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Notes on the Inventory Theory

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Introduction

Inventories are stocks of goods stored for some reaso in spite of the storage costs and the typing up of capital wh: would be invested elsewhere.

There are different reasons for keeping inventories. Frequently they are held because of economics in scale of production or because of the seasonal fluctuations in the prices of raw materials used in production. Uncertainty of future demands and the existance of time lag between the placing of orders and the delivery of goods are also strong reasons for holding inventories. As in the case of keeping cash balances, the motives for maintaining inventories may be summerized in three main kinds: transactions, precautionary, and speculative motives.

In fact, the problem of controlling inventories is a general problem which appears in various fields of study and which is not restricted to the economical management of stocks of commodities as such. Arrow, Karlin, and Scarf [1] point out the wide scope of the subject: "An inventory problem might, for example involve deciding how much typing paper to stock each month for an office, or how many spare parts to keep on hand for a given machine. When production is involved, the inventory problem might require determining how much wheat to plant per year or how much gasoline of a certain variety to have blended. How much water to release from a dam for electricity and irrigation purposes is an inventory problem; how many workers to hire for a given labor force is another. Inventory problems may involve scheduling, production, determining efficient distribution of commodities in certain markets, finding proper replacement policies for old equipments, determining proper prices for goods produced, or combinations of these elements." Scarf [4] states that:" In addition, large areas of economic theory are concerned with similar problems. Any dynamic problem in economic theory is necessarily concerned with stocks, whether these be interpreted as capital, manufacturers inventories, bank reserves or the accumulated savings and assets on an individual consuming unit."

The inventory theory is concerned with questions such as: when to replenish inventories and how much to order for replenishment, such that the relevant costs are minimized? To answer such questions, a mathematical model for the inventory system has to be constructed. By analysing the model and studying its properties, the rules for operating the system in the "best" possible way are obtained. So, if the model considers some cost function as its objective function, these rules will define the inventory policy which minimizes this objective function , i.e., the "optimal" inventory policy.

The costs relivent to inventory problems may be classified into three typs:

1. The "set-up" or ordering cost; which is the cost of ordering, purchasing, or producing the commodity in order to increase or replace inventories.

2. The "holding"or carrying cost; which is the cost of storing the commodity. It includes the opportunity cost.

3. The "shortage" or penalty cost; which is the cost associated with either a delay in meeting demand, or the inability to meet it at all.

In this note several mathematical models of inventory systems will be presented and analysed. We will start by the simple models having deterministic demand. Next, a one-period model where demand is a random variable with a known probability distribution will be discussed followed by the multi and infinite-period models with stochastic demand. All these models are presented under different assumptions concerning the relivent costs, but all of them have in common the property that the organization controlling inventories can decide to either increase the inventory level or leave it as it is. The last section discusses a model where it is also possible to decide to decrease the inventory level (i.e. the disposal of stocks is permitted).

It should be noticed that all the models considered in this note discuss the single commodity case and assume that the inventory system has no control over the demand, on the other hand, it can determine when and in what quantity the inventory level should be changed. It is also assumed that costs remain constant over time.

I. Deterministic Models.

In this section we consider a number of simple inventory models where the demand is assumed to be known with certainty and to have a fixed rate over time. Although this is an unrealistic assumption, these models are studied in order to see how the different kinds of costs interact and affect the decisions about inventory levels.

Case 1. Lead time = o,
No shortage is allowed.

Suppose a retailer expects to sell exactly Z units of a certain commodity during time T and that this commodity has a fixed demand rate, R, over time ($R = \frac{Z}{T}$). Suppose also that the retailer is not allowed to have any unsatisfied demand and that there is no time lag between ordering to replenish inventories and recieving the order, i.e., the lead time equals zero.

Let:

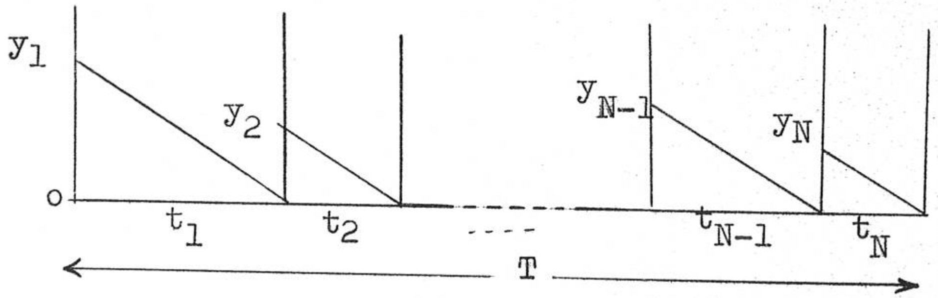
h, ($h \geqslant 0$), denote the holding cost per item per unit of time,

$K+k\xi$, ($K, k \geqslant 0$), denote the set-up cost, i.e., the cost of increasing the inventory level by ξ units ($\xi > 0$).

Now, if the retailer starts by keeping the amount Z in inventory, the holding cost will be high. On the other hand if he decides to keep a small amount in inventory and to replenish the inventory level at different intervals, then he has to pay the set-up costs which depend on the number of orders he places, in addition to the holding costs which will be less than before. So, the retailer's problem is to determine how often he should order and how much should be ordered every time in order to satisfy all the demand and at the same time keep his total costs at a minimum.

Let N be the number of orders placed during the time T, and t_n be the length of the n^{th} period, (i.e., the time between receiving the n^{th} and the $(n+1)^{st}$ order). Then $\sum_{n=1}^{N} t_n = T$. We will assume that the retailer has no initial stock on hand.

It is clear, since the lead time equals zero, that the retailer will not reorder except when the inventory is reduced to zero. If y_n denotes the amount he orders at the beginning of period n, then $\sum_{n=1}^{N} y_n = Z$ (because failure to meet demand is not allowed), and the situation can be summerized by the following diagram:



Notice that the straight lines representing the inventory levels during the different periods have the same negative slope which should equal the constant demand rate R.

Now, what is the optimal number of periods, and what is the optimal value of y_n for each n?

