

## EXISTENCE OF AT LEAST ONE CONTINUOUS SOLUTION OF A COUPLED SYSTEM OF URYSOHN INTEGRAL EQUATIONS

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ABSTRACT. In this work, we are concerning with a coupled system of nonlinear Urysohn functional integral equations. We study the existence of at least one continuous solution. The nonlinear Urysohn functional integral equation will be given as a special case. A coupled system of Hammerstein functional integral equations will be considered as an application.

### 1. INTRODUCTION

It is known that integral equations have many useful applications in describing numerous events and problems of real world and the theory of integral equation is rapidly developing with the help of several tools of functional analysis, topology and fixed point theory.

Consider the coupled system of nonlinear Urysohn functional integral equations

$$x(t) = a_1(t) + \int_0^t f_1(t, s, y(\varphi_1(s)))ds, \quad t \in [0, T] \quad (1)$$

$$y(t) = a_2(t) + \int_0^t f_2(t, s, x(\varphi_2(s)))ds, \quad t \in [0, T]. \quad (2)$$

The existence of at least one solution  $(x, y)$  of the coupled system (1)-(2) will be proved.

The special case, the nonlinear Urysohn functional integral equation

$$x(t) = a_1(t) + \int_0^t f_1(t, s, x(\varphi_1(s)))ds, \quad t \in [0, T] \quad (3)$$

will be considered. Also the existence of the maximal and the minimal solution of (3) will be proved.

The coupled system of Hammerstein functional integral equations

$$x(t) = a_1(t) + \int_0^t k_1(t, s) g_1(s, y(\varphi_1(s)))ds, \quad t \in [0, T] \quad (4)$$

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*Key words and phrases.* Coupled system, Urysohn integral equations, fractional-order integral equations, maximal, minimal existence.

$$y(t) = a_2(t) + \int_0^t k_2(t, s) g_2(s, x(\varphi_2(s))) ds, \quad t \in [0, T] \quad (5)$$

will be considered as applications.

## 2. MAIN RESULTS

Let  $a_i : I = [0, T] \rightarrow R$  be continuous and  $\sup_{t \in I} |a_i(t)| = a_i^*$ .  
Let  $\varphi_i : I \rightarrow I$ ,  $i = 1, 2$  be continuous functions.

Consider the following assumptions

- (i)  $f_i : I \times I \times R \rightarrow R$  are continuous in  $t \in I$  for all  $(s, x) \in I \times R$ , measurable in  $s \in I$  for all  $(t, x) \in I \times R$  and continuous in  $x \in R$  for all  $(t, s) \in I \times I$ ,  $i = 1, 2$ .
- (ii) There exist two integrable functions  $m_i : I \times I \rightarrow R$  and two positive constants  $b_i$ ,  $i = 1, 2$  such that

$$|f_i(t, s, x)| \leq |m_i(t, s)| + b_i|x|.$$

and

$$\int_0^t m_i(t, s) ds \leq M_i, \quad t \in I.$$

Let  $X$  be the Banach space of all order pairs  $(x, y)$  with the norm

$$\|(x, y)\|_X = \|x\| + \|y\| = \sup_{t \in I} |x(t)| + \sup_{t \in I} |y(t)|.$$

Define the operator  $F$  by

$$F(x, y) = (T_1 y, T_2 x)$$

where

$$T_1 y = a_1(t) + \int_0^t f_1(t, s, y(\varphi_1(s))) ds$$

and

$$T_2 x = a_2(t) + \int_0^t f_2(t, s, x(\varphi_2(s))) ds$$

Define the set of functions

$$Q_r = \{(x, y) \in X : \|x\| \leq r_2, \|y\| \leq r_1, r_1 + r_2 = r\},$$

where

$$r_1 = \frac{(a_1 + M_1)}{(1 - b_1 T)} \quad \text{and} \quad r_2 = \frac{(a_2 + M_2)}{(1 - b_2 T)}.$$

**Definition 1.** By a solution of the coupled system (1)-(2) we mean the ordered pair  $(x, y)$  such that  $x, y$  are continuous,  $x, y \in C[0, T]$ . This ordered pair satisfies the coupled system (1)-(2).

Now we prove some lemmas which will be used in proving the main Theorem.

