the referee for his suggestions and comments.

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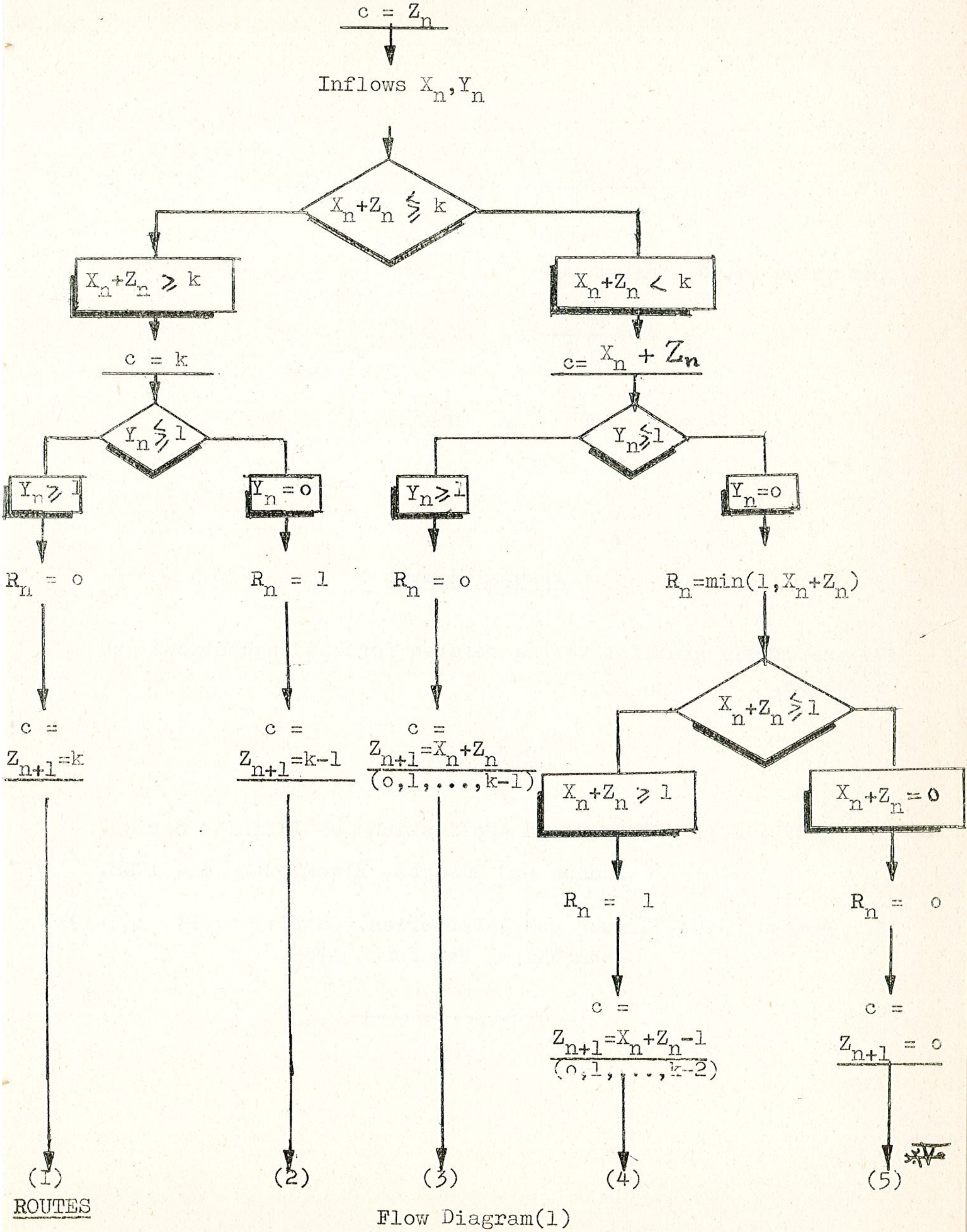
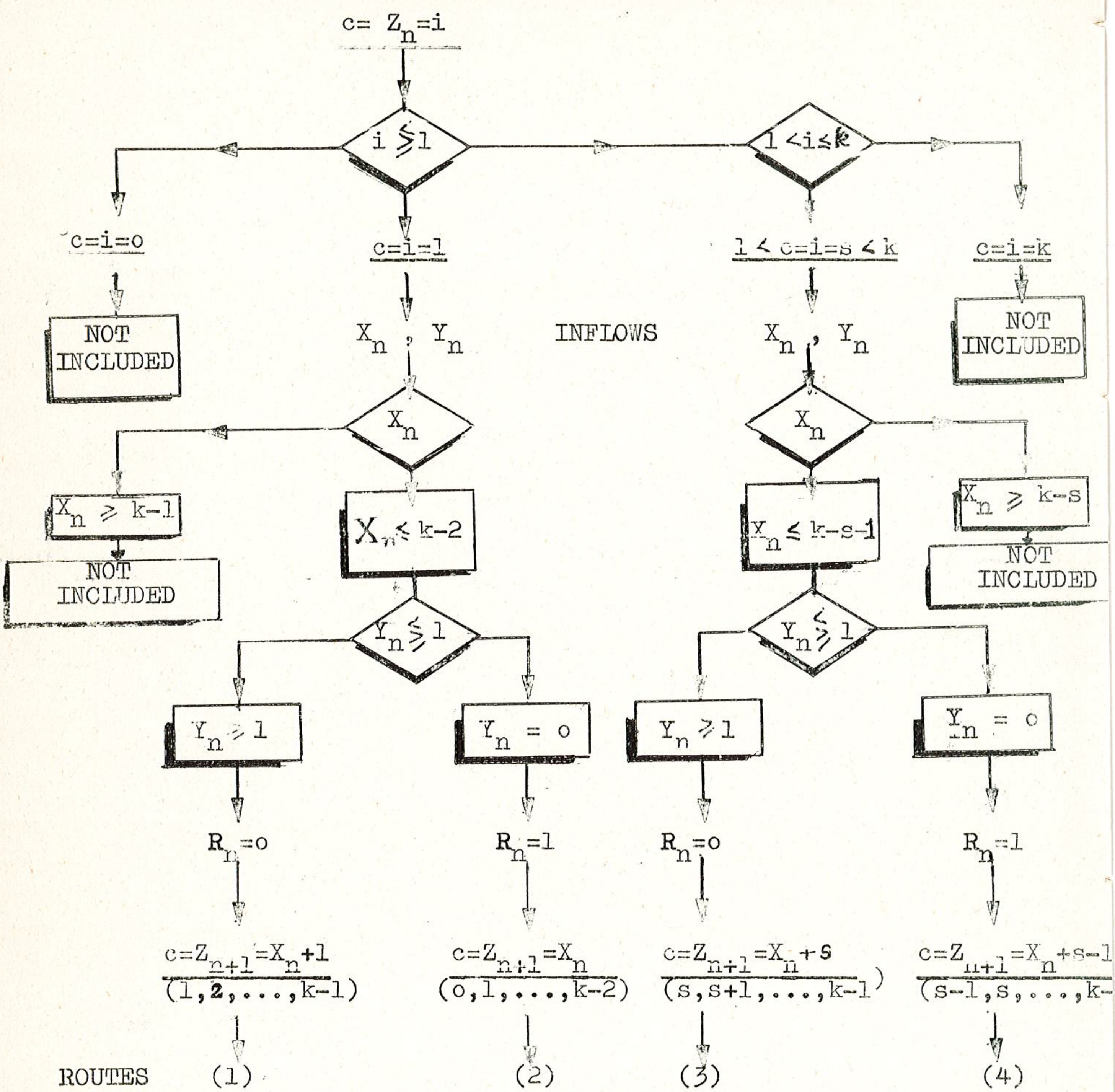


Flow Diagram(1)



Flow Diagram (2)