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# Bayesian and Non-Bayesian Estimation for the Shape Parameters of New Versions of Bivariate Inverse Weibull Distribution based on Progressive Type-II Censoring

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**Abstract:** The inverse Weibull (IW) distribution can be applied to a wide range of situations including applications in ecology, medicine, and reliability. Moreover, IW distribution gives a good fit to survival data such as the times to breakdown of an insulating fluid subject to the action of constant tension. In this paper, two new versions of the bivariate inverse Weibull distribution (BIW) are introduced depending on, the change of the shape parameters as members of a bivariate reversed hazard power parameter family of distributions, which are defined based on Marshal-Olkin and FGM copulas. MlE and Bayesian estimation methods are considered to estimate the unknown parameters for both BIW models based on progressive Type II censoring. Moreover, asymptotic, credible, and bootstrap confidence intervals for the unknown parameters are evaluated in both MLE and Bayesian Estimation for each BIW model. A numerical comparison will be considered for the two BIW models based on real and simulated data in the presence of progressive Type II censored samples.

**Keywords:** Bivariate inverse Weibull distribution; Maximum Likelihood Estimation; Prior distribution; Bayesian Estimation; progressive Type-II censoring; Bootstrap confidence Intervals
Mathematics Subject Classification: 60E05, 62N02, 37M05.

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## 1. Introduction

Studying bivariate distributions is of great importance. For example, in Economic studies; Study the relationship between (years of education and personal income, personal income and expenditure, and inflation and unemployment), Biological studies; Study ( blindness in the left and right eye, the age at death of parent and child in a genetic study, the relation between blood pressure and body weight for a patient and the failure time of the left and right kidney) in engineering studies; analyzing the lifetime of a twin-engine plane, also warranty polices based on failure time and warranty servicing time, as well as, different applications like Shock model, competing risks model, stress model, maintenance model and longevity model.

The inverse Weibull (IW) distribution can be applied to a wide range of situations including applications in ecology, medicine, and reliability. Moreover, IW distribution gives a good fit to survival data such as the times to breakdown of an insulating fluid subject to the action of constant tension. In this paper, two new versions of the bivariate inverse Weibull distribution (BIW) are introduced depending on, the change of the shape parameters as members of a bivariate reversed hazard power parameter family of distributions, which are defined based on Marshal-Olkin idea and FGM copula as will be shown in Section 2.

Failure times are usually not observed for all units. Those units for which the exact failure time is unknown are called censored data. This data contributes valuable information and should not be omitted from the analysis. There are different censoring schemes like type I, type II, random, hybrid, and progressive censoring. Type I, type II, random, and hybrid censoring do not allow any unit to be randomly removed during the experiment. Progressive censoring deals with this disadvantage by allowing units to be randomly removed from the experiment, which results in reducing the cost and time of the experiment.

Let $(X_1, X_2, \ldots, X_n)$ be a random sample from a probability distribution with absolutely continuous cdf $F$. These units are placed on a test at time $t = 0$. At this time of the $i^{th}$ failure, $R_i$, $1 \le i \le m$, many surviving units are randomly withdrawn from the experiment. Thus, if n failures are observed then $R_1 + R_2 + \cdots + R_m$ number of units that progressively censored; hence $n = m + R_1 + R_2 + \cdots + R_m$. The censoring scheme is denoted by the vector $R = (R_1, R_2, \ldots, R_m)$ and $X_{i:m:n}$, $i = 1, \ldots, m$. Is the $i^{th}$ failure time, and called progressive Type II censoring order statistic. The above steps can be extended in bivariate cases as follows:

Suppose that there are $n$ independent pairs of components $(X_{1i}, X_{2i})$, $i = 1 \ldots n$ under experiment, and each of them has bivariate lifetime distribution. During the experiment, immediately after the $i^{th}$ failure is observed, $R_i$ functioning items are randomly removed from the test. The m complete (ordered ) lifetimes thus observed are denoted by $X_{1i:m:n}$, $i = 1, \ldots, m$ with corresponding concomitants variables denoted by $X_{[2i:m:n]}$, $i = 1, \ldots, m$ and hence $X_{[2i:m:n]}$ is called concomitants of progressively type II censored order statistics. Further details on progressive censoring and concomitants can be found by Balakrishnan and Aggarwala [3] and Nagaraja and Abo-Eleneen [15]. Muhammed and Almetwally [14] deal with generalizing the likelihood function in the case of two dependent variables following FGM copula models based on progressive Type II censoring. Moreover, Muhammed (2023) generalizes the likelihood function based on progressive Type II censoring for the Marshal– Olkin bivariate models

The paper is organized as follows. In Section 2, BIW is introduced. Section 3 introduces point and interval estimation of the model parameters under progressive Type-II censoring. Bayesian Inference is presented in Section 4. In Sections 5 and 6 respectively, a simulation study and real data analysis are described. Finally, the paper is concluded in Section 7.

## 2. Models Assumptions and Data Description

### *2.1. Bivariate Inverse Weibull Model*

Recently, Muhammed [9] defined new two versions of the bivariate inverse Weibull distribution based on changing the shape parameters based on the FGM copula and Marshal-Olkin method, these new distributions belong to a semiparametric family of distributions called reversed hazard power parameter family of distributions. Moreover, this family has been modified by Muhammed [10]. we will discuss both Marshal Olkin bivariate inverse Weibull and FGM bivariate inverse Weibull in the following two sub-sections.

#### 2.1.1. Marshall Olkin Bivariate Inverse Weibull distribution

There are many methods for adding a shape parameter to a family of distributions based on the survival and failure functions that produced the so-called, proportional hazard family and proportional reversed hazard family, along the same line the hazard and reversed hazard functions can also be used to adding a power parameter (shape parameter) that are producing two important families of distributions namely, hazard power parameter and reversed hazard power parameter. Muhammed [10] introduced the bivariate extensions of these families based on an idea similar to that of Theorem 3.2 proposed by Marshall and Olkin (1967). These authors introduced a multivariate exponential distribution whose marginals have exponential distributions and proposed a bivariate Weibull distribution. And she introduced the MOBIW as a case study for these semi-parametric families, the proposed Marshall-Olkin Bivariate inverse Weibull (MOBIW) distribution is constructed from three independent univariate IW distributions using the maximization process. The author deduced that the MOBIW distribution is singular, and it can be used quite conveniently if there are ties in the data.

The joint cdf for the Marshall-Olkin Bivariate inverse Weibull (MOBIW) distribution is given as

$$F_{MOBIW}(x_1, x_2) = \exp\left\{-\left[\frac{\lambda}{x_1}\right]^{\alpha_1} - \left[\frac{\lambda}{x_2}\right]^{\alpha_2} - \left[\frac{\lambda}{x_3}\right]^{\alpha_3}\right\}$$

It can be rewritten in the following form

$$F_{MOBIW}(x_1, x_2) = \begin{cases} \exp\left\{-\left[\frac{\lambda}{x_1}\right]^{\alpha_1} - \left[\frac{\lambda}{x_2}\right]^{\alpha_2} - \left[\frac{\lambda}{x_1}\right]^{\alpha_3}\right\}, x_1 < x_2 \\ \exp\left\{-\left[\frac{\lambda}{x_1}\right]^{\alpha_1} - \left[\frac{\lambda}{x_2}\right]^{\alpha_2} - \left[\frac{\lambda}{x_2}\right]^{\alpha_3}\right\}, x_1 > x_2 \\ \exp\left\{-\left[\frac{\lambda}{x}\right]^{\alpha_1} - \left[\frac{\lambda}{x}\right]^{\alpha_2} - \left[\frac{\lambda}{x}\right]^{\alpha_3}\right\}, x_1 = x_2 = x \end{cases}$$

where $x_3 = \min(x_1, x_2)$

The corresponding joint pdf is given as

$$f_{MOBIW}(x_1, x_2) = \begin{cases} f_1(x_1, x_2), x_1 < x_2 \\ f_2(x_1, x_2), x_1 > x_2 \\ f_3(x), x_1 = x_2 = x \end{cases} \tag{2.1}$$

where

$$f_1(x_1, x_2) = \frac{\alpha_2^2 \lambda^2}{(x_1 x_2)^2}\left(\frac{\lambda}{x_2}\right)^{\alpha_2 - 1}\left\{\alpha_1\left(\frac{\lambda}{x_1}\right)^{\alpha_1 - 1} + \alpha_3\left(\frac{\lambda}{x_1}\right)^{\alpha_3 - 1}\right\}\exp\left\{-\left[\frac{\lambda}{x_1}\right]^{\alpha_1} - \left[\frac{\lambda}{x_2}\right]^{\alpha_2} - \left[\frac{\lambda}{x_1}\right]^{\alpha_3}\right\},$$

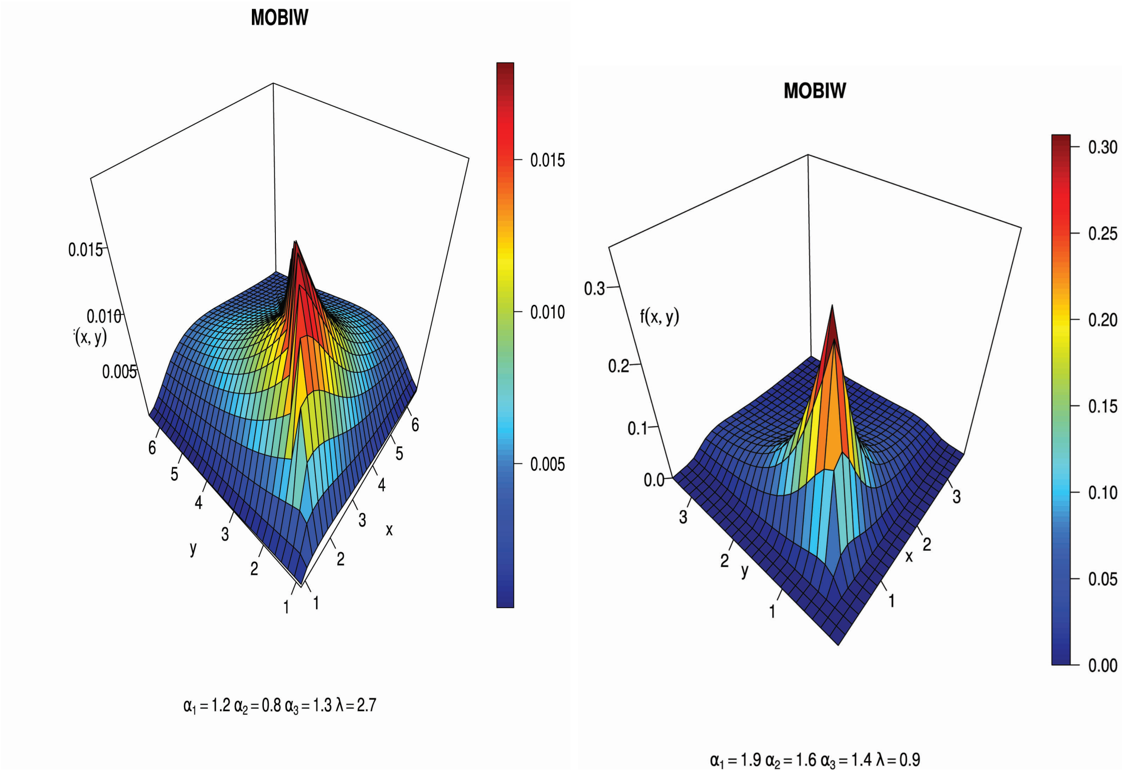


**Figure 1.** Joint pdf for the MOBIW distribution for some parameters values

$$f_2(x_1,x_2)=\frac{\alpha_1^2\lambda^2}{(x_1x_2)^2}\left(\frac{\lambda}{x_1}\right)^{\alpha_1-1}\left\{\alpha_2\left(\frac{\lambda}{x_2}\right)^{\alpha_2-1}+\alpha_3\left(\frac{\lambda}{x_2}\right)^{\alpha_3-1}\right\}\exp\left\{-\left[\frac{\lambda}{x_1}\right]^{\alpha_1}-\left[\frac{\lambda}{x_2}\right]^{\alpha_2}-\left[\frac{\lambda}{x_2}\right]^{\alpha_3}\right\},$$

$$f_3(x)=\frac{\alpha_3\lambda}{x^2}\left(\frac{\lambda}{x}\right)^{\alpha_3-1}\exp\left\{-\left[\frac{\lambda}{x}\right]^{\alpha_1}-\left[\frac{\lambda}{x}\right]^{\alpha_2}-\left[\frac{\lambda}{x}\right]^{\alpha_3}\right\}.$$

Figure 1 shows the joint pdf for the MOBIW distribution for some parameters values, and it is noted that the MOBIW has different shapes when the parameters change.

The joint reversed hazard function for MOBIW distribution can be written as

$$r_{MOBIW}(x_1,x_2)=\begin{cases} r_1(x_1,x_2), x_1<x_2\\ r_2(x_1,x_2), x_1>x_2\\ r_3(x), x_1=x_2=x\end{cases}$$

where

$$r_1(x_1,x_2)=\frac{\alpha_2^2\lambda^2}{(x_1x_2)^2}\left(\frac{\lambda}{x_2}\right)^{\alpha_2-1}\left\{\alpha_1\left(\frac{\lambda}{x_1}\right)^{\alpha_1-1}+\alpha_3\left(\frac{\lambda}{x_1}\right)^{\alpha_3-1}\right\},$$

$$r_2(x_1,x_2)=\frac{\alpha_1^2\lambda^2}{(x_1x_2)^2}\left(\frac{\lambda}{x_1}\right)^{\alpha_1-1}\left\{\alpha_2\left(\frac{\lambda}{x_2}\right)^{\alpha_2-1}+\alpha_3\left(\frac{\lambda}{x_2}\right)^{\alpha_3-1}\right\},$$

$$r_3(x)=\frac{\alpha_3\lambda}{x^2}\left(\frac{\lambda}{x}\right)^{\alpha_3-1}.$$

The marginal cdf and pdf of $X_1$ and $X_2$ are given respectively, as follows

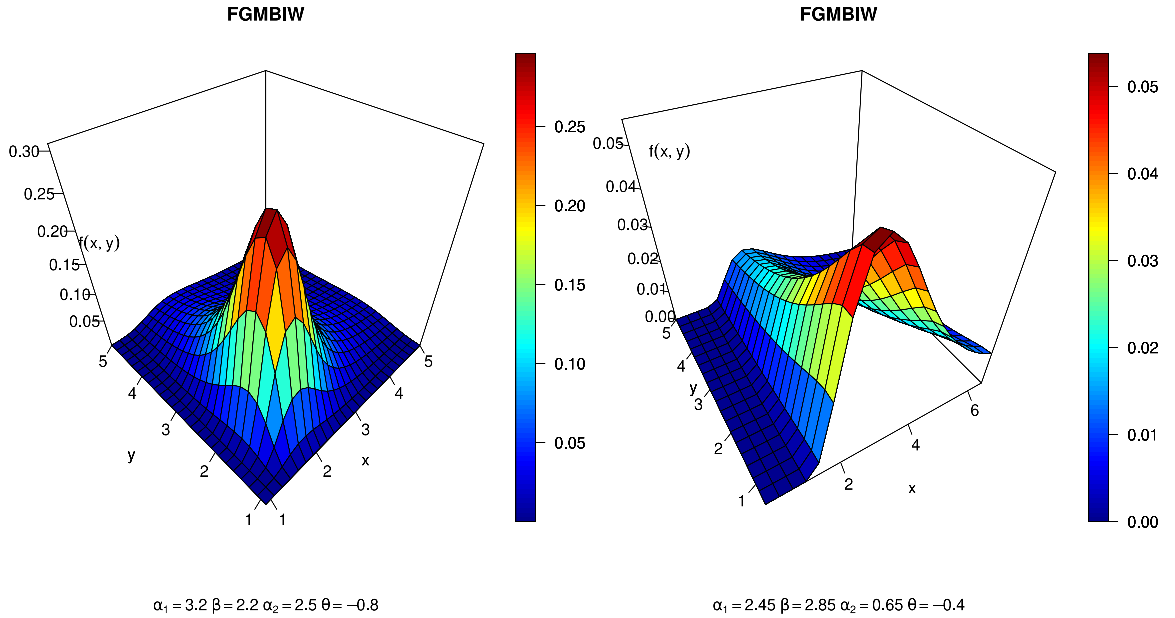


**Figure 2.** Joint pdf for FGMBIW distribution for different parameters values

$$F_{X_i}(x_i) = \exp\left\{-\left(\frac{\lambda}{x_i}\right)^{\alpha_i} - \left(\frac{\lambda}{x_i}\right)^{\alpha_3}\right\}, i = 1, 2.$$

and

$$f_{X_i}(x_i) = \left\{\alpha_i \frac{\lambda}{x_i^2}\left(\frac{\lambda}{x_i}\right)^{\alpha_i - 1} + \alpha_3 \frac{\lambda}{x_i^2}\left(\frac{\lambda}{x_i}\right)^{\alpha_3 - 1}\right\} \exp\left\{-\left(\frac{\lambda}{x_i}\right)^{\alpha_i} - \left(\frac{\lambda}{x_i}\right)^{\alpha_3}\right\}$$

### 2.1.2. FGM Bivariate Inverse Weibull Distribution

bivariate reversed hazard power parameter families of distributions are introduced in terms of reversed hazard rates, which are useful for the analysis of left censored data and in reliability and survival analysis. Unlike the univariate setup, more than one definition based on reversed hazard rates in the bivariate setup is introduced by Muhammed [9] based on different copulas, One of them is the FGM copula. the author considered the FGM bivariate inverse Weibull (FGMBIW) distribution as a case study.