**Lemma 1.** Let the assumptions (i)-(ii) be satisfied, then the set of function

$F : Q_r \rightarrow Q_r$  and the set of function  $FQ_r$  is uniformly bounded.

**Proof.** From our assumptions we have

$$\begin{aligned} |T_1 y| &\leq |a_1(t)| + \left| \int_0^t f_1(t, s, y(\varphi_1(s))) ds \right| \\ &\leq a_1 + \int_0^t m_1(t, s) ds + b_1 \|y\| \int_0^t ds \\ &\leq a_1 + M_1 + b_1 r_1 T \leq r_1, \end{aligned}$$

then

$$\|T_1 y\| \leq r_1.$$

Also

$$\begin{aligned} |T_2 x| &\leq |a_2(t)| + \left| \int_0^t f_2(t, s, x(\varphi_2(s))) ds \right| \\ &\leq a_2 + \int_0^t m_2(t, s) ds + b_2 \|x\| \int_0^t ds \\ &\leq a_2 + M_2 + b_2 r_2 T \leq r_2, \end{aligned}$$

then

$$\|T_2 x\| \leq r_2.$$

Now for  $(x, y) \in Q_r$ , we have

$$\begin{aligned} \|F(x, y)\| &= \|(T_1 y, T_2 x)\| = \|T_1 y\| + \|T_2 x\| \\ &\leq a_1 + M_1 + b_1 r_1 T + a_2 + M_2 + b_2 r_2 T \\ &\leq r_1 + r_2 = r. \end{aligned}$$

This proves that

$$F : Q_r \rightarrow Q_r,$$

and the set of function  $FQ_r$  is uniformly bounded.

**Lemma 2.** Let the assumptions (i)-(ii) be satisfied, then the set of function  $FQ_r$  is equicontinuous.

**Proof.** Let  $t_1, t_2 \in [0, T]$  such that  $|t_2 - t_1| < \delta$ , then

$$\begin{aligned} |T_1 y(t_2) - T_1 y(t_1)| &= |a_1(t_2) - a_1(t_1) + \int_0^{t_2} f_1(t_2, s, y(\varphi_1(s))) ds - \int_0^{t_1} f_1(t_1, s, y(\varphi_1(s))) ds| \\ &= |a_1(t_2) - a_1(t_1) + \int_0^{t_1} f_1(t_2, s, y(\varphi_1(s))) ds \\ &\quad + \int_{t_1}^{t_2} f_1(t_2, s, y(\varphi_1(s))) ds - \int_0^{t_1} f_1(t_1, s, y(\varphi_1(s))) ds|. \end{aligned}$$

and

$$\begin{aligned} |T_1 y(t_2) - T_1 y(t_1)| &\leq |a_1(t_2) - a_1(t_1)| + \int_{t_1}^{t_2} |f_1(t_2, s, y(\varphi_1(s)))| ds \\ &\quad + \int_0^{t_1} |(f_1(t_2, s, y(\varphi_1(s))) - f_1(t_1, s, y(\varphi_1(s))))| ds. \end{aligned} \tag{6}$$

Also

$$\begin{aligned}
|T_2x(t_2) - T_2x(t_1)| &= |a_2(t_2) - a_2(t_1) + \int_0^{t_2} f_2(t_2, s, x(\varphi_2(s)))ds - \int_0^{t_1} f_2(t_1, s, x(\varphi_2(s)))ds| \\
&= |a_2(t_2) - a_2(t_1) + \int_0^{t_1} f_2(t_2, s, x(\varphi_2(s)))ds \\
&\quad + \int_{t_1}^{t_2} f_2(t_2, s, x(\varphi_2(s)))ds - \int_0^{t_1} f_2(t_1, s, x(\varphi_2(s)))ds|
\end{aligned}$$

and

$$\begin{aligned}
|T_2x(t_2) - T_2x(t_1)| &\leq |a_2(t_2) - a_2(t_1)| + \int_{t_1}^{t_2} |f_2(t_2, s, y(\varphi_2(s)))|ds \\
&\quad + \int_0^{t_1} |(f_2(t_2, s, x(\varphi_2(s))) - f_2(t_1, s, x(\varphi_2(s))))|ds. \tag{7}
\end{aligned}$$

Now

$$\begin{aligned}
\|F(x(t_2), y(t_2)) - F(x(t_1), y(t_1))\| &= \|(T_1y(t_2), T_2x(t_2)) - (T_1y(t_1), T_2x(t_1))\| \\
&= \|(T_1y(t_2) - T_1y(t_1), T_2x(t_2) - T_2x(t_1))\| \\
&= \|(T_1y(t_2) - T_1y(t_1))\| + \|T_2x(t_2) - T_2x(t_1)\|.
\end{aligned}$$

Then from (6) and (7) the set of functions  $FQ_r$  is equicontinuous.