Answering these questions is the same as finding the values of N and y_n which minimize the total cost function: $C_T(N, y_n)$

To calculate $C_T(N, y_n)$ notice that the inventory level is y_n at the beginning of period n, and is reduced at a constant rate during the period till it reachs zero by the end of the period. Thus the average inventory level during the period is $\frac{y_n}{2}$.

$\therefore$ The holding cost during period n $= h\, t_n \frac{y_n}{2}$ $n=1,\dots N$

$\therefore$ The total cost during period n $= K+k\, y_n + h\, t_n \frac{y_n}{2}$ $n=1,\dots N$

$\therefore$ $C_T(N, y_n)$, the total cost during time $T = \sum_{n=1}^{N} \left[K + k\, y_n + h\, t_n \frac{y_n}{2}\right]$

$$= NK + k \sum_{n=1}^{N} y_n + \frac{h}{2} \sum_{n=1}^{N} t_n\, y_n$$

$$\therefore\ C_T(N, y_n) = NK + k \sum_{n=1}^{N} y_n + \frac{h}{2} \sum_{n=1}^{N} \frac{y_n^2}{R}$$

where $R = \frac{Z}{T} = \frac{y_n}{t_n}$, for all n.

So, we want to find the values of N and y_n that minimize $C_T(N, y_n)$, which is a convex function, subject to the constraint:

$$\sum_{n=1}^{N} y_n = Z.$$

For a given N, and by using Lagrange multiplier technique, this is equivalent to finding the values of y_n that minimize the new function C, and then finding the optimal value of N.

Let $C = C_T(N, y_n) - \lambda \left(\sum_{n=1}^{N} y_n - Z \right)$,

where λ is Lagrange multiplier.

Putting the first partial derivative $\frac{\partial C}{\partial y_n}$ equal to zero, we get

$$\frac{h}{R}\, y_n^{*} - \lambda = 0 \qquad n = 1, \ldots N.$$

$$\therefore \quad y_n^* = \frac{R\lambda}{h} \qquad n = 1, \ldots N. \qquad \text{(for a given N)}$$

$$\because \quad Z = \sum_{n=1}^{N} y_n$$

$$\therefore \quad Z = \frac{N R \lambda}{h}$$

$$\therefore \quad \lambda = \frac{Z h}{N R}$$

$$\therefore \quad y_n^* = \frac{Z}{N} \qquad n = 1, \ldots. N \qquad \text{(for a given N)}$$

$$\because \quad t_n^* = \frac{Z}{N} \cdot \frac{1}{R} = \frac{T}{N} \qquad n = 1, \ldots N \ \text{(for a given N)}$$

$$\therefore C_T(N, y^*) = NK + kZ + \frac{h Z T}{2 N} \qquad \text{(for a given N)}$$

Putting $\frac{\partial C_T}{\partial N}$ equal to zero, we get the optimal N:

$$\frac{\partial C_T}{\partial N} = K - \frac{h Z T}{2N^2}$$

$$\therefore \quad N^* = \sqrt{\frac{Z h T}{2 K}}$$

Thus, the optimal inventory level at the begining of evry period is given by:

$$y^* = \frac{Z}{N^*} = \sqrt{\frac{2 K Z}{h T}} = \sqrt{\frac{2 K}{h} \cdot R}.$$

(This formula is known by: the Economic Lot Size Formula.) And the optimal policy can be summerized by:

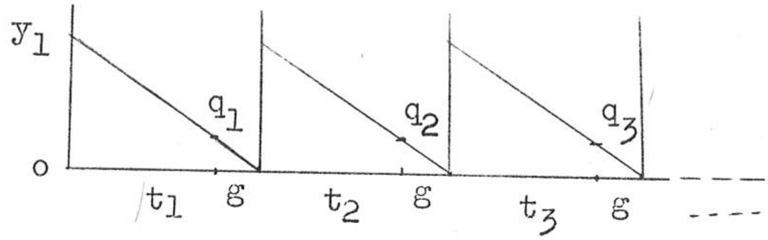
Devide the time T into N^* equal periods, each of length $\frac{T}{N^*}$. At the beginning of every period the retailer will have zero units on hand and should order the amount $y^* = \sqrt{\frac{2K}{h}R}$.

This result shows that, as expected, the optimal inventory level increases when the demand rate, R, and/or the fixed ordering cost, K, increases, while it decreases if the holding cost, h, increases. It also shows that the inventory level should increase only in proportion to the square root of sales (or demand), contrary to the intutive idea of keeping the inventory level as a fixed proportion of sales.

Case 2 : Lead time = $g > 0$
No shortage is allowed.

In stead of assuming that there is no time lag between placing an order and receiving it, we will assume here that there is a fixed lead time of length g , so any order placed at time t will be received at time t + g. How does this assumption affect the optimal inventory policy?

We will use the same notation as in Case 1. Here, y_n denotes the order size as before, and it also denotes the inventory level at the beginning of period n after receiving the order. Let q_n denote the "reorder inventory level" in period n. Applying the same analysis as in Case 1, we get:



The optimal inventory policy is to devide T into $N^* = \sqrt{\frac{Z\ h\ T}{2\ K}}$ equal periods each of length $t^* = \frac{T}{N^*}$. At the beginning of each period, the inventory level before receiving the order should be zero and should be increased to the level $y^* = \sqrt{\frac{2\ K}{n}\ R}$ after receiving the order.

We still have to define the "optimal" reorder level q^*. Since the lead time equals g, then the total demand during the lead time is (g R). So, the optimal policy should specify:

Whenever the inventory level reaches the level $q^* = Rg$, place an order of size y^*.

So, Case 1 is a special case of Case 2, where $g = o$ and consequently $q^* = o$.

[For simplicity g is taken to be less than t].

Case 3 : Lead time = g,

Backlog is allowed.

Here we have the same model as in Case 2 except for relaxing the assumption of not allowing any shortage. In stead, we assume that the retailer may not satisfy the demand when it occurs, but still he has to satisfy it when the commodity next becomes available. this is known by the backlog case. In addition to the costs considered in the previous models, the retailer has to pay penalty costs proportional to the unsatisfied demand.