The joint cdf for the FGM bivariate inverse Weibull (FGMBIW) is given as

$$F_{FGMBIW}(x_1, x_2) = \left\{1 + \theta(1 - e^{-\left[\frac{\lambda}{x_1}\right]^{\alpha_1}})(1 - e^{-\left[\frac{\lambda}{x_2}\right]^{\alpha_2}})\right\} e^{-\left[\frac{\lambda}{x_1}\right]^{\alpha_1} - \left[\frac{\lambda}{x_2}\right]^{\alpha_2}}$$

and the corresponding joint pdf is given as

$$f_{FGMBIW}(x_1, x_2) = \alpha_1 \alpha_2 \frac{\lambda}{x_1^2} \frac{\lambda}{x_2^2}\left\{1 + \theta(1 - 2e^{-\left[\frac{\lambda}{x_1}\right]^{\alpha_1}})(1 - 2e^{-\left[\frac{\lambda}{x_2}\right]^{\alpha_2}})\right\} e^{-\left[\frac{\lambda}{x_1}\right]^{\alpha_1} - \left[\frac{\lambda}{x_2}\right]^{\alpha_2}} \quad (2.2)$$

Figure 2 shows the joint pdf for the FGMBIW distribution with different shapes for some parameters values, The joint reversed hazard function for FGMBIW distribution can be written as

$$r_{FGMBIW}(x_1,x_2)=\alpha_1\alpha_2\frac{\lambda}{x_1^2}\frac{\lambda}{x_2^2}\,\frac{1+\theta(1-2e^{-\left[\frac{\lambda}{x_1}\right]^{\alpha_1}})(1-2e^{-\left[\frac{\lambda}{x_2}\right]^{\alpha_2}})}{1+\theta(1-e^{-\left[\frac{\lambda}{x_1}\right]^{\alpha_1}})(1-e^{-\left[\frac{\lambda}{x_2}\right]^{\alpha_2}})}.$$

Unlike those of MOBIW distribution, the marginal cdf and pdf of FGMBIW distribution follow IW distribution and are given respectively, as follows
$F_{X_i}(x_i)=\exp\left\{-\left(\frac{\lambda}{x_i}\right)^{\alpha_i}\right\}, i=1,2.$and $f_{X_i}(x_i)=\alpha_i\frac{\lambda}{x_i^2}\left(\frac{\lambda}{x_i}\right)^{\alpha_i-1}\exp\left\{-\left(\frac{\lambda}{x_i}\right)^{\alpha_i}\right\}.$

### 2.2. *Data Description*

The likelihood function based on progressive Type-II censoring for bivariate distributions was introduced first by Balakrishnan and Kim [2]. Along the same line, El-Sherpieny et al. (2019) have used the likelihood function based on a Progressive Type-II censoring for FGM bivariate Weibull distribution. Muhammed and Almetwally [14] and Muhammed [11] generalized the likelihood function for the bivariate Marshall Olkin families of distributions and applied it to the classical Marshall Olkin BIW distribution and Marshall Olkin bivariate Dagum distribution respectively. In this sub-section, we will apply the likelihood function for both the MOBIW and FGMBIW models.

Suppose that there are $n$ independent pairs of components$(X_{1i},X_{2i}),\ \ i=1,...n$ under experiment, and each of them has $FGMBIW(\alpha_1,\alpha_2,\lambda,\theta)$ model. Based on a Type-II progressive censoring scheme$(n,m,R_1,...,R_m)$ we have the following observations;

$$[(x_{11:m:n},x_{[21:m:n]}),(x_{12:m:n},x_{[22:m:n]}),...,(x_{1m:m:n},x_{[2m:m:n]})]$$

Where $x_{1i:m:n}$ be the$i^{th}$ order statistic of $X_1$and $X_{[2i:m:n]}$ be its concomitant of $X_2$

Then, according to Balakrishnan and Kim [2], the joint probability of $(X_{1i:m:n},X_{[2i:m:n]}),\ i=1...m$ is given by

$$L(\Theta)=C\prod_{i=1}^{m}f_{(X_{1i:m:n},X_{[2i:m:n]})}(x_{1i:m:n},x_{[2i:m:n]})[1-F_{X_1}(x_{1i:m:n})]^{R_i} \tag{2.3}$$

Suppose that there are $n$ independent pairs of components$(X_{1i},X_{2i}),\ i=1,...n$ under experiment, and each of them has $MOBIW(\alpha_1,\alpha_2,\alpha_3,\lambda)$ model. Then, according to Muhammed and Almetwally [14], the joint probability of $(X_{1i:m:n},X_{[2i:m:n]}),\ i=1...m$ is given by

$$L(\Theta)=C\prod_{i=1}^{m}[f_1(x_{1i:m:n},x_{[2i:m:n]})]^{\delta_{1i}}[f_2(x_{1i:m:n},x_{[2i:m:n]})]^{\delta_{2i}}[f_3(x_{1i:m:n},x_{[2i:m:n]})]^{\delta_{3i}}[1-F_{X_1}(x_{1i:m:n})]^{R_i} \tag{2.4}$$

Where

$$C=n(n-R_1-1)...(n-R_1-R_2-...-m+1),$$

$f_1(.),f_2(.),f_3(.)$ are as given in (2.1) and $F_{X_1}(.)$ is the survival function of $X_1$.

Also $\delta_{ji},\ \ j=1,2,3$ are event indicators such that $\delta_{1i}=\begin{cases}1, & X_{1i:m:n}<X_{[2i:m:n]}\\ 0, & otherwise\end{cases}$, $\delta_{2i}=\begin{cases}1, & X_{1i:m:n}>X_{[2i:m:n]}\\ 0, & otherwise\end{cases}$ and $\delta_{3i}=\begin{cases}1, & X_{1i:m:n}=X_{[2i:m:n]}\\ 0, & otherwise\end{cases}$

That produce $m_1 = \sum_{i=1}^{m} \delta_{1i}$ , $m_2 = \sum_{i=1}^{m} \delta_{2i}$and $m_3 = \sum_{i=1}^{m} \delta_{3i}$such that $m = m_1 + m_2 + m_3$. Throughout this paper, it is assumed that $n, m, R_1, ..., R_m$ are predetermined and fixed. It is noted that the complete case is obtained by setting $R_1 = ... = R_m = 0$ and $n = m$ , the Type II censoring case is obtained If $R_1 = ... = R_{m-1} = 0$ and $R_m = n - m$.

## 3. Maximum Likelihood Estimation

In this section, we discussed the maximum likelihood estimation of parameters for MOBIW and FGMBIW models based on progressively Type-II censored samples. Assume

$$(x_{11:m:n}, x_{[12:m:n]}) < (x_{12:m:n}, x_{[22:m:n]}) < ... < (x_{1m:m:n}, x_{[2m:m:n]})$$

denote progressively Type-II censored sample from $MOBIW(\alpha_1, \alpha_2, \alpha_3, \lambda)$ distribution whose pdf is given in (2.1), for simplicity assume $x_{1i} = x_{1i:m:n}$ and $x_{2i} = x_{[2i:m:n]}$.

The log-likelihood function $l(\Theta) = Log\ L(\Theta)$ is then given in this case as

$$\begin{aligned} l(\Theta) =& m_1 \log\alpha_2 + m_2 \log\alpha_1 + (2m - m_3)\log\lambda + \sum_{i=1}^{m} R_i \log(1 - e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1} - (\frac{\lambda}{x_{1i}})^{\alpha_3}}) \\ &+ (\alpha_2 - 1)\sum_{i=1}^{m} \delta_{1i}\log(\frac{\lambda}{x_{2i}}) + (\alpha_1 - 1)\sum_{i=1}^{m} \delta_{2i}\log(\frac{\lambda}{x_{1i}}) + (\alpha_3 - 1)\sum_{i=1}^{m} \delta_{3i}\log(\frac{\lambda}{x_i}) \\ &+ \sum_{i=1}^{m} \delta_{1i}\log[\alpha_1(\frac{\lambda}{x_{1i}})^{\alpha_1 - 1} + \alpha_3(\frac{\lambda}{x_{1i}})^{\alpha_3 - 1}] + \sum_{i=1}^{m} \delta_{2i}\log[\alpha_2(\frac{\lambda}{x_{2i}})^{\alpha_2 - 1} + \alpha_3(\frac{\lambda}{x_{2i}})^{\alpha_3 - 1} \\ &- \sum_{i=1}^{m} \delta_{1i}[(\frac{\lambda}{x_{1i}})^{\alpha_1} + (\frac{\lambda}{x_{2i}})^{\alpha_2} + (\frac{\lambda}{x_{1i}})^{\alpha_3}] - \sum_{i=1}^{m} \delta_{2i}[(\frac{\lambda}{x_{1i}})^{\alpha_1} + (\frac{\lambda}{x_{2i}})^{\alpha_2} + (\frac{\lambda}{x_{2i}})^{\alpha_3}] \\ &- \sum_{i=1}^{m} \delta_{3i}[(\frac{\lambda}{x_i})^{\alpha_1} + (\frac{\lambda}{x_i})^{\alpha_2} + (\frac{\lambda}{x_i})^{\alpha_3}]. \end{aligned} \tag{3.1}$$

Where $\Theta = (\alpha_1, \alpha_2, \alpha_3, \lambda)$.

The first derivatives of Equation (3.1) with respect to $\Theta = (\alpha_1, \alpha_2, \alpha_3, \lambda)$ are given respectively as follows

$$\begin{aligned} \frac{\partial l(\Theta)}{\partial \alpha_1} = \frac{m_2}{\alpha_1} &+ \sum_{i=1}^{m} R_i\, A(x_{1i}; \alpha_1, \alpha_3, \lambda) + \sum_{i=1}^{m} \delta_{2i}\log(\frac{\lambda}{x_{1i}}) + \sum_{i=1}^{m} \delta_{1i}\, \varphi(x_{1i}; \alpha_1, \alpha_3, \lambda) \\ &- \sum_{i=1}^{m} (\delta_{1i} + \delta_{2i})(\frac{\lambda}{x_{1i}})^{\alpha_1}\log(\frac{\lambda}{x_{1i}}) - \sum_{i=1}^{m} \delta_{3i}(\frac{\lambda}{x_i})^{\alpha_1}\log(\frac{\lambda}{x_i}), \end{aligned}$$

$$\begin{aligned} \frac{\partial l(\Theta)}{\partial \alpha_2} = \frac{m_1}{\alpha_2} + \sum_{i=1}^{m} \delta_{1i}\log(\frac{\lambda}{x_{2i}}) + \sum_{i=1}^{m} \delta_{2i}\, \varphi(x_{2i}; \alpha_2, \alpha_3, \lambda) - \sum_{i=1}^{m} (\delta_{1i} + \delta_{2i})(\frac{\lambda}{x_{2i}})^{\alpha_1}\log(\frac{\lambda}{x_{2i}}) \\ - \sum_{i=1}^{m} \delta_{3i}(\frac{\lambda}{x_i})^{\alpha_2}\log(\frac{\lambda}{x_i}), \end{aligned}$$

$$\frac{\partial l(\Theta)}{\partial \alpha_3} = \frac{m_3}{\alpha_3} + \sum_{i=1}^{m} R_i\ B(x_{1i};\alpha_1,\alpha_3,\lambda) + \sum_{i=1}^{m} \delta_{1i}\ \zeta(x_{1i};\alpha_1,\alpha_3,\lambda) + \sum_{i=1}^{m} \delta_{2i}\ \zeta(x_{2i};\alpha_2,\alpha_3,\lambda)$$
$$+ \sum_{i=1}^{m} \delta_{3i} \log(\frac{\lambda}{x_i}) - \sum_{i=1}^{m} \delta_{1i}(\frac{\lambda}{x_{1i}})^{\alpha_3} \log(\frac{\lambda}{x_{1i}}) - \sum_{i=1}^{m} \delta_{2i}(\frac{\lambda}{x_{2i}})^{\alpha_3} \log(\frac{\lambda}{x_{2i}}) - \sum_{i=1}^{m} \delta_{3i}(\frac{\lambda}{x_i})^{\alpha_3} \log(\frac{\lambda}{x_i}),$$

and

$$\frac{\partial l(\Theta)}{\partial \lambda} = \frac{2m - m_3}{\lambda} + \frac{(\alpha_2-1)m_1}{\lambda} + \frac{(\alpha_1-1)m_2}{\lambda} + \frac{(\alpha_3-1)m_3}{\lambda} + \sum_{i=1}^{m} R_i\ C(x_{1i};\alpha_1,\alpha_3,\lambda)$$
$$+ \sum_{i=1}^{m} \delta_{1i}\ \eta(x_{1i};\alpha_1,\alpha_3,\lambda) + \sum_{i=1}^{m} \delta_{2i}\ \eta(x_{2i};\alpha_2,\alpha_3,\lambda) - \sum_{i=1}^{m} \delta_{1i}\ D_1(x_{1i},x_{2i};\Theta)$$
$$- \sum_{i=1}^{m} \delta_{2i}\ D_2(x_{1i},x_{2i};\Theta) - \sum_{i=1}^{m} \delta_{3i}\ D_3(x_{1i},x_{2i};\Theta).$$

Where

$$\varphi(x;\alpha_1,\alpha_3,\lambda) = \frac{(\frac{\lambda}{x})^{\alpha_1-1} + \alpha_1(\frac{\lambda}{x})^{\alpha_1-1}\log(\frac{\lambda}{x})}{\alpha_1(\frac{\lambda}{x})^{\alpha_1-1} + \alpha_3(\frac{\lambda}{x})^{\alpha_3-1}}, \zeta(x;\alpha_1,\alpha_3,\lambda) = \frac{(\frac{\lambda}{x})^{\alpha_3-1} + \alpha_3(\frac{\lambda}{x})^{\alpha_3-1}\log(\frac{\lambda}{x})}{\alpha_1(\frac{\lambda}{x})^{\alpha_1-1} + \alpha_3(\frac{\lambda}{x})^{\alpha_3-1}},$$

$$\eta(x;\alpha_i,\alpha_3,\lambda) = \frac{\frac{\alpha_1(\alpha_1-1)}{x}(\frac{\lambda}{x})^{\alpha_1-2} + \frac{\alpha_3(\alpha_3-1)}{x}(\frac{\lambda}{x})^{\alpha_3-2}}{\alpha_1(\frac{\lambda}{x})^{\alpha_1-1} + \alpha_3(\frac{\lambda}{x})^{\alpha_3-1}}, A(x;\alpha_1,\alpha_3,\lambda) = \frac{(\frac{\lambda}{x_1})^{\alpha_1}\log(\frac{\lambda}{x_1})e^{-(\frac{\lambda}{x_1})^{\alpha_1}-(\frac{\lambda}{x_1})^{\alpha_3}}}{1 - e^{-(\frac{\lambda}{x_1})^{\alpha_1}-(\frac{\lambda}{x_1})^{\alpha_3}}}$$

$$B(x;\alpha_1,\alpha_3,\lambda) = \frac{(\frac{\lambda}{x})^{\alpha_3}\log(\frac{\lambda}{x})e^{-(\frac{\lambda}{x})^{\alpha_1}-(\frac{\lambda}{x})^{\alpha_3}}}{1 - e^{-(\frac{\lambda}{x})^{\alpha_1}-(\frac{\lambda}{x})^{\alpha_3}}}.$$

In the case of progressively Type-II censored sample drawn from $FGMBIW(\alpha_1,\alpha_2,\lambda,\theta)$ distribution whose pdf is given in (2.2) The log-likelihood function $l(\Omega) = Log\ L(\Omega)$,$\Omega = (\alpha_1,\alpha_2,\lambda,\theta)$ is then given in this case as

$$l(\Theta) = m_1 \log\alpha_2 + m_2 \log\alpha_1 + 2m\log\lambda + \sum_{i=1}^{m} R_i \log(1 - e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}}) - \sum_{i=1}^{m} (\frac{\lambda}{x_{1i}})^{\alpha_1} - (\frac{\lambda}{x_{2i}})^{\alpha_2}$$
$$+ \sum_{i=1}^{m} \log\{1 + \theta(1 - 2e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}})(1 - 2e^{-(\frac{\lambda}{x_{2i}})^{\alpha_2}})\}. \tag{3.2}$$