**Lemma 3.** Let the assumptions (i)-(ii) be satisfied, then the operator  $F$  is continuous in  $Q_r$ .

**Proof.** Let  $(x_n, y_n) \in Q_r$  such that  $(x_n, y_n) \rightarrow (x_0, y_0) \in Q_r$ , then

$$F(x_n(t), y_n(t)) = (a_1(t) + \int_0^t f_1(t, s, y_n(\varphi_1(s)))ds, a_2(t) + \int_0^t f_2(t, s, x_n(\varphi_2(s)))ds)$$

and

$$\lim_{n \rightarrow \infty} T_1y_n(t) = a_1(t) + \lim_{n \rightarrow \infty} \int_0^t f_1(t, s, y_n(\varphi_1(s)))ds.$$

Then from our assumptions we have

$$f_1(t, s, y_n(\varphi_1(s))) \rightarrow f_1(t, s, y_0(\varphi_1(s)))$$

and

$$\begin{aligned}
|f_1(t, s, y_n(\varphi_1(s)))| &\leq m_1(t, s) + b_1|y_n(\varphi_1(s))| \\
&< m_1(t, s) + b_1r_1.
\end{aligned}$$

Applying Lebesgue dominated convergence theorem, we obtain

$$\lim_{n \rightarrow \infty} \int_0^t f_1(t, s, y_n(\varphi_1(s)))ds = \int_0^t f_1(t, s, y_0(\varphi_1(s)))ds.$$

and

$$\lim_{n \rightarrow \infty} T_1y_n(t) = a_1(t) + \int_0^t f_1(t, s, y_0(\varphi_1(s)))ds = T_1y_0(t).$$

By the same way we have

$$\lim_{n \rightarrow \infty} T_2 x_n(t) = a_2(t) + \int_0^t f_2(t, s, x_0(\varphi(s))) ds = T_2 x_0(t).$$

Now we can deduced that

$$F(x_n(t), y_n(t)) \rightarrow F(x_0(t), y_0(t))$$

which implies that the operator  $F$  is continuous.

Now for the existence of at least one solution of the coupled system of integral equations (1)-(2) we have the following theorem.

**Theorem 1.** Let the assumptions (i)-(ii) be satisfied. If

$$b_i T < 1, \quad i = 1, 2,$$

then the coupled system of the integral equations (1)-(2) has at least one solution.

**Proof.** From lemmas (1)-(3) we deduced that  $F$  satisfied the axioms of Schauder fixed point theorem, then the operator  $F$  has a fixed point  $(x, y) \in X$ , then the coupled system of integral equations (1)-(2) has at least one continuous solution.

### 3. URYSOHN FUNCTIONAL INTEGRAL EQUATIONS

Let

$$x = y, \quad f_1 = f_2, \quad \varphi_1 = \varphi_2 \quad \text{and} \quad a_1 = a_2,$$

then the coupled system (1)-(2) will be the Urysohn functional integral equation (3) and we have the following corollary.

**Corollary 1.** Let  $x = y, f_1 = f_2, \varphi_1 = \varphi_2$  and  $a_1 = a_2$  in Theorem 1. Let the assumptions of Theorem 1. be satisfied, then the integral equation (3) has at least one continuous solution  $x \in C[0, T]$ .

