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Statistical Inferences Based on Progressive First-Failure Censoring Scheme of Kumaraswamy Lifetime Distribution

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Abstract:

The problem of statistical inference of Kumaraswamy distribution (KD) based on a progressive first-failure censoring scheme (PFFCS) is discussed in this article. The population parameters as well as the reliability and hazard rate functions are estimated by using the maximum likelihood method for point and interval estimation. Both point and interval-credible estimations of parameters are obtained using the Bayes method. In the Bayes method, we use the Markov chain Monte Carlo (MCMC) technique. The Bayes estimates results are obtained under symmetric and asymmetric loss functions. We also obtained an exact confidence interval (ECI) and an exact joint confidence region (EJCR) of parameters. Real-life data is analyzed for illustrative purposes. By applying Monte Carlo simulation analysis, some comparisons between the different proposed methods are investigated.
keywords: Kumaraswamy distribution, Progressive first-failure-censored scheme, Exact confidence interval, Exact joint confidence region, MCMC technique.

1 Introduction

The Kumaraswamy distribution (KD) was advanced by Kumaraswamy [1]. It is suitable for many natural phenomena, like atmospheric temperatures, hydrological data, etc. Using both grouped and ungrouped data, statical inference of Kumaraswamy distribution was studied by Gholizadeh et al. [2]. The properties of this distribution were studied by Jones [3] and Mitnik [4]. Based on progressive Type-II censored data, the estimating parameters of KD were considered by Feroze and El-Batal [5]. The estimation problem of stress strength under the Kumaraswamy model was considered by Nasser et al. [6]. Mohamed and Wafaa [7] studied lifetime performance for the Kumaraswamy model under PFFCS. Estimation of Kumaraswamy distribution under adaptive Type-II with partially constant-stress, was considered by Saad et al. [8]. The two-parameter Kumaraswamy distribution (KD) shall be referred to as KD(λ,θ). The cumulative distribution function (CDF), probability density function (PDF), reliability, and hazard rate functions are given, respectively:

$$F(x;\lambda,\theta) = 1-\left(1-x^{\theta}\right)^{\lambda},\ (\lambda>0,\theta>0),\ 0<x<1, \tag{1}$$

$$f(x;\lambda,\theta) = \theta\lambda x^{\theta-1}\left(1-x^{\theta}\right)^{\lambda-1},\ (\lambda>0,\theta>0),\ 0<x<1, \tag{2}$$

$$S(t) = \left(1-t^{\theta}\right)^{\lambda},\ (\lambda>0,\theta>0),\ 0<t<1, \tag{3}$$

and

$$\mathbf{H}(t) = \theta\lambda t^{\theta-1}/(1-t^{\theta}),\ (\lambda>0,\theta>0),\ 0<t<1, \tag{4}$$

where λ, $\theta>0$ are shape parameters.
In life-testing experiments, the major objective is to reduce the cost and time, which leads us to what is known as censoring. Censored tests might take many different forms. Type-I progressive censoring system, type-II progressive censoring (Pro-II-C) scheme, and hybrid progressive censoring scheme were all covered in the discussion of the progressive censoring scheme. It may take a while to complete the experiment using highly reliable items tested using the most recent censorship techniques. The first

failure censoring system, see Johnson [9], which groups the test units into many sets with the same number of units and records the first failure in each group, is one of the most important solutions to this issue. The progressive first-failure censoring scheme (PFFCS) was first proposed by Wu and Kus [10]. Progressive censoring is improved and extended by the PFFCS, which is popular in experimental design because of its flexibility. In this scheme, Suppose that N is the number of test units and $n = N/k$ independent groups in the test. The following test procedure:

First, when the first failure (say$X_1^{R_1}$) occurs, we remove the other units in this group and $\mathbf{R}_1$ random groups are eliminated from the test. Secondly, when the Second failure (say$X_2^{R_2}$) occurs, we remove the other units in this group, and $\mathbf{R}_2$ random groups are eliminated from the test, and so on. Finally, when the m-th failure (say$X_m^{R_m}$) occurs, we remove the other units in this group and the rest $\mathbf{R}_m$ groups are eliminated from the test. Hence, data $X_1^{R_1} < X_2^{R_2} < X_3^{R_3} <...... < X_m^{R_m}$ are PFFCS.

Special cases:

There are different censoring schemes as special cases from PFFCS as follows:

(1): First-Failure censored if $\mathbf{R}=(R_1,......,R_m)=0^m)$.

(2): Pro-II-C if $k=1$.

(3): Type II-C if $\mathbf{R}=(R_1,......,R_{m-1},R_m)=(0^{m-1},n-m)$ and $k=1$.