Let:

c , $c \geq o$, denote the shortage cost per unit of unsatisfied demand per unit of time,

$x_n \leq o$ denote the inventory level at the beginning of the n^{th} period, before receiving the order,

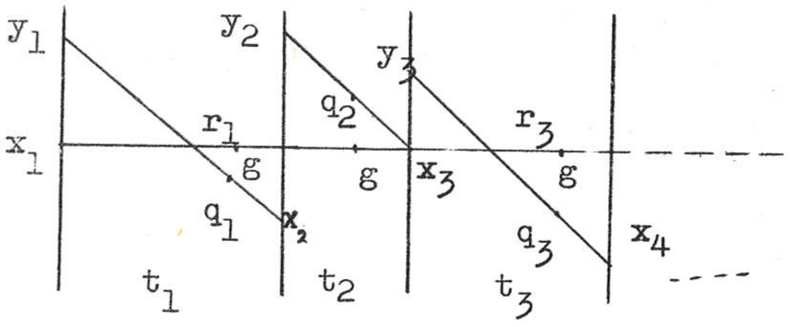
y_n, $y_n \geq x_n$, denote the inventory level at the beginning of the n^{th} period, after receiving the order,

($\therefore$ $y_n - x_n \geq o$ is the amount received at the beginning of period n)

r_n denote the part of period n in which the inventory level is negative.

($\therefore$ $t_n - r_n$ is the part of period n in which the inventory level is positive.)

This model can be represented by the following diagram:



As before, we have:

$$\sum_{n=1}^{N} t_n = T \quad \text{and} \quad \sum_{n=1}^{N} (y_n - x_n) = Z.$$

The average (positive) inventory level during time $t_n - r_n$

$$= \frac{y_n}{2} \qquad n = 1 \ldots N$$

The average shortage (negative) inventory level during time r_n

$$= \frac{x_{n+1}}{2} \; n = o, \ldots . \; N-1$$

$\therefore$ The holding and shortage cost during period n

$$= h \frac{y_n}{2} \; (t_n - r_n) + c \; \frac{-x_{n+1}}{2} r_n$$

$$= h \frac{y_n}{2} \; \frac{y_n}{R} \; + \; c \; \frac{-x_{n+1}}{2} \frac{-x_n}{R}$$

$$= \frac{h}{2} \; \frac{y_n^2}{R} + \frac{c}{2} \; \frac{x_{n+1}^2}{R}$$

$$\therefore C_T(N, y_n, x_{n+1}) = \sum_{n=1}^{N} \left[K + k(y_n - x_n) + \frac{h}{2} \cdot \frac{y_n^2}{R} + \frac{c}{2} \cdot \frac{x_{n+1}^2}{R} \right].$$

$$= NK + kZ + \frac{h}{2R} \sum_{n=1}^{N} y_n^2 + \frac{c}{2R} \sum_{n=o}^{N} x_{n+1}^2$$

we want to find the values of y_n, x_{n+1} and N that minimize

C_T (which is a convex function) subject to the constraint:

$$\sum_{n=1}^{N} (y_n - x_n) = Z.$$

Again, using the Lagrange multiplier technique, and for a fixed N, this is equivalent to minimizing C(N) and then finding the optimal value of N, where

$$C(N) = C_T(N, y_n, x_{n+1}) - \lambda\left[\sum_{n=1}^{N} (y_n - x_n) - Z\right].$$

Putting $\frac{\partial C}{\partial y_n} = o$ and $\frac{\partial C}{\partial x_{n+1}} = o$, yealds:

$$y_n^* = \frac{R\lambda}{h} \quad \text{and} \quad x_n^* = \frac{-R\lambda}{c} \qquad n=1, \ldots N, \quad \text{for a given } N.$$

$$\therefore \quad y_n^* - x_n^* = R\left[\frac{1}{h} + \frac{1}{c}\right] \qquad n=1, \ldots N, \quad \text{given } N.$$

$$\therefore \quad Z = \sum_{n=1}^{N} (y_n^* - x_n^*)$$

$$= N \lambda R \frac{h+c}{hc}$$

$$\therefore \quad \lambda = \frac{Z}{N R} \frac{hc}{h + c}$$

$$\therefore \quad y_n^* = \frac{Z}{N} \frac{c}{h+c} \quad \text{and} \quad x_n^* = -\frac{Z}{N} \frac{h}{h+c} \qquad n=1, \ldots N, \text{ given } N.$$

$$\therefore \quad y_n^* - x_n^* = \frac{Z}{N}$$

$$\therefore \quad t_n^* = \frac{y_n^* - x_n^*}{R}$$

$$= \frac{T}{N} \qquad n=1, \ldots N., \text{ given } N$$

$$\therefore C_T(N, y_n^*, x_n^*) = NK + Zk + \frac{h}{2R} N\left(\frac{Z}{N} \cdot \frac{c}{h+c}\right)^2 + \frac{c}{2R} N\left(\frac{Z}{N} \cdot \frac{-h}{h+c}\right)^2$$

$$= NK + Zk + \frac{Z^2}{2RN} \frac{hc}{h+c}$$

$$= NK + Zk + \frac{ZT}{2N} \frac{hc}{h+c} \qquad \text{for a given } N.$$

To get the optimal value of N, put $\frac{\partial C_T}{\partial N} = 0$.

$$\therefore K - \frac{ZT}{2N^{*2}} \frac{hc}{h+c} = 0$$

$$\therefore N^* = \sqrt{\frac{ZT}{2K} \frac{hc}{h+c}}$$

$$\therefore t^* = \frac{T}{N^*} = \sqrt{\frac{2K}{R} \cdot \frac{h+c}{hc}}$$

$$\therefore y^* - x^* = \frac{Z}{N^*} = \sqrt{2K \frac{h+c}{hc} R}$$

$$\text{and } y^* = \frac{Z}{N^*} \cdot \frac{c}{h+c} = \sqrt{\frac{2cK}{h(h+c)} R}.$$