The first derivatives of Equation (3.2) with respect to $\Omega = (\alpha_1,\alpha_2,\lambda,\theta)$ are given respectively as follows

$$\frac{\partial l(\Omega)}{\partial \alpha_1} = \frac{m}{\alpha_1} + \sum_{i=1}^{m} R_i\ E_1(x_{1i};\alpha_1,\lambda) - \sum_{i=1}^{m} (\frac{\lambda}{x_{1i}})^{\alpha_1} \log(\frac{\lambda}{x_{1i}}) + \sum_{i=1}^{m} E_2(x_{1i},x_{2i};\Omega)\ ,$$

$$\frac{\partial l(\Omega)}{\partial \alpha_2} = \frac{m}{\alpha_2} - \sum_{i=1}^{m} (\frac{\lambda}{x_{2i}})^{\alpha_2} \log(\frac{\lambda}{x_{2i}}) + \sum_{i=1}^{m} E_3(x_{1i},x_{2i};\Omega)\ ,$$

$$\frac{\partial l(\Omega)}{\partial \theta} = \sum_{i=1}^{m} E_4(x_{1i},x_{2i};\alpha_1,\alpha_2,\lambda)\ ,$$

$$\frac{\partial l(\Omega)}{\partial \lambda} = \frac{2m}{\lambda} + \sum_{i=1}^{m} R_i\, E_6(x_{1i};\alpha_1,\lambda) - \sum_{i=1}^{m} \frac{\alpha_1}{x_{1i}}(\frac{\lambda}{x_{1i}})^{\alpha_1-1} + \sum_{i=1}^{m} \frac{\alpha_2}{x_{2i}}(\frac{\lambda}{x_{2i}})^{\alpha_2-1} + \sum_{i=1}^{m} E_5(x_{1i},x_{2i};\Omega)\ .$$

where

$$E_1(x_{1i};\alpha_1,\lambda) = \frac{[\frac{\lambda}{x_{1i}}]^{\alpha_1}\log[\frac{\lambda}{x_{1i}}]e^{-[\frac{\lambda}{x_{1i}}]^{\alpha_1}}}{1-e^{-[\frac{\lambda}{x_{1i}}]^{\alpha_1}}}, E_6(x_{1i};\alpha_1,\lambda) = \frac{(\frac{\lambda}{x_{1i}})^{\alpha_1-1}(\frac{\alpha_1}{x_{1i}})e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}}}{1-e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}}}$$

$$E_2(x_{1i},x_{2i};\Omega) = 2\theta(1-2e^{-(\frac{\lambda}{x_{2i}})^{\alpha_2}})(\frac{\lambda}{x_{1i}})^{\alpha_1}\log(\frac{\lambda}{x_{1i}})e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}} E_3(x_{1i},x_{2i};\Omega) = 2\theta(1-2e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}})(\frac{\lambda}{x_{2i}})^{\alpha_2}\log(\frac{\lambda}{x_{2i}})e^{-(\frac{\lambda}{x_{2i}})^{\alpha_2}}$$

$$E_4(x_{1i},x_{2i};\alpha_1,\alpha_2,\lambda) = (1-2e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}})(1-2e^{-(\frac{\lambda}{x_{2i}})^{\alpha_2}})$$

$$E_5(x_{1i},x_{2i};\Omega) = \theta(1-2e^{-(\frac{\lambda}{x_{2i}})^{\alpha_2}})[\frac{2\alpha_1}{x_{1i}}(\frac{\lambda}{x_{1i}})^{\alpha_1-1}e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}}] + \theta(1-2e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}})[\frac{2\alpha_2}{x_{2i}}(\frac{\lambda}{x_{2i}})^{\alpha_2-1}e^{-(\frac{\lambda}{x_{2i}})^{\alpha_2}}].$$

### *3.1. Asymptotic Confidence Intervals*

To set confidence intervals for the unknown parameters use the asymptotic normal distribution of the MLEs. Concerning the asymptotic variance–covariance matrix of the MLEs of the parameters, it can be approximated by inverting the Fisher information matrix $F$, where it consists of the negative derivatives of the natural logarithm of the likelihood function evaluated at the MLEs of the parameters.

Assume some regularity conditions to be satisfied, a $100(1-\gamma)$ % approximate confidence intervals for MOBIW distribution parameters are given respectively, as following

$\hat{\alpha}_1 \pm z_{\frac{\gamma}{2}}\sqrt{v_{11}},\ \hat{\alpha}_2 \pm z_{\frac{\gamma}{2}}\sqrt{v_{22}},\ \hat{\alpha}_3 \pm z_{\frac{\gamma}{2}}\sqrt{v_{33}}$ and $\hat{\lambda} \pm z_{\frac{\gamma}{2}}\sqrt{v_{44}}$

where $v_{11}$,$v_{22}$,$v_{33}$ and $v_{44}$are the elements on the main diagonal of the variance-covariance matrix of $MOBIW(\alpha_1,\alpha_2,\alpha_3,\lambda)$ and $z_{\frac{\gamma}{2}}$ is the percentile of the standard normal distribution with right tail $\frac{\gamma}{2}$.

and a $100(1-\gamma)$ % approximate confidence intervals for FGMBIW distribution parameters are given respectively, as following

$\hat{\alpha}_1 \pm z_{\frac{\gamma}{2}}\sqrt{V_{11}},\ \hat{\alpha}_2 \pm z_{\frac{\gamma}{2}}\sqrt{V_{22}},\ \hat{\lambda} \pm z_{\frac{\gamma}{2}}\sqrt{V_{33}}$and $\hat{\theta} \pm z_{\frac{\gamma}{2}}\sqrt{V_{44}}$

where $V_{11}$,$V_{22}$,$V_{33}$ and $V_{44}$are the elements on the main diagonal of the variance-covariance matrix for $FGMBIW(\alpha_1,\alpha_2,\lambda,\theta)$and $z_{\frac{\gamma}{2}}$ is the percentile of the standard normal distribution with right tail $\frac{\gamma}{2}$.

### *3.2. Bootstrap Confidence Intervals*

Two parametric bootstrap methods to construct confidence intervals for the unknown parameters will be used in this section. Which are percentile bootstrap confidence interval (B-PCI) and bootstrap-t confidence interval (B-TCI), the following steps are followed to obtain samples for both methods:

1. Obtain the MLEs$\hat{\Theta} = (\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\lambda})$ for the unknown parameters$\Theta = (\alpha_1,\alpha_2,\alpha_3,\lambda)$ based on the original progressively Type II censored sample $(x_{1i},x_{2i}) = (x_{11:m:n},x_{[12:m:n]}) < (x_{12:m:n},x_{[22:m:n]}) < ... < (x_{1m:m:n},x_{[2m:m:n]})$
2. Using $\hat{\Theta} = (\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\lambda})$ to generate a bootstrap sample$(x_{1i},x_{2i})^*, i = 1...m$ where$(x_{1i},x_{2i})^*$ *follow BIW*distribution
3. As in step 1 based on $(x_{1i},x_{2i})^*, i = 1...m$ computing the bootstrap sample estimates of $\hat{\Theta}$say, $\hat{\Theta}^*$.
4. Repeat the above steps 2 and 3 *N=1000* times, then we have $N$ estimate of $\Theta$
5. Order the bootstrap replications of $\hat{\Theta}^*$such that $\hat{\Theta}^*_1 < \hat{\Theta}^*_2 < ... < \hat{\Theta}^*_N$.

**Percentile Bootstrap Confidence Interval (B-PCI):**

Let

$G(\zeta) = P(\hat{\Theta}^* \leq \zeta)$ be cdf $\hat{\Theta}^*$ of Define$\hat{\Theta}^* = G^{-1}(\zeta)$ for given$\zeta$. The approximate bootstrap$100(1-\gamma)\%$ confidence interval of $\hat{\Theta}^*$ is given by$(\hat{\Theta}^*{}_{\frac{\gamma}{2}}, \hat{\Theta}^*{}_{1-\frac{\gamma}{2}})$.

**Bootstrap-t Confidence Interval (B-TCI):**

In step 3 get $\hat{\Theta}^*$and also calculate$var(\hat{\Theta}^*)$ using the observed Fisher information matrix.

Compute the statistic $T_j^* = \frac{\hat{\Theta}_j^* - \hat{\Theta}}{\sqrt{var(\hat{\Theta}_j^*)}}, \ j = 1...N.$

Arrange the bootstrap replications of $T^*$ such that $T_1^* < T_2^* < ... < T_N^*$.

Let $H(\zeta) = P(T^* \leq \zeta)$ be cdf of $T^*$. For a given $\zeta$define

$$\hat{\Theta}_{boot-t} = \hat{\Theta} + \sqrt{var(\hat{\Theta})}\ H^{-1}(\zeta).$$

The approximate $100(1-\gamma)\%$bootstrap confidence interval of $\hat{\Theta}$ will be

$$\left(\hat{\Theta}_{boot-t}\left(\frac{\gamma}{2}\right), \hat{\Theta}_{boot-t}\left(1-\frac{\gamma}{2}\right)\right).$$

## 4. Bayes Estimation

In this section, we consider the Bayesian analysis for the MOBIW and FGMBIW distributions under progressive Type II censoring. We obtained the Bayes estimators under the squared error loss function.

### *4.1. The Case of MOBIW Distribution*

Let $\Theta = (\alpha_1, \alpha_2, \alpha_3, \lambda)$have an independent gamma prior distribution. Assumed that

$$\alpha_1 = Gamma(b_1, d_1), \alpha_2 = Gamma(b_2, d_2),\ \alpha_3 = Gamma(b_3, d_3), \alpha_4 = Gamma(b_4, d_4).$$

Then, the joint prior density of $\Theta$ can be written as

$$\pi(\Theta) \propto \alpha_1^{b_1-1} e^{-d_1\alpha_1} \alpha_2^{b_2-1} e^{-d_2\alpha_2} \alpha_3^{b_3-1} e^{-d_3\alpha_3} \lambda^{b_4-1} e^{-d_4\lambda}; b_1, b_2, b_3, b_4, d_1, d_2, d_3, d_4 > 0 \tag{4.1}$$

The posterior likelihood can be represented to be proportional to the product of the likelihood function of MOBIW distribution and the joint prior densities given in Equation (4.1). That is,

$$\Pi(\Theta|x_{1i}, x_{2[i]}) \propto L(x_{1i}, x_{2[i]}, \Theta)\pi(\Theta)$$

Then, the joint posterior density of $\Theta$ is

$$\Pi_1(\Theta|x_{1i}, x_{2[i]}) = \lambda^{2m-m_3+b_4-1} \prod_{i=1}^{m} (\frac{\lambda}{x_{1i}})^{(\alpha_1-1)\delta_{2i}} (\frac{\lambda}{x_{2i}})^{(\alpha_2-1)\delta_{1i}} (\frac{\lambda}{x_i})^{(\alpha_3-1)\delta_{3i}} [1 - e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1} - (\frac{\lambda}{x_{1i}})^{\alpha_3}}]^{R_i}$$

$$.e^{-(d_1\alpha_1+d_2\alpha_2+d_3\alpha_3+d_4\lambda_1)} \alpha_3^{b_3-1} \prod_{i=1}^{m} [\alpha_1(\frac{\lambda}{x_{1i}})^{\alpha_1-1} + \alpha_3(\frac{\lambda}{x_{1i}})^{\alpha_3-1}]^{\delta_{1i}} [\alpha_2(\frac{\lambda}{x_{2i}})^{\alpha_1-1} + \alpha_3(\frac{\lambda}{x_{1i}})^{\alpha_3-1}]^{\delta_{2i}}$$

$$.\alpha_1^{m_2+b_1-1}\alpha_2^{m_1+b_2-1}e^{-\sum_{i=1}^{m}\delta_{1i}[(\frac{\lambda}{x_{1i}})^{\alpha_1}+(\frac{\lambda}{x_{2i}})^{\alpha_2}+(\frac{\lambda}{x_{1i}})^{\alpha_3}]+\sum_{i=1}^{m}\delta_{2i}[(\frac{\lambda}{x_{1i}})^{\alpha_1}+(\frac{\lambda}{x_{2i}})^{\alpha_2}+(\frac{\lambda}{x_{2i}})^{\alpha_3}]+\sum_{i=1}^{m}\delta_{3i}[(\frac{\lambda}{x_{1i}})^{\alpha_1}+(\frac{\lambda}{x_i})^{\alpha_2}+(\frac{\lambda}{x_i})^{\alpha_3}]} \quad (4.2)$$

We can use the squared error loss function (SELF) to obtain Bayesian estimators of the parameters$\Theta$, that are defined by $l(\tilde{\Theta},\Theta) = (\tilde{\Theta}-\Theta)^2$. The usual estimator of the parameters under the SELF is the posterior mean. Therefore, the Bayesian estimators of the parameters $\Theta$ under SELF denoted by $\tilde{\Theta}$ are given as follows:

$$\tilde{\alpha}_1 = \int_0^\infty \alpha_1 \int_0^\infty \int_0^\infty \int_0^\infty \Pi_1(\Theta|x_{1i}, x_{2[i]})\, d\alpha_2 d\alpha_3 d\lambda d\alpha_1,$$

$$\tilde{\alpha}_2 = \int_0^\infty \alpha_2 \int_0^\infty \int_0^\infty \int_0^\infty \Pi_1(\Theta|x_{1i}, x_{2[i]})\, d\alpha_1 d\alpha_3 d\lambda d\alpha_2,$$

$$\tilde{\alpha}_3 = \int_0^\infty \alpha_3 \int_0^\infty \int_0^\infty \int_0^\infty \Pi_1(\Theta|x_{1i}, x_{2[i]})\, d\alpha_1 d\alpha_2 d\lambda d\alpha_3,$$

and $\tilde{\lambda} = \int_0^\infty \lambda \int_0^\infty \int_0^\infty \int_0^\infty \Pi_1(\Theta|x_{1i}, x_{2[i]})\, d\alpha_1 d\alpha_2 d\alpha_3 d\lambda.$

These integrals are very hard to be solved analytically, so the MCMC approach will be used. For the MOBIW distribution based on progressive Type-II censoring, the full conditional posterior distributions of the parameters are given by

$$\Pi(\alpha_1|\alpha_2,\alpha_3,\lambda,x_{1i},x_{2[i]}) = e^{-d_1\alpha_1}\alpha_1^{m_2+b_1-1}\prod_{i=1}^{m}(\frac{\lambda}{x_{1i}})^{(\alpha_1-1)\delta_{2i}}[1-e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}-(\frac{\lambda}{x_{1i}})^{\alpha_3}}]^{R_i}\prod_{i=1}^{m}[\alpha_1(\frac{\lambda}{x_{1i}})^{\alpha_1-1}+\alpha_3(\frac{\lambda}{x_{1i}})^{\alpha_3-1}]^{\delta_{1i}}$$

$$.e^{-\sum_{i=1}^{m}\delta_{1i}[(\frac{\lambda}{x_{1i}})^{\alpha_1}+(\frac{\lambda}{x_{2i}})^{\alpha_2}+(\frac{\lambda}{x_{1i}})^{\alpha_3}]+\sum_{i=1}^{m}\delta_{2i}[(\frac{\lambda}{x_{1i}})^{\alpha_1}+(\frac{\lambda}{x_{2i}})^{\alpha_2}+(\frac{\lambda}{x_{2i}})^{\alpha_3}]+\sum_{i=1}^{m}\delta_{3i}[(\frac{\lambda}{x_i})^{\alpha_1}+(\frac{\lambda}{x_i})^{\alpha_2}+(\frac{\lambda}{x_i})^{\alpha_3}]},$$

$$\Pi(\alpha_2|\alpha_1,\alpha_3,\lambda,x_{1i},x_{2[i]}) = e^{-d_2\alpha_2}\alpha_2^{m_1+b_2-1}\prod_{i=1}^{m}(\frac{\lambda}{x_{2i}})^{(\alpha_2-1)\delta_{1i}}[\alpha_2(\frac{\lambda}{x_{2i}})^{\alpha_2-1}+\alpha_3(\frac{\lambda}{x_{2i}})^{\alpha_3-1}]^{\delta_{2i}}$$

$$.e^{-\sum_{i=1}^{m}\delta_{1i}[(\frac{\lambda}{x_{1i}})^{\alpha_1}+(\frac{\lambda}{x_{2i}})^{\alpha_2}+(\frac{\lambda}{x_{1i}})^{\alpha_3}]+\sum_{i=1}^{m}\delta_{2i}[(\frac{\lambda}{x_{1i}})^{\alpha_1}+(\frac{\lambda}{x_{2i}})^{\alpha_2}+(\frac{\lambda}{x_{2i}})^{\alpha_3}]+\sum_{i=1}^{m}\delta_{3i}[(\frac{\lambda}{x_i})^{\alpha_1}+(\frac{\lambda}{x_i})^{\alpha_2}+(\frac{\lambda}{x_i})^{\alpha_3}]},$$

$$\Pi(\alpha_3|\alpha_1,\alpha_2,\lambda,x_{1i},x_{2[i]}) = e^{-d_3\alpha_3}\alpha_3^{b_3-1}\prod_{i=1}^{m}(\frac{\lambda}{x_i})^{(\alpha_3-1)\delta_{3i}}[1-e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}-(\frac{\lambda}{x_{1i}})^{\alpha_3}}]^{R_i}$$

$$.\prod_{i=1}^{m}[\alpha_1(\frac{\lambda}{x_{1i}})^{\alpha_1-1}+\alpha_3(\frac{\lambda}{x_{1i}})^{\alpha_3-1}]^{\delta_{1i}}[\alpha_2(\frac{\lambda}{x_{2i}})^{\alpha_2-1}+\alpha_3(\frac{\lambda}{x_{2i}})^{\alpha_3-1}]^{\delta_{1i}}$$

$$.e^{-\sum_{i=1}^{m}\delta_{1i}[(\frac{\lambda}{x_{1i}})^{\alpha_1}+(\frac{\lambda}{x_{2i}})^{\alpha_2}+(\frac{\lambda}{x_{1i}})^{\alpha_3}]+\sum_{i=1}^{m}\delta_{2i}[(\frac{\lambda}{x_{1i}})^{\alpha_1}+(\frac{\lambda}{x_{2i}})^{\alpha_2}+(\frac{\lambda}{x_{2i}})^{\alpha_3}]+\sum_{i=1}^{m}\delta_{3i}[(\frac{\lambda}{x_i})^{\alpha_1}+(\frac{\lambda}{x_i})^{\alpha_2}+(\frac{\lambda}{x_i})^{\alpha_3}]},$$

and

$$\Pi(\lambda|\alpha_1,\alpha_2,\alpha_3,x_{1i},x_{2[i]}) = \lambda^{2m-m_3+b_4-1}e^{-d_4\lambda_1}\prod_{i=1}^{m}(\frac{\lambda}{x_{2i}})^{(\alpha_2-1)\delta_{1i}}(\frac{\lambda}{x_i})^{(\alpha_3-1)\delta_{3i}}[1-e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}-(\frac{\lambda}{x_{1i}})^{\alpha_3}}]^{R_i}$$

$$.\prod_{i=1}^{m}(\frac{\lambda}{x_{1i}})^{(\alpha_1-1)\delta_{2i}}[\alpha_1(\frac{\lambda}{x_{1i}})^{\alpha_1-1}+\alpha_3(\frac{\lambda}{x_{1i}})^{\alpha_3-1}]^{\delta_{1i}}[\alpha_2(\frac{\lambda}{x_{2i}})^{\alpha_2-1}+\alpha_3(\frac{\lambda}{x_{2i}})^{\alpha_3-1}]^{\delta_{2i}}$$

$$.e^{-\sum_{i=1}^{m}\delta_{1i}[(\frac{\lambda}{x_{1i}})^{\alpha_1}+(\frac{\lambda}{x_{2i}})^{\alpha_2}+(\frac{\lambda}{x_{1i}})^{\alpha_3}]+\sum_{i=1}^{m}\delta_{2i}[(\frac{\lambda}{x_{1i}})^{\alpha_1}+(\frac{\lambda}{x_{2i}})^{\alpha_2}+(\frac{\lambda}{x_{2i}})^{\alpha_3}]+\sum_{i=1}^{m}\delta_{3i}[(\frac{\lambda}{x_i})^{\alpha_1}+(\frac{\lambda}{x_i})^{\alpha_2}+(\frac{\lambda}{x_i})^{\alpha_3}]}.$$