(4): Complete sample at $n=m$, $\mathbf{R}=(R_1,......,R_m)=(0^m)$ and $k=1$.

Some authors studied the PFFCS, see [11], [12], [13] and Abou-Elheggag et al. [14].

The likelihood function of PFFCS $x=(x_1,x_2,x_3,...,x_m)$, (Wu and Kus [10])

$$L(\Theta;x)=\delta k^m\prod_{i=1}^{m} f_X(x_{i:};\Theta)[1-F_X(x_{i:};\Theta)]^{k(\mathbf{R}_i+1)-1}, \quad (5)$$

where δ is constant.

The main objective of this article is the development of statistical inferences for KD under PFFCS. Thus, ML estimation is used. Additionally, ECI and EJCR for the parameters are discussed. The Bayes estimates are obtained using the Metropolis-Hastings algorithm within Gibbs sampling. Both point and interval-credible estimations of parameters are obtained using the Bayes method. The Bayes estimates results are obtained under symmetric and asymmetric loss functions. The remaining sections of the article are divided into the following sections: The ML and Bayesian point estimates for the Kumaraswamy model's parameters, reliability, and hazard rate functions are discussed in Sections 2 and 3. The ECI and EJCR for the parameters of the KD using the same censoring are discussed in Section 4. A real-data illustrative example is discussed in Section 5. In Section 6, Monte Carlo simulation studies are formulated to assess and compare the developed results. Finally, a brief conclusion is given in Section 7.

2 ML estimation

The likelihood function (5) is simplified for the given $x=\{x_1^{R_1},x_2^{R_2},x_3^{R_3}...,x_m^{R_m}\}$ and (KD) described by (1) and (2)

$$L(\Theta|\underline{x})=\delta k^m\theta^m\lambda^m\left[\prod_{i=1}^{m}x_i^{\theta-1}(1-x_i^{\theta})^{k\lambda(\mathbf{R}_i+1)-1}\right], \quad (7)$$

where $\Theta=(\lambda,\theta)$ is the parameters vector. So, reducing the natural logarithms of 7 to

$$\ell\propto m\log(\theta)+(\theta-1)\sum_{i=1}^{m}\log(x_i)+m\log(\lambda)+$$
$$\sum_{i=1}^{m}(k\lambda(1+\mathbf{R}_i)-1)\log(1-x_i^{\theta}). \quad (8)$$

By putting the partial derivatives of Eq(8) with respect to parameters vector $\Theta=(\lambda,\theta)$ to zero, the ML estimators can be obtained by soluting the following likelihood equations:

$$\frac{\partial\ell}{\partial\theta}=\sum_{i=1}^{m}\log(x_i)+\frac{m}{\theta}-\theta\sum_{i=1}^{m}(k\lambda(1+\mathbf{R}_i)-1)$$
$$\times\frac{x_i^{\theta}}{(1-x_i^{\theta})}\log(x_i)=0, \quad (9)$$

and

$$\frac{\partial\ell}{\partial\lambda}=\frac{m}{\lambda}+k\sum_{i=1}^{m}(\mathbf{R}_i+1)\log(1-x_i^{\theta})=0. \quad (10)$$

The estimate of λ can be presented from (10) is :

$$\hat{\lambda}_{ML}=\frac{-m}{k\sum_{i=1}^{m}(R_i+1)\log(1-x_i^{\hat{\theta}_{ML}})}, \quad (11)$$

substituting from (11) in (9), we get

$$\sum_{i=1}^{m}\log(x_i)+\frac{m}{\hat{\theta}_{ML}}+\sum_{i=1}^{m}((\frac{\hat{\theta}_{ML}m(1+\mathbf{R}_i)}{\sum_{i=1}^{m}(\mathbf{R}_i+1)\log(1-x_i^{\hat{\theta}_{ML}})})+1)$$
$$\times\frac{x_i^{\hat{\theta}_{ML}}\log(x_i)}{(1-x_i^{\hat{\theta}_{ML}})}=0. \quad (12)$$

The estimate of θ is obtained by solving eq(12). The ML estimate of λ can be obtained from Equation (11) by obtained of θ. Using the invariance property of maximum likelihood estimation (see Mood et al.[15]), and from Eqs (3) and (4), the maximum likelihood estimation of the S(t) and $\mathbf{H}$(t), are $\hat{S}_{ML}(t)=\left(1-t^{\hat{\theta}_{ML}}\right)^{\hat{\lambda}_{ML}}$ and $\hat{\mathbf{H}}_{ML}(t)=\frac{\hat{\theta}_{ML}\hat{\lambda}_{ML}t^{\hat{\theta}_{ML}-1}}{\left(1-t^{\hat{\theta}_{ML}}\right)}$.