Now, since the demand during the lead time is Rg, the optimal reorder level should be:

$$q^* = Rg + x^*$$

$$= Rg - \frac{h}{c} y^* .$$

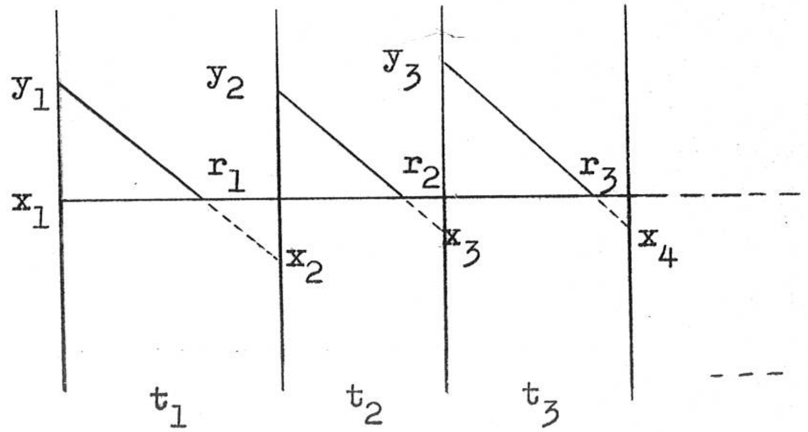
This result shows that the time horizon T should be devided into N^* equal periods each of length t^*. When the inventory level reaches the level q^*, the retailer should place an order of size $(y^* - x^*)$. At the beginning of each period the retailer will have x^* units of unsatisfied demand and should receive the quantity $(y^* - x^*)$ which depends on R, K, c, and h,

It is noticed that as $c \longrightarrow \infty$, $\frac{h+c}{c} = \frac{h}{c} + 1 \longrightarrow 1$. Thus, the form defining $(y^* - x^*)$ is reduced to the form $\sqrt{\frac{2KR}{h}}$, and $x^* \longrightarrow o$, which is the same result as in Cases 1 and 2. This should be true since having $c \longrightarrow \infty$ means that unsatisfied demands are too expensive, hence not allowed to occur, which is exactly the assumption of Cases 1 and 2.

It should also be noticed that k, the proportional ordering cost, does not appear in the formulea defining the optimal policy in Cases 1, 2, and 3. This is also clear since the retailer has to pay the cost (k Z) over the time horizon T whatever is the policy he follows.

Case 4. Lead time = o,
Lost sales Case.

Here as in Case 3, shortages are allowed to take place, but once the demand is not satisfied when it occurs, the retailer will lose it and will be unable to satisfy it later. [This is known by the lost sales case.] This means that the inventory level is not allowed to take any negative values. So, using the same notation as before with $c \geqslant 0$ still denoting the shortage cost per unit of unsatisfied demand per unit of time, and x_n denoting the level of lost sales at the beginning of period n (or by the end of period n-1), we have:



$$\sum_{n=1}^{N} t_n = T \quad , \quad \sum_{n=1}^{N} (y_n - x_n) = Z, \quad \text{and}$$

$$C_T(N, y_n, x_{n+1}) = \sum_{n=1}^{N} \left[K + k\, y_n + h\, \frac{y_n}{2} (t_n - r_n) + c\, \frac{-x_{n+1}}{2} r_n \right]$$

$$= \sum_{n=1}^{N} \left[K + k\, y_n + \frac{h}{2} \cdot \frac{y_n^2}{R} + \frac{c}{2} \cdot \frac{x_{n+1}^2}{R} \right]$$

$$= NK + k \sum y_n + \frac{h}{2R} \sum y_n^2 + \frac{c}{2R} \sum x_{n+1}^2$$

We will assume, for simplicity that the lead time equals zero. Define the function C by:

$$C = C_T(N, y_n, x_{n+1}) - \lambda(\Sigma y_n - Z)$$

Then, the values y_n^* and x_{n+1}^* which minimize C, also minimize C_T for a given N.

$$\frac{\partial C}{\partial y_n} = \frac{h\, y_n}{R} + k - \lambda$$

$$\frac{\partial C}{\partial x_{n+1}} = \frac{c\, x_{n+1}}{R} + \lambda \quad .$$

Putting these derivatives equal to zero, we get:

$$y_n^* = \frac{(\lambda - k)}{h} R \qquad n = 1, \dots N.$$

$$x_{n+1}^* = -\frac{\lambda}{c} R \qquad n = 1, \dots N.$$

$$\therefore Z = \Sigma (y_n^* - x_n^*)$$

$$= -\frac{k}{h} RN + \frac{c+h}{ch} RN\lambda$$

$$\therefore \lambda = \frac{ch}{c+h} \cdot \frac{1}{RN} \left(Z + \frac{k}{h} RN \right)$$

$$\therefore y^* = \frac{c}{c+h} \frac{Z}{N} - \frac{k}{c+h} R$$

$$x^* = \frac{h}{c+h} \frac{Z}{N} + \frac{k}{c+h} R$$

$$\therefore y^* - x^* = \frac{Z}{N} \qquad \text{(for a given N)}$$

$$\therefore t^* = \frac{T}{N} \qquad \text{(for a given N)}$$

$$\therefore\ C_T(N,y^*,x^*) = NK + \frac{kc}{c+h} Z - \frac{k^2}{c+h}\frac{NR}{2} + \frac{ch}{c+h}\frac{Z^2}{2RN} \quad \text{(for a given N)}$$

$$\frac{\partial C_T}{\partial N} = K - \frac{k^2 R}{2(c+h)} - \frac{ch}{c+h}\frac{Z^2}{2RN^2}$$

Equating this with zero, we get:

$$N^* = \sqrt{\frac{ch\ ZT}{2K(c+h)-k^2R}}$$

$$\therefore\ y^* = -\frac{k}{c+h} R + \sqrt{\frac{2cK}{(c+h)h} R - \frac{k^2c}{(c+h)^2h} R^2}$$

$$x^* = \frac{k}{c+h} R + \sqrt{\frac{2hK}{(c+h)c} R - \frac{k^2h}{(c+h)^2c} R^2}$$

This shows that the optimal policy is to devide the interval T into N^* equal periods each of length t^*. At the begining of each period, the retailer should order the amount y^*, and by the end of the period he loses x^* units.