*4.2. The Case of FGMBIW Distribution*

Let $\Psi = (\alpha_1, \alpha_2, \lambda, \theta)$have an independent priors distribution. Assumed that $\alpha_1 \sim Gamma(b_1, d_1)$, $\alpha_2 \sim Gamma(b_2, d_2)$, $\lambda \sim Gamma(b_3, d_3)$ and $\frac{1-\theta}{2} \sim Beta(b_4, d_4)$, Then, the joint prior density of $\Theta$can be written as

$$\pi(\Theta) \propto \alpha_1^{b_1-1} e^{-d_1\alpha_1} \alpha_2^{b_2-1} e^{-d_2\alpha_2} \lambda^{b_3-1} e^{-d_3\lambda} \frac{(1-\theta)^{b_4-1}}{(1+\theta)^{d_4+1}}; b_1, b_2, b_3, b_4, d_1, d_2, d_3, d_4 > 0. \tag{4.3}$$

The posterior likelihood can be represented to be proportional to the product of the likelihood function of FGMBIW distribution and the joint prior densities given in Equation (4.3). The joint posterior density of $\Psi$ is

$$\begin{aligned}\Pi_2(\Psi|x_{1i}, x_{2[i]}) =& \alpha_1^{m+b_1-1} \alpha_2^{m+b_2-1} \lambda^{2m+b_3-1} \frac{(1-\theta)^{b_4-1}}{(1+\theta)^{d_4+1}} e^{-(d_1\alpha_1+d_2\alpha_2+d_3\lambda)} e^{-\sum_{i=1}^{m}(\frac{\lambda}{x_{1i}})^{\alpha_1} - (\frac{\lambda}{x_{2i}})^{\alpha_2}} \\ &. \prod_{i=1}^{m} [1 - e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}}]^{R_i} (1 + \theta[1 - 2e^{(\frac{\lambda}{x_{1i}})^{\alpha_1}}][1 - 2e^{(\frac{\lambda}{x_{2i}})^{\alpha_2}}])\end{aligned} \tag{4.4}$$

We can use the SELF to obtain Bayesian estimators of the parameters$\Psi$. The usual estimator of the parameters under the SELF is the posterior mean. Therefore, the Bayesian estimators of the parameter$\Psi$ under SELF denoted by $\tilde{\Psi}$are given as follows:

$$\tilde{\alpha}_1 = \int_0^\infty \alpha_1 \int_0^\infty \int_{-1}^1 \int_0^\infty \Pi_2(\Psi|x_{1i}, x_{2[i]})\, d\alpha_2 d\lambda d\theta d\alpha_1,$$

$$\tilde{\alpha}_2 = \int_0^\infty \alpha_2 \int_0^\infty \int_{-1}^1 \int_0^\infty \Pi_2(\Psi|x_{1i}, x_{2[i]})\, d\alpha_1 d\lambda d\theta d\alpha_2,$$

$$\tilde{\lambda} = \int_0^\infty \lambda \int_0^\infty \int_{-1}^1 \int_0^\infty \Pi_2(\Psi|x_{1i}, x_{2[i]})\, d\alpha_1 d\alpha_2 d\theta d\lambda,$$

and $\tilde{\theta} = \int_0^\infty \theta \int_0^\infty \int_0^\infty \int_{-1}^1 \Pi_2(\Psi|x_{1i}, x_{2[i]})\, d\alpha_1 d\alpha_2 d\lambda d\theta.$

Again, these integrals are very hard to be solved analytically, so the MCMC approach will be used. For the FGMBIW distribution based on progressive Type-II censoring, the full conditional posterior distributions of the parameters are given by

$$\Pi(\alpha_1|\alpha_2, \lambda, \theta, x_{1i}, x_{2[i]}) = e^{-d_1\alpha_1} \alpha_1^{m+b_1-1} e^{-\sum_{i=1}^{m}(\frac{\lambda}{x_{1i}})^{\alpha_1}} \prod_{i=1}^{m} [1-e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}}]^{R_i} \prod_{i=1}^{m} (1+\theta[1-2e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}}][1-2e^{-(\frac{\lambda}{x_{2i}})^{\alpha_2}}]),$$

$$\Pi(\alpha_2|\alpha_1, \lambda, \theta, x_{1i}, x_{2[i]}) = e^{-d_2\alpha_2} \alpha_2^{m+b_2-1} e^{-\sum_{i=1}^{m}(\frac{\lambda}{x_{2i}})^{\alpha_2}} \prod_{i=1}^{m} (1+\theta[1-2e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}}][1-2e^{-(\frac{\lambda}{x_{2i}})^{\alpha_2}}]),$$

$$\begin{aligned}\Pi(\lambda|\alpha_1, \alpha_2, \theta, x_{1i}, x_{2[i]}) = e^{-d_3\alpha_3} \lambda^{2m+b_3-1} e^{-\sum_{i=1}^{m}(\frac{\lambda}{x_{1i}})^{\alpha_1} = \sum_{i=1}^{m}(\frac{\lambda}{x_{2i}})^{\alpha_2}} \prod_{i=1}^{m} [1-e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}}]^{R_i} \\ . \prod_{i=1}^{m} (1+\theta[1-2e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}}][1-2e^{-(\frac{\lambda}{x_{2i}})^{\alpha_2}}]),\end{aligned} .$$

and

$$\Pi(\theta|\alpha_1, \alpha_2, \lambda, x_{1i}, x_{2[i]}) = \frac{(1-\theta)^{b_4-1}}{(1+\theta)^{d_4+1}} \prod_{i=1}^{m} [1-e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}}]^{R_i} (1+\theta[1-2e^{-(\frac{\lambda}{x_{1i}})^{\alpha_1}}][1-2e^{-(\frac{\lambda}{x_{2i}})^{\alpha_2}}]).$$

### 4.3. MCMC Techniques

The MCMC approach will be used. because the Bayesian integrals are very hard to solve analytically for both the MOBIW and FGMBIW distributions based on progressive Type-II censoring, by using the full conditional posterior distributions of the parameters for both MOBIW and FGMBIW distributions that are given in the two previous subsections.

**Hyper-Parameter Elicitation**: The elicitation of the hyper-parameters will rely on the informative priors. These informative priors will be obtained from the maximum likelihood estimates for $\Theta$ by equating the mean and variance of $\widehat{\Theta}$ with the mean and variance of the considered prior distribution.

An important sub-class of the MCMC techniques is Gibbs sampling and more general Metropolis within Gibbs samplers. Metropolis et al. [7] and Hastings [4] were first introduced this algorithm. In our simulation study presented in Section 6, the Markov chain Monte Carlo (MCMC) procedure is used to generate the full conditional posterior distributions of $\Theta$. We set the number of periods in the MCMC process to be N = 10,000. The Metropolis-Hastings (MH) algorithm generates a sequence of draws from BIW distribution based on progressive Type-II censoring as follows:

1. Start with any initial values $\Theta_l^{\langle 0\rangle}; l = 1, ...4$, satisfying $\pi(\Theta_l^{\langle 0\rangle}) > 0$.
2. Using the initial value, sample a candidate point $\Theta^*$ from the proposal$q(\Theta^*)$.
3. For$t = 0 \ldots N,$ given the candidate point $\Theta^*$, calculate the acceptance probability

$$A_l = \min\left(1, \frac{L(\Theta|x_{1i}, x_{2[i]})\pi(\Theta_l^*)q(\Theta_l)}{L_1(\Theta|x_{1i}, x_{2[i]})\pi(\Theta_l)q(\Theta_l^*)}\right)$$

4. Draw a value of $u$ from the uniform (0,1) distribution, $\Theta_l^{\langle t+1\rangle} = \begin{cases} \Theta_l^* & if\, u \le A_l \\ \Theta_l^{\langle t\rangle} & otherwise \end{cases}$.
5. Repeat steps 2 - 5 $(t + 1)$ times until we get $N$ draws.
6. The Bayes estimate of $\Theta_l$, with respect to the squared error loss function, is$\sum_{t=1}^{N} \frac{\Theta_l^{\langle t-1\rangle}}{N}$ .

Repeat this steps $l$ time to get Bayesian estimate of $\Theta_l$.

According to Chen and Shoa [6], we obtain Bayes credible intervals of the parameters $\Theta_l$ as follows:

1. Arrange$\Theta_l^{\langle j\rangle}$ and $\Theta_l^{[1]}, \Theta_l^{[2]}, ..., \Theta_l^{[L]}$ where $L$ is the length of simulation generated.
2. The $100(1-\gamma)\%$ symmetric credible intervals of$\Theta$ become $\left(\tilde{\Theta}^{[L\frac{\gamma}{2}]}, \tilde{\Theta}^{[L(1-\frac{\gamma}{2})]}\right)$.

## 5. Simulation Study

Monte Carlo simulations are used in this section to compare the output of the various progressive Type-II schemes. The following censoring schemes are considered in our research.

1. I: $R_1 = R_2 = ... = R_{m-1} = 0$ and $R_m = n - m$.
2. II: $R_1 = n - m$ and $R_2 = R_3 = ... = R_{m-1} = 0$.
3. III:$R_1 = R_2 = ... = R_m = 0$.

It is noted that, Scheme I is a special case of progressive Type-II censoring which is named a Type-II censoring scheme. Scheme III is a special case of progressive Type-II censoring which is named as complete scheme.

We censored the sample for$X_1$ and $X_2$ is communicant of $X_1$. After generating $n$ sample of bivariate IW $(x_{1i}, x_{2[i]})$based on Marshall-Olkin method and FGM copula which are discussed in several articles as Muhammed [8], Almetwally and Muhammed [1], Chesneau [12, 13], we generate a bivariate censored sample by using these steps:

The first step is to sort$x_{1i}$, where $x_{11:n} < x_{12:n} < ... < x_{1n:n}$, and $x_{[2i]}$ it is communicant, that can be denoted as$(x_{1i:n}, x_{[2i:n]})$.

The second step, determine the scheme:

In Scheme I, the bivariate progressive Type-II censored sample $(X_{1i:m:n}, X_{[2i:m:n]})$, $i = 1...m$ is obtained by$[(x_{11:m:n}, x_{[21:m:n]}), (x_{12:m:n}, x_{[22:m:n]}), ..., (x_{1m:m:n}, x_{[2m:m:n]})]$

In scheme II, the bivariate progressive Type-II censored sample$(x_{1i:m:n}, x_{[2i:m:n]})$ is obtained by selecting the first failure $(x_{11:m:n}, x_{[21:m:n]})$and removing $n - m$ randomly from surviving samples $n - 1$.

In scheme III, the bivariate progressive Type-II censored sample $(x_{1i:m:n}, x_{[2i:m:n]})$ is obtained by selecting the $n$ sample $(x_{1i:m:n}, x_{[2i:m:n]})$ as a complete sample.

the R software is used to estimate the parameters of a bivariate IW distribution based on progressive Type-II censored schemes.

The actual parameters are chosen as the following cases for the generated random variables:

In MOBIW, Cases 1 and 2 are considered as follows:

Case 1: $(\alpha_1 = 1.9, \alpha_2 = 1.6, \alpha_3 = 1.4, \lambda = 2.2\ )$

Case 2: $(\alpha_1 = 0.7, \alpha_2 = 0.8, \alpha_3 = 1.3, \lambda = 0.7)$

In FGMBIW, Cases 3 and 4 are considered as follows:

Case 3: $(\alpha_1 = 1.6, \alpha_2 = 1.2, \lambda = 1.4, \theta = 0.5)$

Case 4: $(\alpha_1 = 3.2, \alpha_2 = 2.5, \lambda = 2.2, \theta = 0.8)$

For different sample sizes $n = 30$, 70, and 100. For different censored sample size $m = 24$, and 27 when n=30, $m = 50$, and 60 when n=70, and $m = 80$, and 90 when n=100. The following formulas are used to measure the Bias, MSE, and duration of confidence interval (L.CI) for each model:

$$Bias = (\hat{\Theta} - \Theta), \tag{5.1}$$

$$MSE = Mean(\hat{\Theta} - \Theta)^2. \tag{5.2}$$

and

$$L.CI = Upper.CI - Lower.CI \tag{5.3}$$

The number of repeated samples was limited to 10,000. From Tables 1-6 (where Tables 1-6 contain a result for MOBIW distribution and Tables 1-6 contain a result for FGMBIW distribution), the following remarks can be deduced:

1. The MSE, Bias, and CI length of the considered parameters decreases as the sample size grows.
2. The values of the Bias, MSE, and Length of CI for the parameters of the BIW distribution decrease as the number of stages (m) increases.
3. Scheme II, with fixed sample size values, in the majority, provides a more reliable outcome than other schemes in terms of reducing MSE, Bias, and CI duration.
4. For the majority of BIW distribution parameters, Bayesian estimates are more efficient than MLE.

5. The Bootstrap CIs method is more robust than the conventional ACI method for estimating intervals for BIW distribution parameters.
6. For most factors, the BT CI outperforms the BP CI.

## 6. Data Analysis

Infection recurrence period kidney data were collected for 30 patients by McGilchrist and Aisbett [5]. This data reflects the infection recurrence period for kidney patients, with $X_1$: referring to the first recurrence time and being as follows: 8, 23, 22, 447, 30, 24, 7, 511, 53, 15, 7, 141, 96, 149, 536, 17, 185, 292, 22, 15, 152, 402, 13, 39, 12, 113, 132, 34, 2, 130. $X_2$: to second recurrence time and being as follows: 6, 13, 28, 318, 12, 245, 9, 30, 196, 154, 333, 8, 38, 70, 25, 4, 117, 114, 159, 108, 362, 24, 66, 46, 40, 201, 156, 30, 25, 26. For the IW marginal distributions, Table 7 shows the MLE, standard error (SE), Kolmogorov-Smirnov distance (KSD), p-value, and different measures as Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC). The fitted pdf and approximate cdf are shown in Figures 3 and 4 and help our findings (Kolmogorov-Smirnov-test) in Table 7.