3 Bayes Estimation

This section considers estimating the unknown parameters of KD(λ,θ) using Bayesian methods for a given progressive first-failure scheme. We obtained a Bayes estimate under the balanced squared error loss ($BSEL$) function and the balanced linear exponential loss (BLINEXL) function. According to gamma prior distributions, λ and θ are thought to be independent. So, the prior distributions of the parameters vector $\Theta=(\lambda,\theta)$ are suggested to be

$$h_{\lambda}(\lambda)\propto\lambda^{a_1-1}e^{-b_1\lambda},\ \lambda>0, \tag{13}$$

and

$$h_{\theta}(\theta)\propto\theta^{a_2-1}e^{-b_2\theta},\ \theta>0. \tag{14}$$

The corresponding posterior distribution of λ and θ is given by

$$h^*_{\lambda,\theta}(\lambda,\theta|\underline{x})=\frac{h_{\lambda}(\lambda)h_{\theta}(\theta)L(\lambda,\theta|\underline{x})}{\int_0^{\infty}\int_0^{\infty}h_{\lambda}(\lambda)h_{\theta}(\theta)L(\lambda,\theta|\underline{x})d\lambda\ d\theta}. \tag{15}$$

Under $BSEL$ function the Bayes estimate of $\Psi=\{\lambda,\theta,S(t)$ or $\mathbf{H}(t)\}$, (see Soliman et.al. [16]), is written as

$$\hat{\Psi}_{BS}=\omega\hat{\Psi}_{ML}+(1-\omega)\int_0^{\infty}\int_0^{\infty}\Psi\ h^*_{\lambda,\theta}(\lambda,\theta|\underline{x})d\lambda\ d\theta, \tag{16}$$

where $\hat{\Psi}_{ML}$ is the MLE of Ψ.
Moreover, Under BLINEXL function the Bayes estimate of function Ψ (see Soliman et.al.[16]), is given by

$$\hat{\Psi}_{BL}=\frac{-1}{c}\log[\omega e^{-c\hat{\Psi}_{ML}}+(1-\omega)\int_0^{\infty}\int_0^{\infty}e^{-c\Psi}\ h^*_{\lambda,\theta}(\lambda,\theta|\underline{x})d\lambda\ d\theta. \tag{17}$$

Note that Eqs. (16) and (17) cannot be obtained analytically, therefore, we adopt the MCMC method for approximating (16) and (17). For this purpose, the posterior density function (15) can be rewritten as

$$h^*_{\lambda,\theta}(\lambda,\theta|\underline{x})\propto\theta^{m+a_2-1}\lambda^{m+a_1-1}e^{(-b_2\theta-b_1\lambda)}\prod_{i=1}^{m}x_i^{\theta-1}[1-x_i^{\theta}]^{k\lambda(\mathbf{R}i+1)-1}. \tag{18}$$

The conditional PDFs of the posterior distribution are given by

$$h^*_{\lambda}(\lambda|\theta,\underline{x})\propto\lambda^{m+a_1-1}e^{-\lambda(b_1-k\sum_{i=1}^{m}(\mathbf{R}i+1)log(1-x_i^{\theta}))}, \tag{19}$$

and

$$h^*_{\theta}(\theta|\lambda,\underline{x})\propto\theta^{m+a_2-1}e^{-\theta(b_2-\sum_{i=1}^{m}log(1-x_i^{\theta}))}\prod_{i=1}^{m}x_i^{\theta-1}. \tag{20}$$

The function (20) has shown that, the conditional PDF of the parameters λ is reduced to gamma distribution and (19) is proper function of the parameter θ similar to the normal distribution.
The procedures of this Metropolis-Hastings algorithm within the Gibbs sampling is executed as follows:

(1): Begin by initial $(\lambda^{(0)},\theta^{(0)})=(\hat{\lambda}_{ML},\hat{\theta}_{ML})$, and $I=1$.
(2): The value $\lambda^{(I)}$ is generated from
$Gamma\left(m+a_1,b_1-k\sum_{i=1}^{m}(\mathbf{R}_i+1)log(1-x_i^{\theta})\right)$.
(3): The value $\theta^{(I)}$ is generated from $h^*_{\theta}(\theta^{(I-1)}|\lambda^{(I)},\underline{x})$ by Metropolis-Hastings algorithms with normal $N\left(\theta^{(I-1)},v(\hat{\theta}_{ML})\right)$), the proposal distribution.
(4): Substitute $\lambda^{(I)}$ and $\theta^{(I)}$ into Eqs. (3) and (4) to calculate $S^{(I)}(t)$ and $H^{(I)}(t)$.