Notice that putting k = o reduces y^* to the form: $y^* = \sqrt{\frac{2cK}{(c+h)c} R}$ which is the same as the result of Case 3. This has to be true since k = o means that the proportional ordering cost does not affect the inventory level, consequently the optimal decision depends on the other costs c, h, and K in the same way as in Case 2.

II. Stochastic One-period Model

In the previous models it has been assumed that the demand rate is known with certainty . In most practical problems, this is not the case. It is more realistic to consider the demand as a random variable having a known probability distribution.

We will start by the one-period model, where the inventory management has to decide, once and for all, the inventory level that it should have at the begining of a given period. It is assumed here that unsatisfied demands are permitted and that the lead time is zero.

Let h, c, k, and K denote the relevant costs as defined in the previous models.

Let x_1 , denote the initial inventory level (before ordering),

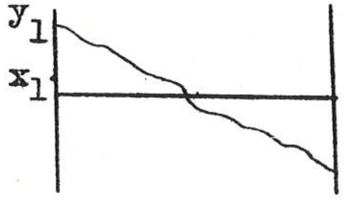
y_1, $y_1 \geqslant x_1$ denote the inventory level right after replenishment,

Z , $Z \geqslant 0$ denote the quantity demanded which is a random variable with a continuous probability distribution having known density p(z),

$L(y_1)$ denote the expected holding and shortage costs,

$C_{x_1}(y_1)$ denote the total expected cost if x_1 is the initial inventory level and y_1 is the inventory level after ordering.

Notice that in this model we can no longer talk about the "total cost", instead we should consider the "total expected cost" and try to find out the policy which minimizes it.



$\therefore$ $L(y_1)$ = expected holding and shortage cost,

$$\therefore L(y_1) = \begin{cases} h\int_0^{y_1}(y_1-z)\,p(z)\,dz + c\int_{y_1}^{\infty}(z-y_1)p(z)dz & \text{if } y_1 > 0 \\ c\int_0^{\infty}(z-y_1)\,p(z)\,dz & \text{if } y_1 \leq 0 \end{cases}$$

which is a convex function.

$$\therefore C_{x_1}(y_1) = K\,\delta(y_1-x_1) + k\,(y_1-x_1) + L(y_1)$$

$$= K\,\delta(y_1-x_1) - kx_1 + G_1(y_1)$$

where $\delta(\xi) = \begin{cases} 1 & \text{if } \xi > 0 \\ 0 & \text{if } \xi = 0 \end{cases}$, and

$G_1(y_1) = k\,y_1 + L(y_1)$.

Since $L(y_1)$ is convex, then $G_1(y_1)$ is a convex function too. Suppose $G_1(y_1)$ reaches its minimum at a point $y_1 = S_1$. Then S_1 must be defined by : $\left.\frac{d\, G_1(y_1)}{d\, y_1}\right|_{y_1=S_1} = o$

But $\frac{d\, G_1(y_1)}{d\, y_1} = k + \frac{d}{dy_1} L(y_1) = k + h \int_o^{y_1} p(z)dz - c \int_{y_1}^{\infty} p(z)dz$

Then S_1 is defined by: $k + L'(S_1) = o$

$$\text{i.e.} \quad k + h \int_o^{S_1} p(z)dz - c \int_{S_1}^{\infty} p(z)dz = o$$

$$\therefore \int_o^{S_1} p(z)dz = 1 - \int_{S_1}^{\infty} p(z)\, dz$$

$$\therefore \quad k + h - (h+c) \int_{S_1}^{\infty} p(z)dz = o$$

$$\therefore \quad \int_{S_1}^{\infty} p(z)dz = \frac{k+h}{h+c} .$$

Now, we will describe the optimal inventory policy in two different cases:

Case 1 : if $K = o$

Case 2 : if $K > o$

Case 1. If $K = o$:

In this case, $C_{x_1}(y_1) = -k\, x_1 + G_1(y_1)$, and the optimal policy is given by:

If $x_1 < S_1$: order the quantity $S_1 - x_1$, i.e. increase the inventory level to the level S_1.

If $x_1 \geqslant S_1$: do not order,

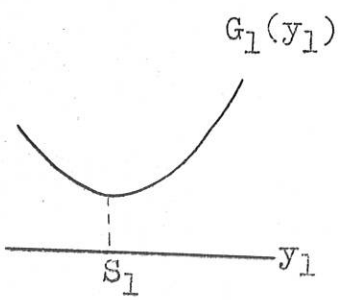
i.e. leave the inventory level as it is,

where S_1 is defined by: $\int_{S_1}^{\infty} p(z)dz = \frac{k+h}{h+c}$

Proof:

To prove that this policy minimizes the total expected costs, we consider the different regions that x_1 may fall in and compare the values of $C_{x_1}(y_1)$ for different y_1's.

i. If $x_1 \leqslant S_1$:



$\because$ $G_1(y_1)$ reaches its minimum at S_1

$\therefore$ $G_1(y_1) > G_1(S_1)$ for all y_1

$\therefore$ $-k\,x_1 + G_1(y_1) > -k\,S_1 + G_1(S_1)$ for all y_1

$\therefore$ $C_{x_1}(y_1) > C_{x_1}(S_1)$ for all y_1

$\therefore$ $C_{x_1}(y_1)$ reaches its minimum at $y_1 = S_1$

ii. If $x_1 > S_1$:

Since $G_1(y_1)$ is convex and reaches its minimum at S_1, and since $x_1 > S_1$, then

$G_1(y_1) > G_1(x_1)$ for all $y_1 > x_1$

$\therefore \quad ky_1 + L(y_1) > kx_1 + L(x_1)$ for all $y_1 > x_1$

$\therefore \quad k(y_1-x_1) + L(y_1) > L(x_1)$ for all $y_1 > x_1$

$\therefore \quad C_{x_1}(y_1) > C_{x_1}(x_1)$ for all $y_1 > x_1$

$\therefore \quad C_{x_1}(y_1)$ reaches its minimum at $y_1 = x_1$

<u>Case 2</u>: If $K > 0$:

Let s_1 be defined by: $s_1 \leqslant S_1$ and $G_1(s_1)=G_1(S_1)+K$. Then the optimal policy in this case is given by:

If $x_1 < s_1$; order the quantity (S_1-x_1), i.e. increase the inventory level up to S_1

If $x_1 \geqslant s_1$; do not order, i.e. leave the inventory level as it is.