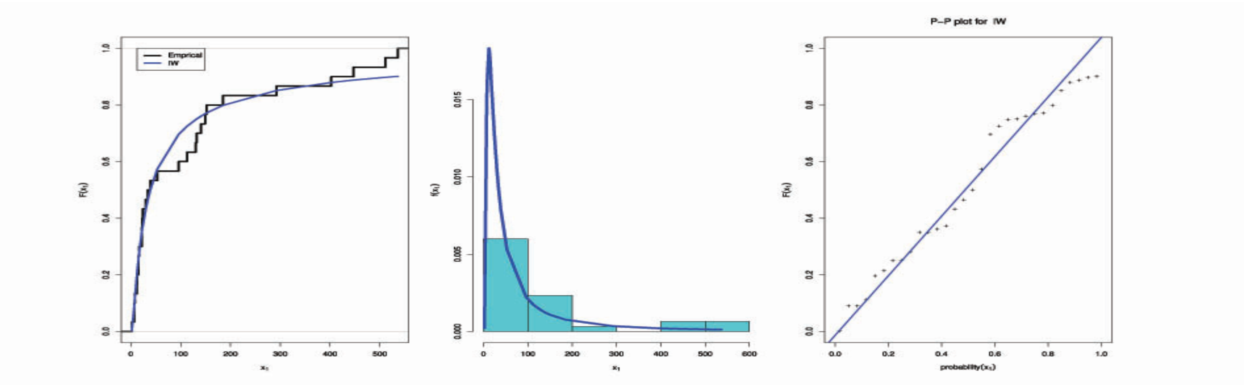


**Figure 3.** cdf, pdf and PP-Plot for IW distribution for first recurrence time of Infection period of kidney data

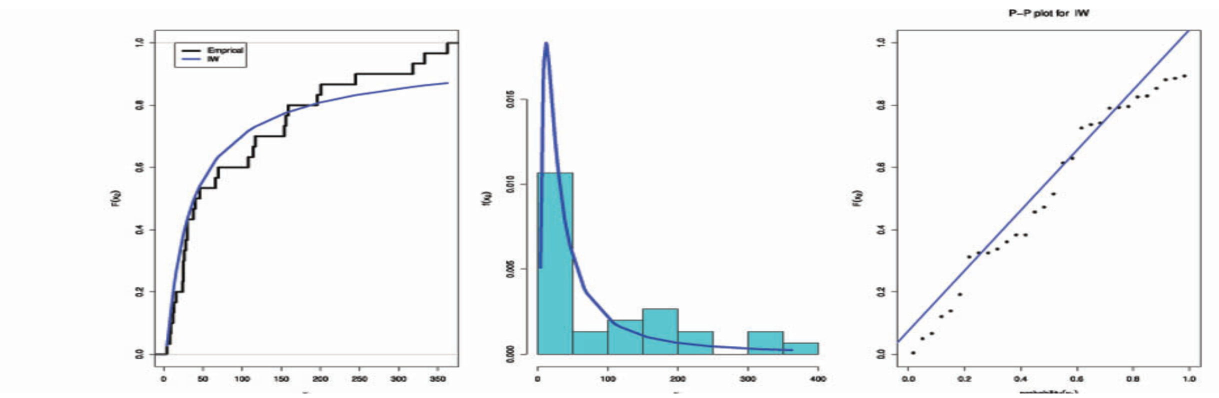


**Figure 4.** cdf, pdf and PP-Plot for IW distribution for second recurrence time of Infection period of kidney data

**Table 1.** Bias, MSE, and length of CI for parameters of MOBIW distribution: Case 1 when n=30, and 70

| Case 1 | | | | MLE | | | | | Bayesian | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| n | R | m | | Bias | MSE | L.CI | BP | BT | Bias | MSE | L.CI | BP | BT |
| 30 | I | 24 | $\alpha_1$ | 0.2271 | 0.7285 | 3.2268 | 1.381 | 1.3845 | 0.2042 | 0.5295 | 2.4245 | 0.6226 | 0.6151 |
| | | | $\alpha_2$ | 0.1406 | 0.2141 | 1.7292 | 0.2439 | 0.2444 | 0.1533 | 0.21 | 1.5865 | 0.3606 | 0.3594 |
| | | | $\alpha_3$ | 0.1646 | 0.3032 | 2.061 | 0.2838 | 0.2908 | 0.1809 | 0.2689 | 1.7564 | 0.46 | 0.4635 |
| | | | $\lambda$ | 0.0286 | 0.0461 | 0.8346 | 0.1622 | 0.1671 | 0.0269 | 0.0453 | 0.8045 | 0.1881 | 0.1888 |
| | | 27 | $\alpha_1$ | 0.1463 | 0.4555 | 2.5841 | 0.3739 | 0.3783 | 0.1519 | 0.3996 | 2.1466 | 0.3765 | 0.3776 |
| | | | $\alpha_2$ | 0.1057 | 0.1893 | 1.6554 | 0.3501 | 0.3512 | 0.1239 | 0.1863 | 1.4735 | 0.227 | 0.2302 |
| | | | $\alpha_3$ | 0.1508 | 0.2272 | 1.7736 | 0.3437 | 0.3455 | 0.1585 | 0.2111 | 1.5433 | 0.3648 | 0.359 |
| | | | $\lambda$ | 0.0336 | 0.0421 | 0.7936 | 0.1772 | 0.1765 | 0.0317 | 0.0417 | 0.7919 | 0.1905 | 0.1968 |
| | II | 24 | $\alpha_1$ | 0.2012 | 0.3924 | 2.3266 | 0.3134 | 0.3047 | 0.2073 | 0.3543 | 1.8245 | 0.2585 | 0.2584 |
| | | | $\alpha_2$ | 0.1307 | 0.2439 | 1.868 | 0.2573 | 0.2675 | 0.1497 | 0.2447 | 1.6909 | 0.4242 | 0.4226 |
| | | | $\alpha_3$ | 0.1097 | 0.1567 | 1.4915 | 0.27 | 0.2619 | 0.1213 | 0.1494 | 1.2726 | 0.3321 | 0.3354 |
| | | | $\lambda$ | 0.0385 | 0.0551 | 0.9079 | 0.1526 | 0.1471 | 0.0406 | 0.055 | 0.8941 | 0.1542 | 0.1623 |
| | | 27 | $\alpha_1$ | 0.1483 | 0.2983 | 2.0616 | 0.3223 | 0.3204 | 0.1623 | 0.2922 | 1.865 | 0.3922 | 0.392 |
| | | | $\alpha_2$ | 0.1107 | 0.184 | 1.6255 | 0.2575 | 0.2586 | 0.1288 | 0.1889 | 1.4927 | 0.4769 | 0.4682 |
| | | | $\alpha_3$ | 0.0626 | 0.1137 | 1.2995 | 0.2871 | 0.2867 | 0.0734 | 0.1085 | 1.186 | 0.2176 | 0.2192 |
| | | | $\lambda$ | 0.0297 | 0.0479 | 0.8509 | 0.1308 | 0.133 | 0.0325 | 0.0481 | 0.8202 | 0.1491 | 0.1509 |
| | III | 30 | $\alpha_1$ | 0.1554 | 0.3231 | 2.1444 | 0.3047 | 0.3063 | 0.166 | 0.2826 | 1.7467 | 0.3817 | 0.3704 |
| | | | $\alpha_2$ | 0.1212 | 0.1935 | 1.6584 | 0.4256 | 0.4183 | 0.1341 | 0.1927 | 1.4435 | 0.2576 | 0.2588 |
| | | | $\alpha_3$ | 0.0778 | 0.1143 | 1.2906 | 0.3132 | 0.3068 | 0.0874 | 0.1122 | 1.1895 | 0.1614 | 0.1612 |
| | | | $\lambda$ | 0.0418 | 0.0441 | 0.807 | 0.1726 | 0.1736 | 0.0429 | 0.0441 | 0.7828 | 0.1263 | 0.1259 |
| 70 | I | 50 | $\alpha_1$ | 0.0605 | 0.2805 | 2.0634 | 0.2703 | 0.276 | 0.063 | 0.243 | 1.8849 | 0.3224 | 0.3217 |
| | | | $\alpha_2$ | 0.0435 | 0.0732 | 1.0471 | 0.1517 | 0.1531 | 0.0521 | 0.0728 | 0.9985 | 0.1463 | 0.1528 |
| | | | $\alpha_3$ | 0.1335 | 0.217 | 1.7504 | 0.3048 | 0.3022 | 0.1477 | 0.1972 | 1.5215 | 0.3011 | 0.2966 |
| | | | $\lambda$ | 0.0115 | 0.0185 | 0.5311 | 0.0743 | 0.0735 | 0.0078 | 0.0181 | 0.5227 | 0.0624 | 0.0627 |
| | | 60 | $\alpha_1$ | 0.0448 | 0.1583 | 1.5503 | 0.2039 | 0.2034 | 0.0528 | 0.1469 | 1.4233 | 0.1738 | 0.1739 |
| | | | $\alpha_2$ | 0.0405 | 0.0641 | 0.9799 | 0.1352 | 0.1328 | 0.048 | 0.0647 | 0.9406 | 0.1444 | 0.142 |
| | | | $\alpha_3$ | 0.0792 | 0.0953 | 1.17 | 0.1627 | 0.172 | 0.0895 | 0.0871 | 1.0897 | 0.1274 | 0.1293 |
| | | | $\lambda$ | 0.0099 | 0.0187 | 0.5352 | 0.081 | 0.0858 | 0.0088 | 0.0186 | 0.5223 | 0.0637 | 0.0634 |
| | II | 50 | $\alpha_1$ | 0.0638 | 0.1149 | 1.3056 | 0.1691 | 0.1668 | 0.0721 | 0.1142 | 1.2094 | 0.2376 | 0.2331 |
| | | | $\alpha_2$ | 0.0546 | 0.0828 | 1.1079 | 0.152 | 0.1507 | 0.0669 | 0.0805 | 1.0644 | 0.1299 | 0.1298 |
| | | | $\alpha_3$ | 0.0346 | 0.0582 | 0.9367 | 0.1431 | 0.1462 | 0.0411 | 0.0568 | 0.9016 | 0.1428 | 0.1444 |
| | | | $\lambda$ | 0.0166 | 0.0234 | 0.597 | 0.092 | 0.0921 | 0.0183 | 0.0234 | 0.5864 | 0.0937 | 0.095 |
| | | 60 | $\alpha_1$ | 0.0844 | 0.1075 | 1.2428 | 0.1455 | 0.1445 | 0.0883 | 0.1043 | 1.2006 | 0.1675 | 0.1665 |
| | | | $\alpha_2$ | 0.0503 | 0.0681 | 1.0042 | 0.1436 | 0.1419 | 0.0613 | 0.067 | 0.9697 | 0.1343 | 0.1417 |
| | | | $\alpha_3$ | 0.0398 | 0.0481 | 0.8461 | 0.1095 | 0.1116 | 0.0469 | 0.0469 | 0.8252 | 0.1049 | 0.1044 |
| | | | $\lambda$ | 0.0091 | 0.0182 | 0.5278 | 0.0608 | 0.0623 | 0.0107 | 0.0178 | 0.5184 | 0.0635 | 0.0651 |
| | III | 70 | $\alpha_1$ | 0.0598 | 0.081 | 1.091 | 0.1188 | 0.1138 | 0.0637 | 0.0801 | 1.0894 | 0.1356 | 0.1332 |
| | | | $\alpha_2$ | 0.034 | 0.0522 | 0.8865 | 0.1119 | 0.1127 | 0.0398 | 0.0518 | 0.8468 | 0.1156 | 0.1143 |
| | | | $\alpha_3$ | 0.0394 | 0.0424 | 0.7931 | 0.0899 | 0.0923 | 0.0455 | 0.0413 | 0.7655 | 0.0726 | 0.0719 |
| | | | $\lambda$ | 0.0007 | 0.0177 | 0.5225 | 0.0641 | 0.0669 | 0.0016 | 0.0176 | 0.525 | 0.0586 | 0.0569 |

**Table 2.** Bias, MSE, and length of CI for parameters of MOBIW distribution: Case 2 when n=30 and 70

| Case 2 | | | | MLE | | | | | Bayesian | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| n | R | m | | Bias | MSE | L.CI | BP | BT | Bias | MSE | L.CI | BP | BT |
| 30 | I | 24 | $\alpha_1$ | 0.1335 | 0.2923 | 2.0546 | 0.3466 | 0.3459 | 0.142 | 0.2299 | 1.615 | 0.5979 | 0.5713 |
| | | | $\alpha_2$ | 0.0503 | 0.0487 | 0.8427 | 0.1578 | 0.1625 | 0.0653 | 0.0453 | 0.7878 | 0.137 | 0.1359 |
| | | | $\alpha_3$ | 0.1487 | 0.2463 | 1.8571 | 0.3935 | 0.394 | 0.1407 | 0.2134 | 1.5624 | 0.2778 | 0.2809 |
| | | | $\lambda$ | 0.0152 | 0.0095 | 0.3771 | 0.0764 | 0.0757 | 0.0165 | 0.0094 | 0.3681 | 0.0624 | 0.0608 |
| | | 27 | $\alpha_1$ | 0.1262 | 0.2017 | 1.6902 | 0.3577 | 0.3603 | 0.1387 | 0.1836 | 1.398 | 0.3119 | 0.3095 |
| | | | $\alpha_2$ | 0.0522 | 0.0447 | 0.8032 | 0.136 | 0.1318 | 0.063 | 0.0405 | 0.7574 | 0.1311 | 0.1316 |
| | | | $\alpha_3$ | 0.1162 | 0.1613 | 1.508 | 0.3165 | 0.3104 | 0.1121 | 0.1542 | 1.3381 | 0.235 | 0.2384 |
| | | | $\lambda$ | 0.0162 | 0.009 | 0.3674 | 0.0799 | 0.0817 | 0.018 | 0.0089 | 0.3557 | 0.0647 | 0.064 |
| | II | 24 | $\alpha_1$ | 0.1054 | 0.1388 | 1.4012 | 0.2245 | 0.2222 | 0.1174 | 0.1347 | 1.233 | 0.395 | 0.3956 |
| | | | $\alpha_2$ | 0.05 | 0.0511 | 0.8644 | 0.1523 | 0.1557 | 0.06 | 0.0481 | 0.7507 | 0.1854 | 0.1831 |
| | | | $\alpha_3$ | 0.1084 | 0.141 | 1.4098 | 0.2989 | 0.3013 | 0.1086 | 0.1359 | 1.3269 | 0.2528 | 0.251 |
| | | | $\lambda$ | 0.0213 | 0.0104 | 0.3914 | 0.088 | 0.0869 | 0.0262 | 0.0102 | 0.3828 | 0.0745 | 0.0721 |
| | | 27 | $\alpha_1$ | 0.0845 | 0.1162 | 1.2951 | 0.3046 | 0.3077 | 0.0939 | 0.1161 | 1.1518 | 0.1961 | 0.1964 |
| | | | $\alpha_2$ | 0.0535 | 0.0451 | 0.8058 | 0.1724 | 0.1702 | 0.0646 | 0.0447 | 0.7597 | 0.0818 | 0.08 |
| | | | $\alpha_3$ | 0.0798 | 0.1023 | 1.2149 | 0.1483 | 0.151 | 0.0832 | 0.1013 | 1.1136 | 0.1391 | 0.1403 |
| | | | $\lambda$ | 0.0188 | 0.0103 | 0.3904 | 0.078 | 0.0779 | 0.0215 | 0.0102 | 0.3888 | 0.0607 | 0.061 |
| | III | 30 | $\alpha_1$ | 0.0755 | 0.105 | 1.2361 | 0.2628 | 0.2571 | 0.0857 | 0.1016 | 1.0856 | 0.21 | 0.208 |
| | | | $\alpha_2$ | 0.039 | 0.0355 | 0.7232 | 0.1416 | 0.1387 | 0.0451 | 0.0348 | 0.6873 | 0.1121 | 0.114 |
| | | | $\alpha_3$ | 0.0942 | 0.0936 | 1.1415 | 0.2443 | 0.2454 | 0.0962 | 0.0905 | 1.0153 | 0.2322 | 0.2335 |
| | | | $\lambda$ | 0.0197 | 0.0097 | 0.3777 | 0.0759 | 0.0766 | 0.0227 | 0.0096 | 0.3727 | 0.0665 | 0.067 |
| 70 | I | 50 | $\alpha_1$ | 0.085 | 0.1435 | 1.4481 | 0.2798 | 0.2825 | 0.1 | 0.128 | 1.2408 | 0.1864 | 0.1871 |
| | | | $\alpha_2$ | 0.0265 | 0.0195 | 0.5382 | 0.0783 | 0.0797 | 0.0334 | 0.0192 | 0.5155 | 0.0752 | 0.0745 |
| | | | $\alpha_3$ | 0.0788 | 0.1304 | 1.3824 | 0.1419 | 0.1496 | 0.0769 | 0.1122 | 1.2242 | 0.2078 | 0.2167 |
| | | | $\lambda$ | 0.0064 | 0.0041 | 0.2498 | 0.0297 | 0.0301 | 0.0062 | 0.0041 | 0.253 | 0.034 | 0.0336 |
| | | 60 | $\alpha_1$ | 0.0667 | 0.086 | 1.1199 | 0.1265 | 0.127 | 0.0795 | 0.0822 | 0.973 | 0.1422 | 0.1424 |
| | | | $\alpha_2$ | 0.0181 | 0.0155 | 0.4835 | 0.0658 | 0.0675 | 0.0254 | 0.0146 | 0.4646 | 0.0521 | 0.0529 |
| | | | $\alpha_3$ | 0.0487 | 0.0741 | 1.0503 | 0.1092 | 0.1078 | 0.0498 | 0.068 | 0.9615 | 0.1199 | 0.1209 |
| | | | $\lambda$ | 0.0053 | 0.0039 | 0.2447 | 0.0342 | 0.0341 | 0.0056 | 0.0039 | 0.2433 | 0.029 | 0.0287 |
| | II | 50 | $\alpha_1$ | 0.0409 | 0.0487 | 0.851 | 0.1392 | 0.1398 | 0.0473 | 0.0477 | 0.7904 | 0.1129 | 0.1132 |
| | | | $\alpha_2$ | 0.0264 | 0.0183 | 0.5207 | 0.0791 | 0.0757 | 0.0331 | 0.0182 | 0.4793 | 0.0705 | 0.0701 |
| | | | $\alpha_3$ | 0.0508 | 0.0484 | 0.8394 | 0.1088 | 0.1055 | 0.0528 | 0.0482 | 0.81 | 0.1037 | 0.1057 |
| | | | $\lambda$ | 0.007 | 0.0046 | 0.2646 | 0.0419 | 0.0414 | 0.0092 | 0.0045 | 0.2539 | 0.043 | 0.0404 |
| | | 60 | $\alpha_1$ | 0.0407 | 0.0394 | 0.7622 | 0.0959 | 0.1011 | 0.0467 | 0.0388 | 0.7265 | 0.0879 | 0.0857 |
| | | | $\alpha_2$ | 0.0218 | 0.0164 | 0.4947 | 0.0728 | 0.0718 | 0.0279 | 0.0161 | 0.4608 | 0.0591 | 0.0599 |
| | | | $\alpha_3$ | 0.0432 | 0.0377 | 0.7426 | 0.0949 | 0.0951 | 0.0448 | 0.0368 | 0.716 | 0.0911 | 0.0905 |
| | | | $\lambda$ | 0.0053 | 0.004 | 0.2466 | 0.0346 | 0.0342 | 0.0071 | 0.004 | 0.2363 | 0.0262 | 0.0255 |
| | III | 70 | $\alpha_1$ | 0.0393 | 0.0365 | 0.7334 | 0.0819 | 0.0811 | 0.0454 | 0.0363 | 0.6843 | 0.075 | 0.0743 |
| | | | $\alpha_2$ | 0.0157 | 0.0134 | 0.45 | 0.0546 | 0.0543 | 0.0212 | 0.0114 | 0.4411 | 0.057 | 0.0557 |
| | | | $\alpha_3$ | 0.035 | 0.033 | 0.6996 | 0.0715 | 0.0734 | 0.0369 | 0.033 | 0.6719 | 0.0751 | 0.0744 |
| | | | $\lambda$ | 0.0108 | 0.0033 | 0.222 | 0.0299 | 0.0308 | 0.0123 | 0.0032 | 0.2155 | 0.0271 | 0.0283 |