### 4. COUPLED SYSTEM OF HAMMERSTEIN FUNCTIONAL INTEGRAL EQUATIONS

Let

$$f_1(t, s, y) = k_1(t, s) g_1(s, y(\varphi_1(s))), \quad \text{and} \quad f_2(t, s, x) = k_2(t, s) g_2(s, x(\varphi_2(s))).$$

Then the coupled system (1)-(2) will be the coupled system of Hammerstein functional integral equations (5)-(6)

$$\begin{aligned} x(t) &= a_1(t) + \int_0^t k_1(t, s) g_1(s, y(\varphi_1(s))) ds, \quad t \in [0, T] \\ y(t) &= a_2(t) + \int_0^t k_2(t, s) g_2(s, x(\varphi_2(s))) ds, \quad t \in [0, T] \end{aligned}$$

Consider the following assumptions (iii)  $g_i : I \times R \rightarrow R$  are measurable in  $s \in I$  for all  $x \in R$  and continuous in  $x \in R$  for all  $s \in I$  and there exist two functions

$m_i^* \in L_1[0, T]$  and two positive constants  $b_i^* > 0, i = 1, 2$  such that

$$\begin{aligned} |g_1(t, y)| &\leq m_1^*(t) + b_1^*|x| \\ |g_2(t, x)| &\leq m_2^*(t) + b_2^*|y| \end{aligned}$$

(iv)  $k_i : I \times R \rightarrow R$  are continuous in  $t \in I$  for every  $s \in I$  and measurable in  $s \in I$  for all  $t \in I$  such that

$$\sup_{t \in I} \int_0^t |k_i(t, s)| |m_i^*(s)| ds \leq K_i, \quad t \in I.$$

Now we have the following corollary.

**Corollary 2.** Let the assumptions (iii)-(iv) be satisfied.

$$b_i T < 1, \quad i = 1, 2,$$

then the coupled system of integral equations (5)-(6) has at least one continuous solution.

Let

$$x = y, \quad g_1 = g_2, \quad \varphi_1 = \varphi_2, \quad a_1 = a_2 \quad \text{and} \quad k_1 = k_2,$$

then the coupled system (5)-(6) transformed to the Hammerstein functional integral equation

$$x(t) = a_1(t) + \int_0^t k(t, s) g_1(s, y(\varphi_1(s))) ds, \quad t \in I \quad (8)$$

and we have the following corollary

**Corollary 3.** Let  $x = y, g_1 = g_2, \varphi_1 = \varphi_2, a_1 = a_2$  and  $k_1 = k_2$ . If the assumption of Corollary 2 are satisfied then the functional integral equation (8) has at least one continuous solution.

## 5. MAXIMAL AND MINIMAL SOLUTIONS

**Definition 2.** Let  $q$  be a solution of (3), then  $q$  is said to be a maximal solution of (3) if for every solution of (3) satisfies the inequality  $x(t) < q(t), \quad t \in I$ .

A minimal solution  $s$  can be defined by similar way by reversing the above inequality i.e.  $x(t) > s(t), \quad t \in I$ .

The following lemma will be used later.

**Lemma 4.** Let  $f_1 \in L_1$  and  $x, y$  be continuous functions on  $I$  satisfying

$$\begin{aligned} x(t) &\leq a_1(t) + \int_0^t f_1(t, s, x(\varphi_1(s))) ds, \quad t \in I. \\ y(t) &\geq a_1(t) + \int_0^t f_1(t, s, y(\varphi_1(s))) ds, \quad t \in I, \end{aligned}$$

and one of them is strict. If  $f_1(t, s, x)$  is monotonic nondecreasing in  $x$ , then

$$x(t) < y(t), \quad t \in I. \quad (9)$$

**proof.** Let the conclusion (12) be false, then there exists  $t_1$  such that

$$x(t_1) = y(t_1), \quad t_1 > 0$$

and

$$x(t) < y(t), \quad 0 < t < t_1.$$

From the monotonicity of  $f_1$  in  $x$ , we get

$$\begin{aligned} x(t_1) &\leq a_1(t_1) + \int_0^{t_1} f_1(t_1, s, x(\varphi_1(s))) ds \\ &< a_1(t_1) + \int_0^{t_1} f_1(t_1, s, y(\varphi_1(s))) ds \\ &< y(t_1), \end{aligned}$$

which contradicts the fact  $x(t_1) = y(t_1)$ , then  $x(t) < y(t)$ ,  $t \in I$ .

For the existence of the maximal and minimal solutions we have the following theorem.

**Theorem 2.** Let the assumption of Theorem 1 be satisfied. If  $f$  is nondecreasing in  $x$  on  $I$ , then there exist maximal and minimal solutions of the integral equation (3).