<u>Proof</u>:

Again, we consider the different regions that x_1 may fall in and find out the values of y_1 at which $C_{x_1}(y_1)$ reaches its minimum.

i. If $x_1 < s_1$:

$\because \quad G_1(y_1)$ reaches its minimum at S_1

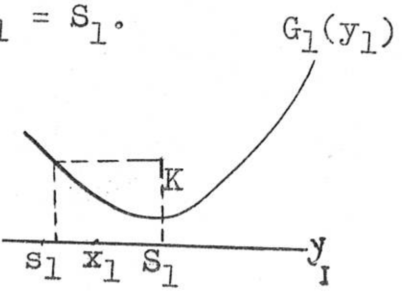
$\therefore \quad G_1(y_1) > G_1(S_1)$ for all y_1

$\therefore \quad K - kx_1 + G_1(y_1) > K-kx_1 + G_1(S_1)$ for all y_1

$\therefore \quad C_{x_1}(y_1) > C_{x_1}(S_1)$ for all y_1

$\therefore \quad C_{x_1}(y_1)$ reachesits minimum at $y_1 = S_1$.

ii. If $s_1 \leqslant x_1 \leqslant S_1$:



It is clear from the figure that:

$K + G_1(y_1) \geqslant G_1(x_1)$ for all $y_1 > x_1$

$\therefore \quad K + ky_1 + L(y_1) \geqslant kx_1 + L(x_1)$ for all $y_1 > x_1$

$\therefore \quad K + k(y_1-x_1)+L(y_1) \geqslant L(x_1)$ for all $y_1 > x_1$

$\therefore \quad C_{x_1}(y_1) \geqslant C_{x_1}(x_1)$ for all $y_1 > x_1$

$\therefore \quad C_{x_1}(y_1)$ reaches its minimum at $y_1 = x_1$

iii. If $x_1 > S_1$:

$\because \quad G_1(y_1)$ is convex and reachs its minimum at S_1

$\therefore \quad G_1(y_1) > G_1(x_1)$ for all $y_1 > x_1$

$\therefore \quad K + G_1(y_1) > G_1(x_1)$ for all $y_1 > x_1$

$\therefore \quad K + k(y_1-x_1) + L(y_1) > L(x_1)$ for all $y_1 > x_1$

$\therefore\ C_{x_1}(y_1) > C_{x_1}(x_1)$ for all $y_1 > x_1$

$\therefore\ C_{x_1}(y_1)$ reaches its minimum at $y_1 = x_1$

[Note: All the given results hold true if L(y) is any convex function.]

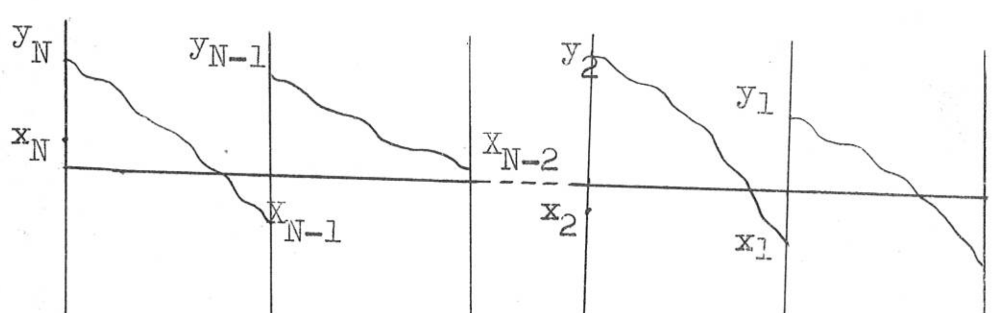
The policy described here is called the (S , s) policy. It shows that the order size should not be less than $(S_1 - s_1)$ which depends on K (by the definition of s_1). So, this optimal policy shows that the effect of the economies of scale is having $(S_1 - s_1)$ as the minimum order size.

III. Stochastic Multiperiod Model

Consider now the N-period model with independent identically distributed demands having common distribution with continuous density $p(z)$. In this model, the inventory management can take decisions concerning inventory levels at the beginning of each of N equal periods. It is assumed that backlogs, but not lost sales, are allowed and that the lead time is zero.

The solution for this problem is not to use the optimal one-period policy N times. Smaller costs may be achieved by viewing the problem from a dynamic programming point of view. After numbering the periods in a backword order, so that the beginning of period n implies that there are n periods left in the horizon, we try to find the sequence of inventory decisions which minimizes the total expected discounted cost over the N periods.

The situation may be presented in the following diagram:



where:

x_n is the initial inventory level at the beginning of period n, before taking any decision,

$y_n \geq x_n$ is the inventory level at the beginning of period n after taking the decision.

Using the same notation as before, we get:

The total expected cost in period n

$$= K \; (y_n - x_n) + k(y_n - x_n) + L(y_n)$$

Let $C_n(x_n)$ be the minimum expected discounted cost over n periods, if x_n is the initial inventory level, and an optimal policy is followed.

$$\therefore C_n(x_n) = \min_{y_n \geq x_n} \left[K\,\delta\,(y_n - x_n) + k(y_n - x_n) + L(y_n) + \alpha \int_0^\infty C_{n-1}(y_n - z)p(z)dz \right]$$

$$= \min_{y_n \geq x_n} \left[K\,\delta\,(y_n - x_n) - k\,x_n + G_n(y_n) \right] \quad ,$$

$C_o(x_o) = o.$

where: $o < \alpha < 1$ is the discount factor, and

$$G_n(y_n) = k\,y_n + L(y_n) + \alpha \int_0^\infty C_{n-1}(y_n - z)p(z)dz.$$

As before, we will describe the optimal policy in two cases:

Case 1. $K = o$.