**Table 3.** Bias, MSE, and length of CI for parameters of MOBIW distribution: Case 1 and 2 when n=100

| $n$ = 100 | | | | MLE | | | | | Bayesian | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Case | $R$ | m | | Bias | MSE | L.CI | BP | BT | Bias | MSE | L.CI | BP | BT |
| 1 | I | 80 | $\alpha_1$ | 0.0262 | 0.1499 | 1.5151 | 0.1587 | 0.1658 | 0.026 | 0.1312 | 1.3878 | 0.1695 | 0.1711 |
| | | | $\alpha_2$ | 0.0386 | 0.0448 | 0.8161 | 0.0903 | 0.0904 | 0.0406 | 0.0446 | 0.7948 | 0.1033 | 0.1055 |
| | | | $\alpha_3$ | 0.0727 | 0.0867 | 1.1193 | 0.1493 | 0.1495 | 0.0861 | 0.081 | 0.9805 | 0.127 | 0.1278 |
| | | | $\lambda$ | 0.0066 | 0.0132 | 0.4497 | 0.0526 | 0.0527 | 0.0028 | 0.0129 | 0.4554 | 0.049 | 0.05 |
| | | 90 | $\alpha_1$ | 0.0263 | 0.0928 | 1.1902 | 0.102 | 0.1018 | 0.0252 | 0.0797 | 1.0694 | 0.1219 | 0.1221 |
| | | | $\alpha_2$ | 0.0248 | 0.0405 | 0.7832 | 0.0858 | 0.0831 | 0.026 | 0.0383 | 0.7214 | 0.0862 | 0.0871 |
| | | | $\alpha_3$ | 0.0466 | 0.0512 | 0.868 | 0.1064 | 0.1078 | 0.0539 | 0.0489 | 0.7897 | 0.0902 | 0.0924 |
| | | | $\lambda$ | 0.0112 | 0.0117 | 0.4228 | 0.0418 | 0.042 | 0.0064 | 0.0125 | 0.4136 | 0.0522 | 0.0523 |
| | II | 80 | $\alpha_1$ | 0.0388 | 0.0664 | 0.999 | 0.1199 | 0.1159 | 0.0454 | 0.0679 | 0.9631 | 0.1052 | 0.1043 |
| | | | $\alpha_2$ | 0.0422 | 0.0509 | 0.869 | 0.0968 | 0.0949 | 0.0477 | 0.0507 | 0.8446 | 0.0844 | 0.0848 |
| | | | $\alpha_3$ | 0.0288 | 0.0326 | 0.6994 | 0.0805 | 0.0808 | 0.0329 | 0.0322 | 0.6901 | 0.0728 | 0.0725 |
| | | | $\lambda$ | 0.0096 | 0.0144 | 0.4696 | 0.0528 | 0.0524 | 0.0109 | 0.0144 | 0.4766 | 0.0471 | 0.0475 |
| | | 90 | $\alpha_1$ | 0.0483 | 0.0602 | 0.9435 | 0.0917 | 0.0908 | 0.0493 | 0.0562 | 0.8957 | 0.0975 | 0.0997 |
| | | | $\alpha_2$ | 0.0309 | 0.0403 | 0.7775 | 0.0798 | 0.0839 | 0.0316 | 0.0382 | 0.7449 | 0.0819 | 0.082 |
| | | | $\alpha_3$ | 0.0212 | 0.0272 | 0.6416 | 0.0611 | 0.062 | 0.0222 | 0.0262 | 0.611 | 0.0592 | 0.0597 |
| | | | $\lambda$ | 0.0068 | 0.0123 | 0.4337 | 0.0446 | 0.046 | 0.0038 | 0.0123 | 0.4384 | 0.0534 | 0.0548 |
| | III | 100 | $\alpha_1$ | 0.035 | 0.0553 | 0.9116 | 0.0895 | 0.0911 | 0.0196 | 0.044 | 0.795 | 0.0847 | 0.0853 |
| | | | $\alpha_2$ | 0.0318 | 0.0402 | 0.7766 | 0.0734 | 0.0752 | 0.0208 | 0.0357 | 0.71 | 0.0747 | 0.0732 |
| | | | $\alpha_3$ | 0.0261 | 0.0273 | 0.6394 | 0.0629 | 0.0621 | 0.0167 | 0.0238 | 0.5882 | 0.0607 | 0.0632 |
| | | | $\lambda$ | 0.0166 | 0.0123 | 0.4293 | 0.0383 | 0.0384 | -0.0009 | 0.0121 | 0.4346 | 0.0381 | 0.038 |
| 2 | I | 80 | $\alpha_1$ | 0.0487 | 0.0749 | 1.0561 | 0.1428 | 0.1401 | 0.0614 | 0.0703 | 0.9427 | 0.1173 | 0.1191 |
| | | | $\alpha_2$ | 0.0127 | 0.0111 | 0.4107 | 0.0447 | 0.0442 | 0.0172 | 0.0111 | 0.4008 | 0.0422 | 0.0426 |
| | | | $\alpha_3$ | 0.0461 | 0.0661 | 0.9919 | 0.1078 | 0.1089 | 0.0424 | 0.0594 | 0.9155 | 0.0998 | 0.0977 |
| | | | $\lambda$ | 0.0062 | 0.0027 | 0.2012 | 0.0253 | 0.0265 | 0.0065 | 0.0027 | 0.1958 | 0.0245 | 0.0249 |
| | | 90 | $\alpha_1$ | 0.0497 | 0.0455 | 0.8133 | 0.0876 | 0.0864 | 0.0593 | 0.044 | 0.765 | 0.0835 | 0.0821 |
| | | | $\alpha_2$ | 0.0157 | 0.0101 | 0.3896 | 0.0389 | 0.0403 | 0.0198 | 0.0099 | 0.3892 | 0.0378 | 0.0372 |
| | | | $\alpha_3$ | 0.0209 | 0.0363 | 0.7425 | 0.0736 | 0.0745 | 0.0225 | 0.0339 | 0.6651 | 0.0703 | 0.0718 |
| | | | $\lambda$ | 0.0045 | 0.0024 | 0.1926 | 0.0193 | 0.0202 | 0.0052 | 0.0022 | 0.1916 | 0.0194 | 0.0195 |
| | II | 80 | $\alpha_1$ | 0.0344 | 0.0281 | 0.6436 | 0.0719 | 0.0751 | 0.0398 | 0.0278 | 0.6047 | 0.0679 | 0.0663 |
| | | | $\alpha_2$ | 0.0184 | 0.0119 | 0.4219 | 0.0463 | 0.046 | 0.0234 | 0.0117 | 0.4066 | 0.0461 | 0.0456 |
| | | | $\alpha_3$ | 0.0178 | 0.027 | 0.6409 | 0.067 | 0.0677 | 0.0203 | 0.0269 | 0.627 | 0.0562 | 0.0569 |
| | | | $\lambda$ | 0.0062 | 0.003 | 0.2119 | 0.0265 | 0.0256 | 0.0079 | 0.0029 | 0.2045 | 0.0258 | 0.0251 |
| | | 90 | $\alpha_1$ | 0.0304 | 0.0237 | 0.5916 | 0.068 | 0.0679 | 0.0351 | 0.0224 | 0.5528 | 0.0686 | 0.0697 |
| | | | $\alpha_2$ | 0.0153 | 0.0103 | 0.3944 | 0.0417 | 0.0426 | 0.0199 | 0.0102 | 0.3971 | 0.0406 | 0.0391 |
| | | | $\alpha_3$ | 0.0181 | 0.0209 | 0.5627 | 0.0553 | 0.0541 | 0.0192 | 0.0208 | 0.5505 | 0.061 | 0.0609 |
| | | | $\lambda$ | 0.0039 | 0.0026 | 0.1999 | 0.0232 | 0.0232 | 0.0053 | 0.0026 | 0.1917 | 0.0197 | 0.0198 |
| | III | 100 | $\alpha_1$ | 0.0228 | 0.0207 | 0.557 | 0.0616 | 0.0614 | 0.0279 | 0.0192 | 0.5314 | 0.0512 | 0.0526 |
| | | | $\alpha_2$ | 0.0148 | 0.009 | 0.3674 | 0.0388 | 0.0386 | 0.019 | 0.0089 | 0.3559 | 0.0366 | 0.0369 |
| | | | $\alpha_3$ | 0.0277 | 0.0214 | 0.5639 | 0.0634 | 0.0625 | 0.0287 | 0.0213 | 0.5537 | 0.0544 | 0.0569 |
| | | | $\lambda$ | 0.0048 | 0.0024 | 0.1929 | 0.0207 | 0.0209 | 0.006 | 0.0024 | 0.1879 | 0.0184 | 0.0186 |

**Table 4.** Bias, MSE, and length of CI for parameters of MOBIW distribution: Case 3 when n=30 and 70

| Case 3 | | | | MLE | | | | | Bayesian | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| n | $R$ | m | | Bias | MSE | L.CI | BP | BT | Bias | MSE | L.CI | BP | BT |
| 30 | I | 24 | $\alpha_1$ | 0.088 | 0.1025 | 1.2074 | 0.1992 | 0.199 | 0.0472 | 0.0266 | 0.5537 | 0.0864 | 0.0862 |
| | | | $\alpha_2$ | 0.0465 | 0.0694 | 1.0171 | 0.1865 | 0.1865 | 0.0098 | 0.0069 | 0.2905 | 0.0556 | 0.0559 |
| | | | $\alpha_3$ | 0.0284 | 0.0505 | 0.8742 | 0.146 | 0.1484 | 0.0246 | 0.0487 | 0.3979 | 0.1254 | 0.1138 |
| | | | $\lambda$ | 0.1683 | 1.6565 | 4.7876 | 1.3187 | 1.2711 | -0.0878 | 0.114 | 1.1511 | 0.2015 | 0.2036 |
| | | 27 | $\alpha_1$ | 0.0718 | 0.0767 | 1.0494 | 0.2055 | 0.2032 | 0.0297 | 0.0227 | 0.511 | 0.0628 | 0.0639 |
| | | | $\alpha_2$ | 0.02 | 0.033 | 0.7076 | 0.1289 | 0.1338 | 0.0119 | 0.0065 | 0.2968 | 0.0563 | 0.0586 |
| | | | $\alpha_3$ | 0.0455 | 0.0389 | 0.753 | 0.1499 | 0.1521 | 0.029 | 0.0116 | 0.3845 | 0.0501 | 0.0496 |
| | | | $\lambda$ | -0.0401 | 0.4958 | 2.5311 | 0.5554 | 0.5433 | -0.0909 | 0.1006 | 1.0708 | 0.2409 | 0.2343 |
| | II | 24 | $\alpha_1$ | 0.0708 | 0.0905 | 1.1467 | 0.2749 | 0.266 | 0.0322 | 0.0253 | 0.5229 | 0.0898 | 0.0894 |
| | | | $\alpha_2$ | 0.0124 | 0.0451 | 0.8311 | 0.1734 | 0.174 | 0.0066 | 0.0111 | 0.3101 | 0.0652 | 0.0652 |
| | | | $\alpha_3$ | 0.0572 | 0.0548 | 0.8905 | 0.1888 | 0.1932 | 0.0344 | 0.0167 | 0.4349 | 0.0673 | 0.0678 |
| | | | $\lambda$ | 0.0103 | 0.5502 | 2.7092 | 0.2565 | 0.2597 | -0.105 | 0.1126 | 1.1258 | 0.2381 | 0.239 |
| | | 27 | $\alpha_1$ | 0.0612 | 0.0738 | 1.0384 | 0.2396 | 0.2416 | 0.042 | 0.0522 | 0.5165 | 0.1045 | 0.1051 |
| | | | $\alpha_2$ | 0.0052 | 0.0316 | 0.6964 | 0.1665 | 0.1576 | 0.0055 | 0.0101 | 0.2885 | 0.0604 | 0.0602 |
| | | | $\alpha_3$ | 0.0497 | 0.0458 | 0.8167 | 0.1924 | 0.1934 | 0.0297 | 0.0139 | 0.4025 | 0.067 | 0.0671 |
| | | | $\lambda$ | 0.0145 | 0.5077 | 2.6143 | 0.7406 | 0.7164 | -0.1055 | 0.0996 | 1.0488 | 0.2112 | 0.2073 |
| | III | 30 | $\alpha_1$ | 0.0551 | 0.0709 | 1.0214 | 0.1684 | 0.1674 | 0.0272 | 0.018 | 0.469 | 0.0808 | 0.0798 |
| | | | $\alpha_2$ | 0.0149 | 0.0264 | 0.6349 | 0.1013 | 0.1012 | 0.009 | 0.0055 | 0.2798 | 0.0562 | 0.0566 |
| | | | $\alpha_3$ | 0.052 | 0.0357 | 0.7128 | 0.1187 | 0.1208 | 0.03 | 0.0104 | 0.3723 | 0.0626 | 0.0611 |
| | | | $\lambda$ | 0.023 | 0.4075 | 2.3124 | 1.4782 | 1.4797 | -0.1017 | 0.0948 | 1.0309 | 0.1788 | 0.1795 |
| 70 | I | 50 | $\alpha_1$ | 0.0323 | 0.0343 | 0.7155 | 0.1023 | 0.102 | 0.0153 | 0.0082 | 0.3384 | 0.0529 | 0.0525 |
| | | | $\alpha_2$ | 0.0162 | 0.0142 | 0.4637 | 0.0986 | 0.1009 | 0.006 | 0.0027 | 0.2061 | 0.0331 | 0.0324 |
| | | | $\alpha_3$ | 0.0237 | 0.0218 | 0.5718 | 0.0782 | 0.0794 | 0.0183 | 0.0059 | 0.2851 | 0.0446 | 0.043 |
| | | | $\lambda$ | 0.0113 | 0.3134 | 2.031 | 0.3175 | 0.3146 | -0.0696 | 0.0745 | 0.9759 | 0.1496 | 0.1503 |
| | | 60 | $\alpha_1$ | 0.0372 | 0.0314 | 0.6797 | 0.0748 | 0.0766 | 0.0161 | 0.008 | 0.3366 | 0.0404 | 0.0409 |
| | | | $\alpha_2$ | 0.0072 | 0.0114 | 0.4175 | 0.0518 | 0.0515 | 0.0045 | 0.0027 | 0.1953 | 0.0259 | 0.0262 |
| | | | $\alpha_3$ | 0.0204 | 0.0154 | 0.4804 | 0.0661 | 0.0662 | 0.0119 | 0.004 | 0.2294 | 0.024 | 0.0247 |
| | | | $\lambda$ | -0.0329 | 0.1963 | 1.5977 | 0.2385 | 0.2371 | -0.0873 | 0.0703 | 0.921 | 0.13 | 0.1286 |
| | II | 50 | $\alpha_1$ | 0.0345 | 0.0298 | 0.6634 | 0.092 | 0.0909 | 0.0168 | 0.0077 | 0.3407 | 0.0475 | 0.0476 |
| | | | $\alpha_2$ | 0.0073 | 0.0117 | 0.423 | 0.063 | 0.0631 | 0.0065 | 0.0029 | 0.2072 | 0.0265 | 0.0277 |
| | | | $\alpha_3$ | 0.0228 | 0.0195 | 0.5404 | 0.0734 | 0.072 | 0.0128 | 0.0051 | 0.2661 | 0.0394 | 0.0389 |
| | | | $\lambda$ | -0.0238 | 0.1808 | 1.543 | 0.3735 | 0.3853 | -0.0961 | 0.0656 | 0.849 | 0.1201 | 0.1232 |
| | | 60 | $\alpha_1$ | 0.0255 | 0.0308 | 0.6815 | 0.0761 | 0.0788 | 0.013 | 0.0083 | 0.333 | 0.0479 | 0.0486 |
| | | | $\alpha_2$ | 0.008 | 0.0104 | 0.3995 | 0.0484 | 0.049 | 0.0062 | 0.0026 | 0.197 | 0.0172 | 0.0176 |
| | | | $\alpha_3$ | 0.0224 | 0.0159 | 0.4865 | 0.0618 | 0.0619 | 0.0126 | 0.0043 | 0.2363 | 0.0377 | 0.0373 |
| | | | $\lambda$ | -0.0229 | 0.1409 | 1.3697 | 0.1957 | 0.1927 | -0.0944 | 0.0548 | 0.7872 | 0.1034 | 0.1032 |
| | III | 70 | $\alpha_1$ | 0.0335 | 0.0245 | 0.5999 | 0.0759 | 0.0777 | 0.0149 | 0.0064 | 0.2966 | 0.0383 | 0.038 |
| | | | $\alpha_2$ | 0.0036 | 0.0099 | 0.3895 | 0.0445 | 0.0449 | 0.0038 | 0.0025 | 0.1896 | 0.0244 | 0.0243 |
| | | | $\alpha_3$ | 0.0174 | 0.013 | 0.4411 | 0.0478 | 0.0469 | 0.009 | 0.0035 | 0.2165 | 0.0252 | 0.0257 |
| | | | $\lambda$ | -0.0161 | 0.1211 | 1.2688 | 0.1645 | 0.16 | -0.099 | 0.0546 | 0.7762 | 0.0858 | 0.0857 |