**proof.** Firstly we shall prove the existence of the maximal solution of (3).

Let  $\epsilon > 0$  be given and consider the integral equation

$$x_\epsilon(t) = a_1(t) + \int_0^t f_1 \epsilon(t, s, x_\epsilon(\varphi_1(s))) ds, \quad t \in I. \quad (10)$$

where

$$f_1 \epsilon(t, s, x_\epsilon(\varphi(s))) = f_1(t, s, x_\epsilon(\varphi(s))) + \epsilon$$

Clearly the function  $f_1 \epsilon(t, s, x_\epsilon(\varphi(s)))$  satisfy assumptions (i) (or  $(i^*)$ ) of Theorem 1. and therefore equation (13) has at least one positive continuous solution  $x_\epsilon(t) \in C[0, T]$ .

Let  $\epsilon_1$  and  $\epsilon_2$  such that  $0 < \epsilon_2 < \epsilon_1 < \epsilon$ , then

$$\begin{aligned} x_{\epsilon_2}(t) &= a_1(t) + \int_0^t f_1 \epsilon_2(t, s, x_{\epsilon_2}(\varphi_1(s))) ds \\ &= a_1(t) + \int_0^t f_1(t, s, x_{\epsilon_2}(\varphi_1(s)) + \epsilon_2) ds \end{aligned} \quad (11)$$

and

$$\begin{aligned} x_{\epsilon_1}(t) &= a_1(t) + \int_0^t f_1 \epsilon_1(t, s, x_{\epsilon_1}(\varphi_1(s))) ds \\ &= a_1(t) + \int_0^t f_1(t, s, x_{\epsilon_1}(\varphi_1(s)) + \epsilon_1) ds \\ &> a_1(t) + \int_0^t f_1(t, s, x_{\epsilon_1}(\varphi_1(s)) + \epsilon_2) ds \end{aligned} \quad (12)$$

Applying Lemma 4. on (14) and (15), we have

$$x_{\epsilon_2}(t) < x_{\epsilon_1}(t), \quad t \in I.$$

As shown before the family of functions  $x_\epsilon(t)$  is equi-continuous and uniformly bounded. Hence by Arzela- Ascoli Theorem, there exists a decreasing sequence  $\epsilon_n$  such that  $\epsilon \rightarrow 0$  as  $n \rightarrow \infty$ , and

$$\lim_{n \rightarrow \infty} x_{\epsilon_n}(t)$$

exist uniformly in  $I$ .

Denote this limit by  $q$ , then from the continuity of the function  $f_{1\epsilon}(t, s, x_\epsilon(\varphi(s)))$  in the third argument, we get

$$q(t) = \lim_{n \rightarrow \infty} x_{\epsilon_n}(t) = a_1(t) + \int_0^t f_1(t, s, x(\varphi_1(s))) ds$$

which implies that  $q$  is a solution of (3).

Finally, we shall show that  $q$  is the maximal solution of (3). To do this, let  $x$  be any solution of (3). Then

$$\begin{aligned} x_\epsilon(t) &= a_1(t) + \int_0^t f_{1\epsilon}(t, s, x_\epsilon(\varphi_1(s))) ds \\ &= a_1(t) + \int_0^t (f_1(t, s, x_\epsilon(\varphi_1(s)) + \epsilon) ds \\ &> a_1(t) + \int_0^t f_1(t, s, x_\epsilon(\varphi_1(s))) ds \end{aligned}$$

Also applying Lemma 4. we have

$$x(t) = a_1(t) + \int_0^t f_1(t, s, x(\varphi_1(s))) ds \Rightarrow x(t) < x_\epsilon(t) \quad \text{for } t \in I,$$

from the uniqueness of the maximal solution, it is clear that  $x_\epsilon(t)$  tends to  $q(t)$  uniformly in  $t \in I$  as  $\epsilon \rightarrow 0$ .

By similar way as done above we set

$$f_{1\epsilon}(t, s, x_\epsilon(\varphi_1(s))) = f_1(t, s, x(\varphi_1(s))) - \epsilon$$

and prove the existence of the minimal solution.

For the maximal and minimal solutions of the fractional integral equation (11) we have the following theorem.

**Theorem 3.** If the assumption  $(ii^*)$  is satisfied. If  $g$  is nondecreasing in  $x$  on  $I$ , then there exist maximal and minimal solutions of the integral equation of fractional orders (11).

**proof.** The proof follows from the results of Theorem 2.

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