In this case $C_n(x_n)$ is given by:

$$C_n(x_n) = \min_{y_n \geq x_n} \left[-k\, x_n + G_n(y_n) \right], \quad \text{and;}$$

The optimal policy in period n (n=1...N) is of the form:

If $x_n < S_n$; order the amount$(S_n - x_n)$, i.e. increase the inventory level to S_n.

If $x_n \geqslant S_n$; do not order, i.e. leave the inventory level as it is.

where S_n is defined by:

$$G_n(S_n) \leqslant G_n(y_n) \quad \text{for all } y_n.$$

Proof:

For n = 1 :

The result has been proved for the one period case.

For n = 2 : Since the optimal policy in period 1 is of the given from then $C_1(x_1) = \begin{cases} k(S_1 - x_1) + L(S_1) & x_1 \leqslant S_1 \\ L(x_1) & x_1 > S_1 \end{cases}$

$\because$ $L(\cdot)$ is convex

$\therefore$ $C_1(x_1)$ is convex

Now,

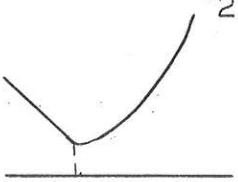
$$\therefore \quad G_2(y_2) = ky_2 + L(y_2) + \alpha \int_0^{\infty} C_1(y_2-z)p(z)\,dz$$

$\therefore$ $G_2(y_2)$ is also convex. It reaches its minimum at S_2, where S_2 is defined by:

$$\left. \frac{d\, G_2(y_2)}{dy_2} \right|_{y_2=S_2} = 0$$

i.e., $G_2(S_2) \leq G_2(y_2)$ for all y_2

$$\therefore \quad C_2(x_2) = \min_{y_2 \geqslant x_2} \left[-k\,x_2 + G_2(y_2) \right]$$



$$= \begin{cases} -k\,x_2 + G_2(S_2) & \text{if } x_2 < S_2 \\ -k\,x_2 + G_2(x_2) & \text{if } x_2 \geqslant S_2 \end{cases}$$

$\therefore$ The optimal policy in period 2 is to order up to S_2 if $x_2 < S_2$, and to leave the inventory level as it is if $x_2 \geqslant S_2$.

<u>For n > 2</u> (An induction proof)

Assume that all the results proved for n = 2 hold true for all periods up to and including period n-1, we will prove that they are true for period n:

Since the optimal policy in period n-1 is as described,

$$\therefore \quad C_{n-1}(x_{n-1}) = \begin{cases} k(S_{n-1}-x_{n-1}) + L(S_{n-1}) + \alpha\int_0^{\infty} C_{n-1}(S_{n-1}-z)p(z)dz & \text{if } x_{n-1} \leq S_{n-1} \\ L(x_{n-1}) + \alpha\int_0^{\infty} C_{n-1}(x_{n-1}-z)p(z)dz & \text{if } x_{n-1} \geqslant S_{n-1} \end{cases}$$

$\therefore$ $C_{n-2}(\cdot)$ is convex, by the induction assumption

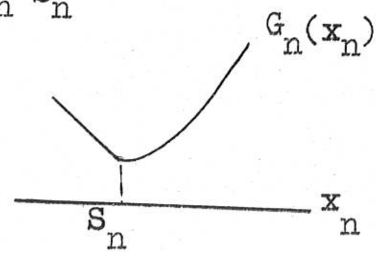
$\therefore$ $C_{n-1}(\cdot)$ is convex

$\therefore$ $G_n(y_n)$ is also convex; it reaches its unique minimum at S_n, where S_n is given by:

$$\frac{d}{dy_n} G_n(y_n) \Big|_{y_n=S_n} = 0$$

$$\therefore \quad C_n(x_n) = \min_{y_n \geq x_n} \left[-kx_n + G_n(y_n) \right]$$

$$= \begin{cases} -kx_n + G_n(S_n) & \text{if } x_n < S_n \\ -kx_n + G_n(x_n) & \text{if } x_n > S_n. \end{cases}$$



This shows that the optimal policy for period n is of the given form.

Case 2: $K > 0$.

Let s_n be defined by: $s_n \leq S_n$ and $G_n(s_n) = G_n(S_n) + K$,

Where S_n is still defined by $G(S_n) \leq G(y_n)$ for all y_n.
[For simplicity assume s_n is unique. Otherwise define it to be the smallest value that satisfies the definition.]

Then, the optimal policy in this case is given by:

If $x_n < s_n$; order the amount $(S_n - x_n)$; i.e. increase the inventory level to S_n;

If $x_n \geqslant s_n$; do not order, i.e.
leave the inventory level as it is.

Proof:

For n = 1 : This result has been proved for the one-period case.

For n = 2 : To carry on the proof for $n \geqslant 2$, we have to introduce a new definition:

Definition. A function f(.) is "right-hand K-convex" iff:

$$f(x+a) - f(x) - af'(x) + K \geqslant 0$$

(If K=o, then f(.) is a convex function)

Now, since the optimal policy in period 1 is of the given form,

then:
$$C_1(x_1) = \begin{cases} K - kx_1 + G_1(S_1) & \text{if } x_1 \leqslant s_1 \\ -kx_1 + G_1(x_1) & \text{if } x_1 \geqslant s_1 \end{cases}$$

This function is right-hand K-convex. In order to prove this, consider the different regions that x_1 may fall in:

i. if $x_1 > s_1$:
$$C_1(x_1) = -kx_1 + G_1(x_1)$$
$= L(x_1)$ which is a convex function

ii. if $x_1 < s_1 < x_1 + a$:

$$C_1(x_1+a) - C_1(x_1) - a\,C_1'(x_1) + K$$

$$= L(x_1+a)-K-k(S_1-x_1)-L(S_1)+ak+K$$

$$= L(x_1+a) + k(x_1+a) - L(S_1)-kS_1$$

$$= G(x_1+a) - G(S_1)$$

$$\geqslant o \text{ by definition of } S_1$$

iii. if $x_1 \leqslant x_1 + a \leqslant s_1$

$$C_1(x_1+a)-C_1(x_1)-a\,C_1'(x_1)+K$$

$$= K-k(x_1+a)+G_1(S_1)-K+kx_1-G_1(S_1)+ak+K$$

$$= K > o$$

This completes the proof that $C_1(x_1)$ is right-hand K-convex.