**Table 5.** Bias, MSE, and length of CI for parameters of MOBIW distribution: Case 4 when n=30 and 70

| Case 4 | | | | MLE | | | | | Bayesian | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| n | $R$ | m | | Bias | MSE | L.CI | BP | BT | Bias | MSE | L.CI | BP | BT |
| 30 | I | 24 | $\alpha_1$ | 0.1479 | 0.3932 | 2.3898 | 0.4057 | 0.4119 | 0.0886 | 0.1452 | 1.1075 | 0.3651 | 0.3605 |
| | | | $\alpha_2$ | 0.0393 | 0.0379 | 0.7474 | 0.1072 | 0.1054 | 0.0006 | 0.0269 | 0.2533 | 0.056 | 0.0566 |
| | | | $\alpha_3$ | 0.0682 | 0.2586 | 1.9764 | 0.3715 | 0.3843 | 0.0856 | 0.08 | 0.9069 | 0.1292 | 0.1293 |
| | | | $\lambda$ | 0.3094 | 2.2569 | 5.6207 | 1.1409 | 1.0847 | -0.4272 | 0.2677 | 1.0667 | 0.2633 | 0.2668 |
| | | 27 | $\alpha_1$ | 0.1527 | 0.3505 | 2.2435 | 0.468 | 0.4664 | 0.0844 | 0.0927 | 1.0522 | 0.2986 | 0.3046 |
| | | | $\alpha_2$ | 0.0162 | 0.0276 | 0.649 | 0.0847 | 0.0845 | -0.0052 | 0.0126 | 0.2458 | 0.0517 | 0.0523 |
| | | | $\alpha_3$ | 0.0977 | 0.2039 | 1.7289 | 0.383 | 0.3844 | 0.0768 | 0.0614 | 0.8222 | 0.1853 | 0.1804 |
| | | | $\lambda$ | 0.1328 | 0.9356 | 3.612 | 0.8061 | 0.7914 | -0.429 | 0.2429 | 0.9585 | 0.1549 | 0.1569 |
| | II | 24 | $\alpha_1$ | 0.1419 | 0.3669 | 2.3095 | 0.5533 | 0.5507 | 0.0852 | 0.1041 | 1.0804 | 0.2395 | 0.2452 |
| | | | $\alpha_2$ | 0.0106 | 0.0216 | 0.5749 | 0.095 | 0.096 | 0.0104 | 0.0199 | 0.2671 | 0.0496 | 0.0506 |
| | | | $\alpha_3$ | 0.0947 | 0.227 | 1.8311 | 0.2889 | 0.2887 | 0.0657 | 0.0708 | 0.8483 | 0.1757 | 0.1728 |
| | | | $\lambda$ | 0.175 | 1.1045 | 3.9452 | 0.5673 | 0.5677 | -0.4332 | 0.2519 | 0.9541 | 0.2019 | 0.2013 |
| | | 27 | $\alpha_1$ | 0.1443 | 0.3071 | 2.0982 | 0.3691 | 0.3566 | 0.0839 | 0.0861 | 0.9856 | 0.1822 | 0.1829 |
| | | | $\alpha_2$ | 0.014 | 0.0199 | 0.55 | 0.115 | 0.1178 | 0.0011 | 0.0085 | 0.2395 | 0.0432 | 0.0433 |
| | | | $\alpha_3$ | 0.097 | 0.1932 | 1.6812 | 0.2465 | 0.2405 | 0.0674 | 0.0583 | 0.8356 | 0.2362 | 0.2333 |
| | | | $\lambda$ | 0.1067 | 0.6856 | 3.0868 | 1.6316 | 1.5289 | -0.4554 | 0.2679 | 0.9275 | 0.1534 | 0.1565 |
| | III | 30 | $\alpha_1$ | 0.1323 | 0.2885 | 2.0417 | 0.4629 | 0.4747 | 0.0759 | 0.0795 | 0.9736 | 0.2056 | 0.2045 |
| | | | $\alpha_2$ | 0.0137 | 0.018 | 0.5231 | 0.0965 | 0.0984 | 0.0058 | 0.0043 | 0.2443 | 0.0416 | 0.0423 |
| | | | $\alpha_3$ | 0.0948 | 0.1586 | 1.5168 | 0.3677 | 0.376 | 0.0791 | 0.1246 | 0.7417 | 0.1364 | 0.1388 |
| | | | $\lambda$ | 0.1172 | 0.5528 | 2.7476 | 0.4753 | 0.4733 | -0.4469 | 0.2514 | 0.8816 | 0.1557 | 0.1536 |
| 70 | I | 50 | $\alpha_1$ | 0.0739 | 0.1608 | 1.5458 | 0.1872 | 0.1846 | 0.0526 | 0.0405 | 0.6733 | 0.1256 | 0.1227 |
| | | | $\alpha_2$ | 0.0126 | 0.0141 | 0.4629 | 0.054 | 0.0541 | -0.0098 | 0.0018 | 0.164 | 0.0243 | 0.0235 |
| | | | $\alpha_3$ | 0.0383 | 0.115 | 1.3214 | 0.1632 | 0.161 | 0.0609 | 0.0338 | 0.6491 | 0.0956 | 0.0939 |
| | | | $\lambda$ | 0.0563 | 0.3953 | 2.3048 | 0.4922 | 0.5091 | -0.41 | 0.2116 | 0.7927 | 0.1101 | 0.1077 |
| | | 60 | $\alpha_1$ | 0.0696 | 0.1245 | 1.3569 | 0.168 | 0.1695 | 0.0443 | 0.0343 | 0.6593 | 0.0739 | 0.0741 |
| | | | $\alpha_2$ | 0.0063 | 0.0077 | 0.3429 | 0.0632 | 0.0631 | -0.0049 | 0.0017 | 0.1576 | 0.0191 | 0.0187 |
| | | | $\alpha_3$ | 0.0263 | 0.0677 | 1.0153 | 0.1111 | 0.1094 | 0.0337 | 0.0193 | 0.4997 | 0.0586 | 0.0587 |
| | | | $\lambda$ | 0.0127 | 0.2142 | 1.6889 | 0.2568 | 0.2561 | -0.4259 | 0.2114 | 0.698 | 0.0741 | 0.0716 |
| | II | 50 | $\alpha_1$ | 0.0697 | 0.133 | 1.4037 | 0.1909 | 0.1895 | 0.0437 | 0.0365 | 0.6992 | 0.1041 | 0.1037 |
| | | | $\alpha_2$ | 0.0072 | 0.0082 | 0.3542 | 0.0425 | 0.0435 | 0.0037 | 0.002 | 0.1678 | 0.0212 | 0.0217 |
| | | | $\alpha_3$ | 0.0477 | 0.087 | 1.1413 | 0.1497 | 0.1498 | 0.037 | 0.0255 | 0.5558 | 0.0818 | 0.0834 |
| | | | $\lambda$ | 0.0369 | 0.1755 | 1.54 | 0.263 | 0.2673 | -0.0436 | 0.1217 | 0.6441 | 0.1048 | 0.1067 |
| | | 60 | $\alpha_1$ | 0.0515 | 0.1083 | 1.2746 | 0.1573 | 0.1634 | 0.0351 | 0.0293 | 0.6498 | 0.0775 | 0.0768 |
| | | | $\alpha_2$ | 0.0013 | 0.0065 | 0.3152 | 0.0398 | 0.0423 | 0.0001 | 0.0017 | 0.1611 | 0.0187 | 0.0189 |
| | | | $\alpha_3$ | 0.04 | 0.0699 | 1.0251 | 0.1164 | 0.1148 | 0.031 | 0.0197 | 0.507 | 0.0553 | 0.0553 |
| | | | $\lambda$ | 0.0063 | 0.1478 | 1.4048 | 0.2255 | 0.2301 | -0.0045 | 0.1221 | 0.5858 | 0.0871 | 0.0902 |
| | III | 70 | $\alpha_1$ | 0.071 | 0.1028 | 1.2263 | 0.1418 | 0.1466 | 0.041 | 0.0285 | 0.6303 | 0.0653 | 0.067 |
| | | | $\alpha_2$ | 0.0003 | 0.0064 | 0.3135 | 0.0439 | 0.0436 | -0.0006 | 0.0016 | 0.1536 | 0.0196 | 0.0195 |
| | | | $\alpha_3$ | 0.0164 | 0.053 | 0.9003 | 0.115 | 0.1156 | 0.016 | 0.0144 | 0.4568 | 0.0464 | 0.0462 |
| | | | $\lambda$ | 0.0116 | 0.1123 | 1.2249 | 0.18 | 0.1834 | -0.0104 | 0.1021 | 0.5221 | 0.0578 | 0.0573 |

**Table 6.** Bias, MSE, and length of CI for parameters of MOBIW distribution: Case 3 and 4 when n=100

| $n$ = 100 | | | | MLE | | | | | Bayesian | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Case | $R$ | m | | Bias | MSE | L.CI | BP | BT | Bias | MSE | L.CI | BP | BT |
| 3 | I | 80 | $\alpha_1$ | 0.0239 | 0.0217 | 0.5695 | 0.0678 | 0.0692 | 0.0111 | 0.0055 | 0.2784 | 0.0334 | 0.032 |
| | | | $\alpha_2$ | 0.0065 | 0.0081 | 0.3515 | 0.0394 | 0.0407 | 0.0024 | 0.0019 | 0.1705 | 0.0196 | 0.0203 |
| | | | $\alpha_3$ | 0.0186 | 0.0126 | 0.4344 | 0.0465 | 0.0467 | 0.0119 | 0.0034 | 0.2147 | 0.0252 | 0.0247 |
| | | | $\lambda$ | -0.0093 | 0.1424 | 1.3588 | 0.1475 | 0.1517 | -0.0752 | 0.0583 | 0.8191 | 0.0984 | 0.0976 |
| | | 90 | $\alpha_1$ | 0.02 | 0.0177 | 0.5159 | 0.0485 | 0.0489 | 0.009 | 0.0044 | 0.2531 | 0.0262 | 0.0263 |
| | | | $\alpha_2$ | 0.008 | 0.0074 | 0.3355 | 0.035 | 0.0341 | 0.0041 | 0.0018 | 0.16 | 0.0168 | 0.0172 |
| | | | $\alpha_3$ | 0.009 | 0.0093 | 0.3766 | 0.0376 | 0.0374 | 0.0062 | 0.0025 | 0.1874 | 0.0211 | 0.0204 |
| | | | $\lambda$ | -0.0129 | 0.1077 | 1.1926 | 0.149 | 0.1528 | -0.0795 | 0.0498 | 0.7605 | 0.0845 | 0.0867 |
| | II | 80 | $\alpha_1$ | 0.0274 | 0.0215 | 0.5652 | 0.064 | 0.0645 | 0.0128 | 0.0055 | 0.2764 | 0.0309 | 0.0306 |
| | | | $\alpha_2$ | -0.002 | 0.0078 | 0.3468 | 0.0341 | 0.0329 | 0.0009 | 0.002 | 0.1714 | 0.021 | 0.0208 |
| | | | $\alpha_3$ | 0.0198 | 0.0122 | 0.4255 | 0.0522 | 0.0521 | 0.0103 | 0.0033 | 0.2092 | 0.0206 | 0.0204 |
| | | | $\lambda$ | 0.0032 | 0.1016 | 1.1637 | 0.157 | 0.1565 | -0.0727 | 0.0439 | 0.7143 | 0.0806 | 0.081 |
| | | 90 | $\alpha_1$ | 0.0137 | 0.0185 | 0.5307 | 0.0459 | 0.0474 | 0.0063 | 0.0049 | 0.2534 | 0.0308 | 0.0309 |
| | | | $\alpha_2$ | 0.004 | 0.0073 | 0.3352 | 0.0428 | 0.0431 | 0.0033 | 0.0019 | 0.1657 | 0.0196 | 0.0196 |
| | | | $\alpha_3$ | 0.0149 | 0.0099 | 0.3866 | 0.0382 | 0.0392 | 0.0088 | 0.0027 | 0.1999 | 0.0216 | 0.0218 |
| | | | $\lambda$ | 0.0143 | 0.0824 | 1.0448 | 0.1269 | 0.1303 | -0.0712 | 0.0381 | 0.6736 | 0.0717 | 0.0733 |
| | III | 100 | $\alpha_1$ | 0.0238 | 0.0168 | 0.4995 | 0.0479 | 0.0473 | 0.0105 | 0.0044 | 0.2414 | 0.0266 | 0.0269 |
| | | | $\alpha_2$ | 0.0025 | 0.006 | 0.3029 | 0.0315 | 0.0311 | 0.0024 | 0.0015 | 0.1537 | 0.017 | 0.0168 |
| | | | $\alpha_3$ | 0.0144 | 0.0095 | 0.3786 | 0.0401 | 0.0398 | 0.0075 | 0.0025 | 0.1893 | 0.0183 | 0.0187 |
| | | | $\lambda$ | -0.0152 | 0.0781 | 1.0234 | 0.1202 | 0.1202 | -0.0876 | 0.0396 | 0.6538 | 0.0699 | 0.0696 |
| 4 | I | 80 | $\alpha_1$ | 0.048 | 0.0839 | 1.1202 | 0.1248 | 0.1228 | 0.0346 | 0.0231 | 0.5658 | 0.0706 | 0.0705 |
| | | | $\alpha_2$ | 0.0045 | 0.0056 | 0.2942 | 0.0313 | 0.0317 | -0.0073 | 0.0013 | 0.1384 | 0.0178 | 0.0181 |
| | | | $\alpha_3$ | 0.0289 | 0.0568 | 0.9276 | 0.114 | 0.1147 | 0.0364 | 0.0158 | 0.4416 | 0.0503 | 0.0503 |
| | | | $\lambda$ | 0.0055 | 0.1543 | 1.4177 | 0.165 | 0.1656 | -0.0042 | 0.0621 | 0.6093 | 0.0704 | 0.0707 |
| | | 90 | $\alpha_1$ | 0.0579 | 0.0723 | 1.0295 | 0.0984 | 0.0992 | 0.0369 | 0.0206 | 0.5142 | 0.0528 | 0.0512 |
| | | | $\alpha_2$ | -0.0003 | 0.0043 | 0.2565 | 0.0296 | 0.0312 | -0.0042 | 0.0011 | 0.1308 | 0.0129 | 0.0131 |
| | | | $\alpha_3$ | 0.0353 | 0.0432 | 0.8032 | 0.077 | 0.075 | 0.0294 | 0.0126 | 0.4137 | 0.0473 | 0.0491 |
| | | | $\lambda$ | -0.002 | 0.1142 | 1.2277 | 0.1616 | 0.1604 | -0.002 | 0.012 | 0.5001 | 0.0547 | 0.0544 |
| | II | 80 | $\alpha_1$ | 0.064 | 0.0894 | 1.1454 | 0.1219 | 0.1238 | 0.0412 | 0.026 | 0.6022 | 0.0593 | 0.0613 |
| | | | $\alpha_2$ | 0.0007 | 0.005 | 0.2765 | 0.0323 | 0.0319 | 0.0009 | 0.0013 | 0.1403 | 0.0156 | 0.0161 |
| | | | $\alpha_3$ | 0.0327 | 0.0502 | 0.8691 | 0.0997 | 0.0997 | 0.0251 | 0.014 | 0.439 | 0.05 | 0.0511 |
| | | | $\lambda$ | -0.0049 | 0.0959 | 1.1248 | 0.1566 | 0.1598 | -0.0044 | 0.0821 | 0.5079 | 0.0525 | 0.0525 |
| | | 90 | $\alpha_1$ | 0.0426 | 0.0729 | 1.0453 | 0.1 | 0.1038 | 0.0296 | 0.0197 | 0.5099 | 0.0584 | 0.0586 |
| | | | $\alpha_2$ | 0.0071 | 0.0047 | 0.2681 | 0.0278 | 0.0277 | 0.0038 | 0.0012 | 0.1373 | 0.0149 | 0.0149 |
| | | | $\alpha_3$ | 0.0236 | 0.0427 | 0.8055 | 0.0832 | 0.0829 | 0.021 | 0.0122 | 0.4272 | 0.0485 | 0.0478 |
| | | | $\lambda$ | 0.0235 | 0.0915 | 1.0977 | 0.122 | 0.1258 | -0.0194 | 0.082 | 0.4725 | 0.053 | 0.0548 |
| | III | 100 | $\alpha_1$ | 0.0452 | 0.0634 | 0.9717 | 0.0974 | 0.1017 | 0.0285 | 0.0176 | 0.4732 | 0.0554 | 0.0551 |
| | | | $\alpha_2$ | -0.0008 | 0.0039 | 0.2454 | 0.0229 | 0.0236 | -0.0005 | 0.001 | 0.1275 | 0.0116 | 0.0121 |
| | | | $\alpha_3$ | 0.0312 | 0.0421 | 0.7958 | 0.0829 | 0.0839 | 0.0242 | 0.0118 | 0.4104 | 0.0408 | 0.0415 |
| | | | $\lambda$ | -0.0102 | 0.0755 | 1.0019 | 0.121 | 0.1171 | -0.4466 | 0.0091 | 0.0106 | 0.0388 | 0.0388 |

**Table 7.** MLE, SE, different measures of criteria, KSD, and p-values for marginal distributions for Infection recurrence period of kidney data

| | | $\alpha$ | $\lambda$ | AIC | CAIC | BIC | HQIC | KS | P-Value |
|---|---|---|---|---|---|---|---|---|---|
| $x_1$ | estimate | 0.7234 | 0.0977 | 348.0859 | 348.5304 | 350.8883 | 348.9824 | 0.13 | 0.6916 |
| | SE | 23.5222 | 6.2894 | | | | | | |
| $x_2$ | estimate | 0.8604 | 0.1166 | 342.4596 | 342.9041 | 345.262 | 343.3561 | 0.1269 | 0.7194 |
| | SE | 28.6175 | 6.435 | | | | | | |
| $max(x_1, x_2)$ | estimate | 0.878 | 0.119 | 380.2013 | 380.6458 | 383.0037 | 381.0978 | 0.1863 | 0.2196 |
| | SE | 56.1674 | 12.3866 | | | | | | |

**Table 8.** MLE of FGMBIW and MOBIW Distribution for Infection recurrence period of kidney data

| | | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\lambda$ | AIC | CAIC | BIC | HQIC |
|---|---|---|---|---|---|---|---|---|---|
| FGMBIW | estimate | 0.7135 | 25.8529 | 0.8746 | 0.5774 | 689.8392 | 691.4392 | 695.444 | 691.6322 |
| | SE | 0.0933 | 4.8921 | 0.1182 | 0.5522 | | | | |
| MOBIW | estimate | 0.6051 | 0.7172 | 0.922 | 11.4695 | 780.4511 | 782.0511 | 786.0559 | 782.2441 |
| | SE | 0.1134 | 0.1383 | 0.1896 | 2.3362 | | | | |

By Table 8 and Figures 3 and 4, we can conclude that the data fit IW distribution and BIW distribution based on FGM copula and MO method can be used.