$\therefore$ $G_2(y_2)$ is right-hand K-convex and reaches its minimum at S_2 where S_2 is defined by:

$$G_2(S_2) \leqslant G_2(y_2) \quad \text{for all } y_2$$

$$\therefore G_2(x_2) = \min_{y_2 \geqslant x_2}\left[K\,\delta(y_2-x_2)-kx_2 + G_2(y_2)\right]$$

$$= \begin{cases} K-kx_2 + G_2(S_2) & \text{if } x_2 < s_2 \\ -kx_2 + G_2(x_2) & \text{if } x_2 \geqslant s_2 \end{cases}$$

[The detailed proof for this last result goes in exactly the same steps as in the one period model except for using the property of the right-hand K-convextry of $G_2(y_2)$ in place of the convextry property.]

For $n > 2$:

Using the right-hand K-convexty and the induction proof (as in Case 1, $n > 2$) we can prove that the optimal policy for period n is of the given form.

[Note: The fact that $G_n(x_n)$ is right-hand K-convex implies that, although it might have different local minimums, yet the/oscillations are never large enough to cause a divergence from the (S_n, s_n) policy.]

IV. Stochastic Infinits-Period Model

Instead of having a finite number of periods, we will assume that the retailer has an infinite horizon and wants to know the inventory policy that he should follow at the beginning of every period, in order to minimize the total expected discounted costs over the whole horizon. We are still assuming that the lead time is zero and that unfulfilled demands are backlogged.

If C(x) denotes the total expected discounted costs if an optimal policy is followed, then it should satisfy the functional relation:

$$C(x)=\min_{y \geq x}\left[K\,\delta(y-x)+k(y-x)+L(y)+\alpha\int_0^\infty C(y-z)p(z)dz\right].$$

The analysis of this case follows directly from the n-period analysis by considering the properties of the sequence of functions: $\left\{C_n(x)\right\}_{n=0}^{n=\infty}$. It had been proved that this sequance is montone increasing and bounded from above. It had been also proved that the sequence converges uniformly for all x in any finite interval. The limit function C(x) is right-hand K-convex and is the unique solution to the functional equation:

$$C(x) = \min_{y \geq x}\left[K\,\delta(y-x) + k(y-x)+L(y)+\alpha\int_0^\infty C(y-z)p(z)dz\right].$$

These properties are enough to prove that the optimal policy for the infinite-period problem is of the (S,s) type, and is given by

if $x < s$: order the quantity S-x, i.e. increase inventories to the level S ,

if $x \geqslant s$: do not order, i.e. leave the inventory level as it is.

where: S is defined by: $G(S) \leqslant G(y)$ for all y ,

$s \leqslant S$ and $G(s) = G(S) + K$.

(as usual $G(y) = ky + L(y) + \alpha \int_0^\infty C(y-z)p(z)dz$,

and s is assumed to be unique).

In the special case where K = o, we have S = s, and it can be proved that $S = \lim_{n \to \infty} S_n$ and that

$$k(1-\alpha) + \frac{d}{dy} L(y) = 0 \quad \text{at } y = S.$$

As for the Case where K > o, the computations of the values of S and s are quite difficult.

V. Stochastic Model with the possibility of deciding to reduce the inventory level.

In this section we assume that at the beginning of each period, the inventory management has the choice between replenishing and depleting the inventory level. So, if x_n is the inventory level at the beginning of period n before taking any decision, and if y_n is the inventory level after taking the decision, then y_n may take any value greater than, less than, or equivalent to x_n. $(y_n \gtreqless x_n)$.

Let $d \leq o$ represent the cost of decreasing the inventory level by one unit,

$$a\xi = \begin{cases} k\xi & \text{if } \xi > o \\ o & \text{if } \xi = o \\ d\xi & \text{if } \xi < o \end{cases},$$

$$\delta\xi = \begin{cases} 1 & \text{if } \xi > o \\ o & \text{if } \xi \leq o \end{cases}.$$

Then $K\delta(y-x) + a(y-x)$ represents the "set-up" costs under the new assumption. The total expected cost during period n is $K\delta(y-x) + a(y-x) + L(y)$, and the total expected discounted cost over n-periods if the optimal policy is followed is:

$$C_n(x_n) = \min_{y_n} \left[K\delta(y_n-x_n) + a(y_n-x_n) + L(y_n) + \alpha\int_0^\infty C_{n-1}(y_n-z)p(z)dz \right].$$

[We still have the backlog case with lead time = o]

Under these assumptions, it had been proved that the optimal policy is given by:

At the beginning of period of:

if $x_n < s_n$ increase the inventories to the level S_n,

if $s_n \leq x_n \leq u_n$ decrease the inventories to the level $\underline{y}_n(x)$, $[s_n < \underline{y}_n(x) \leq x_n]$,

if $u_n < x_n$ decrease the inventories to the level u_n.

Where u_n is defined by:

$$d(u_n-x_n)+L(u_n)+\alpha\int_0^\infty C_{n-1}(u_n-z)p(z)dz \leq d(y_n-x_n)+L(y_n)+\alpha\int_0^\infty C_{n-1}(y_n-z)p(z)dz$$

for all y_n

S_n is defined by:

$$k(S_n-x_n)+L(S_n)+\alpha\int_0^\infty C_{n-1}(S_n-z)p(z)dz \leq k(y_n-x_n)+L(y_n)+\alpha\int_0^\infty C_{n-1}(y_n-z)p(z)dz$$

for all y_n

$s_n \leq S_n$ and is defined by:

$$k(s_n-x_n)+L(s_n) + \alpha\int_0^\infty C_{n-1}(s_n-z)p(z)dz = K+k(S_n-x_n)+L(S_n)+\alpha\int_0^\infty C_{n-1}(S_n-z)p(z)dz$$

($s_n = S_n$ if $K = o$)

For the infinte period problem, the optimal policy will have the same form.

The proof of these results is similar to the proofs in parts II, III, and IV, It depends on the right-hand K-convexty of $C_n(x)$ and the convergence properties of this sequence of functions as n tends to infinity.

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