Table 8 shows MLE, SE, and different measures of criteria for parameters of FGMBIW and MOBIW distribution. Table 9 shows the Bayesian estimation and SE for parameters of FGMBIW and MOBIW distribution.

The MCMC iterations and the kernel histograms of the posterior samples of the parameters $\Theta_1, \Theta_2, \Theta_3$, and $\Theta_4$ are plotted in Figures 5 and 6. Figures 7 and 8 show posterior distributions of the four parameters which are symmetric normal.

**Table 9.** Bayesian estimation of FGMBIW and MOBIW Distribution for Infection recurrence period of kidney data

| | | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\lambda$ |
|---|---|---|---|---|---|
| FGMBIW | Estimate | 0.6733 | 25.8413 | 0.8792 | 0.5965 |
| | SE | 0.091 | 0.1975 | 0.1013 | 0.3928 |
| MOBIW | Estimate | 0.6113 | 0.7282 | 0.9095 | 11.8886 |
| | SE | 0.1115 | 0.131 | 0.1792 | 2.2037 |

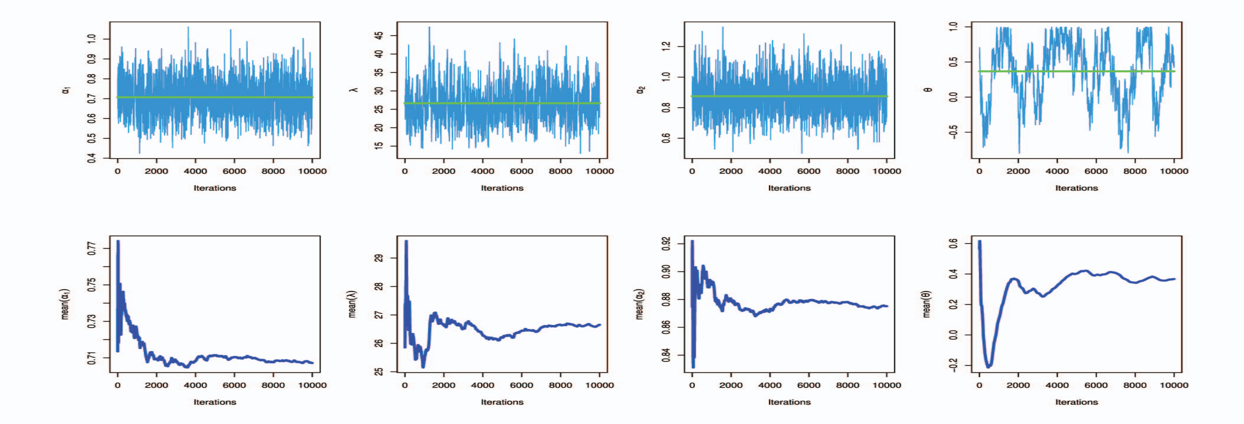

**Figure 5.** Convergence of MCMC Estimation of FGMBIW Parameters for Real Data

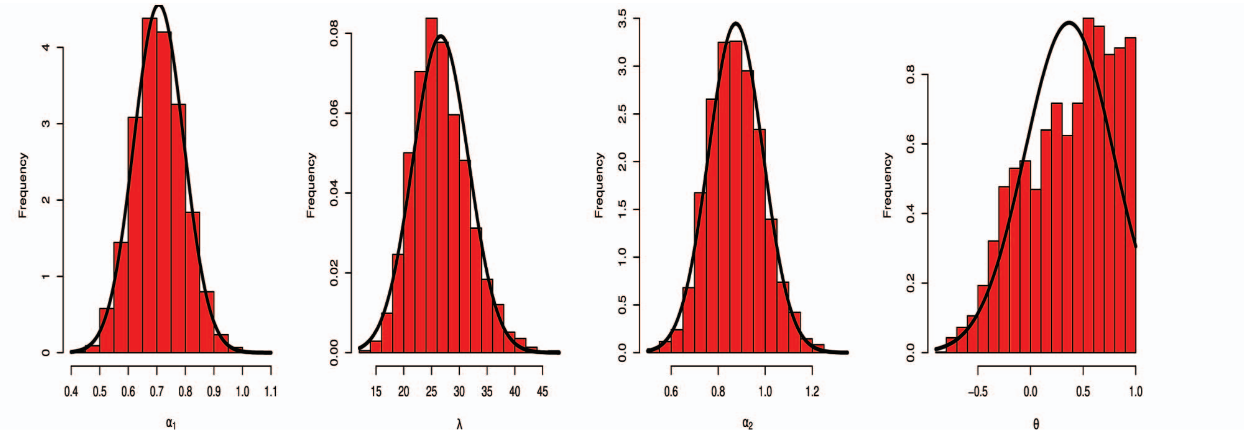

**Figure 7.** Histogram and proposal distribution (normal) of result MCMC Estimation of FGMBIW Parameters for Infection recurrence period of kidney data

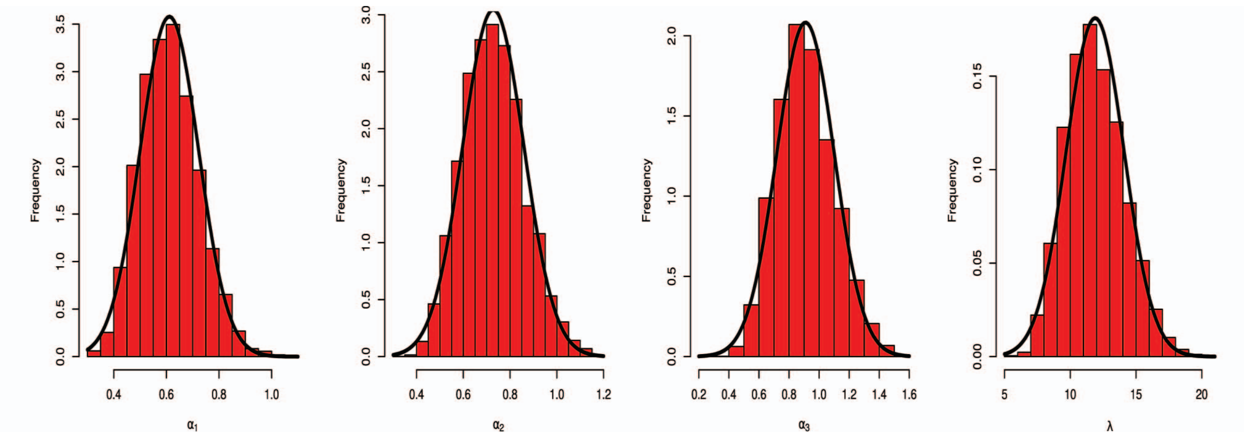

**Figure 8.** Histogram and proposal distribution (normal) of result MCMC Estimation of MOBIW Parameters for Infection recurrence period of kidney data

In scheme 2, when m=20, the first recurrence time of the infection period of the kidney can be as follows: 2, 7, 8, 12, 13, 15, 23, 24, 39, 53, 96, 130, 132, 149, 152, 185, 292, 402, 511, 536. The second

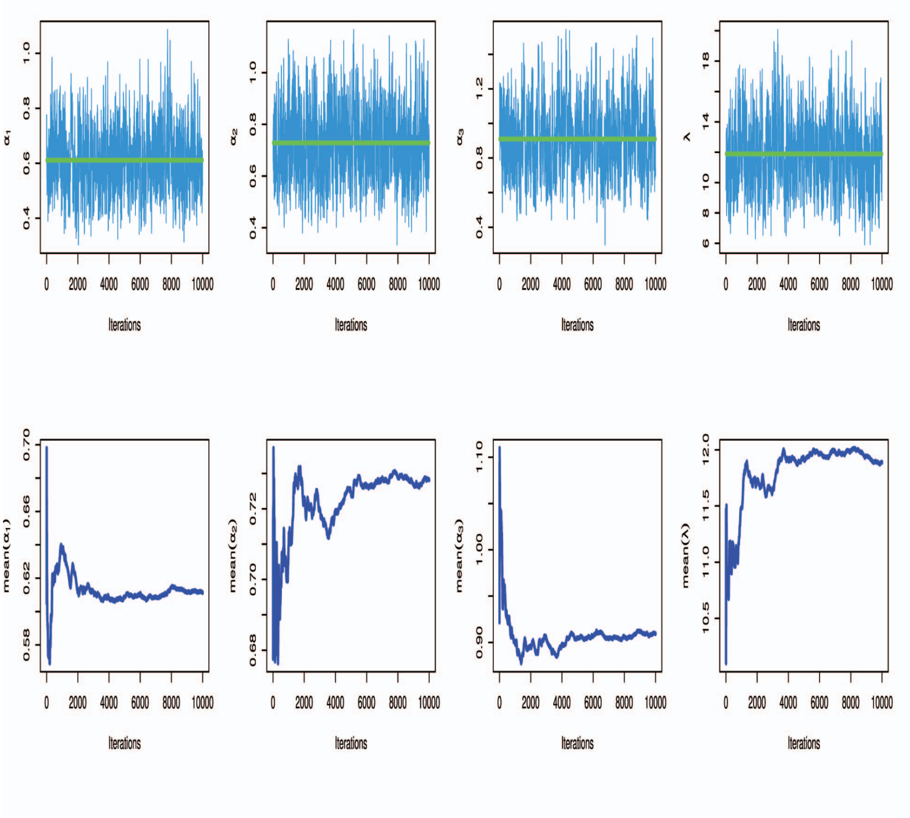

**Figure 6.** Convergence of MCMC Estimation of MOBIW Parameters for Infection recurrence period of kidney data

recurrence time of the infection period of the kidney can be as follows: 25, 333, 16, 40, 66, 108, 13, 245, 46, 196, 38, 26, 156, 70, 362, 117, 114, 24, 30, 25.

In scheme 2, when m=25, the first recurrence time of the infection period of the kidney can be as follows: 2, 7, 8, 12, 13, 15, 17, 22, 23, 24, 34, 39, 53, 96, 130, 132, 141, 149, 152, 185, 292, 402, 447, 511, 536. The second recurrence time of the Infection period of the kidney can be as follows: 25, 333, 16, 40, 66, 108, 4, 28, 13, 245, 30, 46, 196, 38, 26, 156, 8, 70, 362, 117, 114, 24, 318, 30, 25.

In scheme 2, when m=28, the first recurrence time of the Infection period of the kidney can be as follows: 2, 7, 7, 8, 12, 13, 15, 15, 17, 22, 23, 24, 30, 34, 39, 53, 96, 130, 132, 141, 149, 152, 185, 292, 402, 447, 511, 536. The second recurrence time of the Infection period of the kidney can be as follows: 25, 9, 333, 16, 40, 66, 154, 108, 4, 28, 13, 245, 12, 30, 46, 196, 38, 26, 156, 8, 70, 362, 117, 114, 24, 318, 30, 25.

Table 10 shows the MLE and Bayesian estimation for the parameters of FGMBIW and MOBIW distributions based on progressive Type-II censored samples with different schemes for the infection recurrence period of kidney data. We conclude that the Bayesian estimation method is the best estimation method under the SE loss function. When m increases then the SE decreases.

**Table 10.** Estimation for BIW Model under Progressive Type-II Censoring Schemes for Infection recurrence period of kidney data

| | | | | MLE | | | | Bayesian | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| R | m | | | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\lambda$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\lambda$ |
| I | 20 | FGMBIW | estimate | 0.6138 | 27.1061 | 0.8438 | 0.1113 | 0.6126 | 27.4786 | 0.8435 | 0.4552 |
| | | | SE | 0.0959 | 6.033 | 0.1382 | 0.5919 | 0.0907 | 5.7877 | 0.1273 | 0.5148 |
| | | MOBIW | estimate | 0.5498 | 0.8595 | 0.3984 | 21.6522 | 0.5994 | 0.8666 | 0.0743 | 24.4272 |
| | | | SE | 0.1136 | 0.1476 | 0.2493 | 5.2001 | 0.1102 | 0.1436 | 0.1834 | 5.1588 |
| | 25 | FGMBIW | estimate | 0.6727 | 27.0506 | 0.8282 | 0.3807 | 0.6765 | 26.8106 | 0.828 | 0.6037 |
| | | | SE | 0.0938 | 5.348 | 0.1218 | 0.435 | 0.0902 | 5.2892 | 0.1206 | 0.4322 |
| | | MOBIW | estimate | 0.6079 | 0.7864 | 0.4639 | 17.2423 | 0.5979 | 0.7913 | 0.4682 | 16.4585 |
| | | | SE | 0.1256 | 0.1385 | 0.148 | 3.7904 | 0.1241 | 0.1366 | 0.1431 | 3.6727 |
| | 28 | FGMBIW | estimate | 0.6818 | 27.6578 | 0.8315 | 0.3256 | 0.683 | 27.262 | 0.8331 | 0.724 |
| | | | SE | 0.0912 | 5.393 | 0.1146 | 0.4012 | 0.0902 | 5.287 | 0.1106 | 0.3937 |
| | | MOBIW | estimate | 0.5985 | 0.7561 | 0.6593 | 14.5212 | 0.5933 | 0.7549 | 0.6542 | 13.9468 |
| | | | SE | 0.1138 | 0.149 | 0.149 | 3.0556 | 0.1103 | 0.1449 | 0.1444 | 3.0054 |
| II | 20 | FGMBIW | estimate | 0.6022 | 36.0904 | 1.2024 | 0.0684 | 0.606 | 35.4914 | 1.2057 | 0.0222 |
| | | | SE | 0.0986 | 6.6262 | 0.2252 | 0.0482 | 0.0893 | 6.3422 | 0.2119 | 0.0396 |
| | | MOBIW | estimate | 0.5449 | 0.8881 | 0.7727 | 15.9399 | 0.5375 | 0.8396 | 0.8135 | 14.6863 |
| | | | SE | 0.1205 | 0.2234 | 0.1731 | 4.1038 | 0.1118 | 0.2211 | 0.1693 | 4.0096 |
| | 25 | FGMBIW | estimate | 0.674 | 28.428 | 0.8772 | 0.1875 | 0.6737 | 27.5389 | 0.886 | 0.4797 |
| | | | SE | 0.0906 | 5.6651 | 0.1259 | 0.5431 | 0.0843 | 5.5776 | 0.1201 | 0.5347 |
| | | MOBIW | estimate | 0.5844 | 0.7509 | 0.827 | 12.3256 | 0.5713 | 0.7521 | 0.848 | 12.1181 |
| | | | SE | 0.107 | 0.1549 | 0.1678 | 2.5198 | 0.1046 | 0.1515 | 0.1617 | 2.4327 |
| | 28 | FGMBIW | estimate | 0.7012 | 24.814 | 0.8925 | 0.2355 | 0.7051 | 24.6257 | 0.8955 | 0.5967 |
| | | | SE | 0.0927 | 4.1467 | 0.1202 | 0.4633 | 0.0904 | 4.1007 | 0.1196 | 0.4616 |
| | | MOBIW | estimate | 0.5973 | 0.7717 | 0.917 | 11.2308 | 0.5881 | 0.7432 | 0.9108 | 10.9731 |
| | | | SE | 0.1076 | 0.1543 | 0.1859 | 2.2553 | 0.1039 | 0.1509 | 0.1796 | 2.1051 |

## 7. Conclusion

In this paper, two versions of the bivariate inverse Weibull distribution based on the FGM copula and the Marshal-Olkin method are introduced. Estimation based on progressive Type-II censoring is considered for both MOBIW and FGMBIW distributions. Both Maximum likelihood and Bayesian estimation approaches are considered to estimate the unknown parameters of MOBIW and FGMBIW distributions based on progressive Type-II censored samples with different censoring schemes. Moreover, asymptotic, credible, and bootstrap confidence intervals for the unknown parameters are evaluated in both MLE and Bayesian Estimation. Moreover, simulation studies and a real data set are carried out for illustrative purposes.


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