$$S^{(I)}(t)=\left(1-t^{\theta^{(I)}}\right)^{\lambda^{(I)}},$$

and

$$H^{(I)}(t)=\theta^{(I)}\lambda^{(I)}t^{\theta^{(I)}-1}/(1-t^{\theta^{(I)}}),\ 0<t<1.$$

(5): Put $I=I+1$.
(6): Reiterate items 2 - 5, $\mathbf{N}$ times.
(7): Choose M as the burn-in period, then Bayes estimates of $\Psi(\lambda,\theta)=\lambda$, θ, $S(t)$ and $H(t)$, in relation to BSEL and BLINEXL functions, respectively, are

$$\hat{\Psi}_{BS}=\omega\hat{\Psi}(\lambda,\theta)_{ML}+\frac{(1-\omega)}{\mathbf{N}-M}\sum_{i=M+1}^{\mathbf{N}}\Psi(\lambda^{(I)},\theta^{(I)}),$$

and

$$\hat{\Psi}_{BL}=\frac{-1}{c}\times\log\left(\omega e^{-c\hat{\Psi}(\lambda,\theta)_{ML}}+\frac{(1-\omega)}{\mathbf{N}-M}\sum_{i=M+1}^{\mathbf{N}}e^{-c\Psi(\lambda^{(I)},\theta^{(I)})}\right).$$

(8): To obtain credible intervals (CI_s) of $\Psi(\lambda,\theta)=\lambda$, θ, $S(t)$ and $\mathbf{H}(t)$, order $\theta^{(I)}$, $\lambda^{(I)}$, $S^{(I)}(t)$ and $\mathbf{H}^{(I)}(t)$, $I=M+1,M+2,...,\mathbf{N}$ as $\Psi^{(1)}<\Psi^{(2)}<\Psi^{(3)}<...<\Psi^{(\mathbf{N}-M)}$.
Hence, the $100(1-\alpha)\%$ (CI_s) of $\Psi=\lambda$, θ, $S(t)$ and $\mathbf{H}(t)$ are $(\Psi^{((\mathbf{N}-\mathbf{M})\alpha/2)},\Psi^{(\mathbf{N}-M)(1-\alpha/2))})$.

4 Interval estimation

The ECI and EJCR for the parameters of Kumaraswamy distribution under PFFCS are discussed in this section.

4.1 Exact confidence interval estimation (ECI)

We have $x = \{x_1^{\mathbf{R_1}}, x_2^{\mathbf{R_2}}, x_3^{\mathbf{R_3}} ..., x_m^{\mathbf{R_m}}\}$ are PFFCS sample from KD(λ, θ), let

$$U_i^{\mathbf{R}} = k\lambda \log(1-x_i^{\theta})^{-1},\ i = 1,2,3,...,m. \quad (22)$$

We observe that: $U_1^{\mathbf{R}} < U_2^{\mathbf{R}} < U_3^{\mathbf{R}} < ... < U_m^{\mathbf{R}}$ is the progressive censored of exponential distribution Exp(1). Take this conversion into consideration:

$$\begin{aligned} w_1 &= nU_1^{\mathbf{R}} \\ w_2 &= (n-\mathbf{R}_1-1)(U_2^{\mathbf{R}}-U_1^{\mathbf{R}}) \\ w_3 &= (n-\mathbf{R}_1-\mathbf{R}_2-2)(U_3^{\mathbf{R}}-U_2^{\mathbf{R}}) \\ &\vdots \\ w_m &= (n-\mathbf{R}_1-\cdots-\mathbf{R}_{m-1}-m+1)(U_m^{\mathbf{R}}-U_{m-1}^{\mathbf{R}}) \end{aligned} \quad (23)$$

We observed that w_1, w_2, w_3, ..., w_m were order from an exponential distribution Exp(1), see Soliman et al. [11]. Hence,

$$\Omega_1 = 2w_1 = 2nU_1^{\mathbf{R}}, \quad (24)$$

where $\Omega_1 \sim \chi^2(2)$, and

$$\Omega_2 = 2\sum_{i=2}^{m} w_i = 2\sum_{i=1}^{m}(\mathbf{R}_i+1)(U_i^{\mathbf{R}}-U_1^{\mathbf{R}}), \quad (25)$$

where $\Omega_2 \sim \chi^2(2m-2)$, In addition, obvious that θ_1 and θ_2 are independent.

$$\eta_1(\theta) = \frac{\Omega_2}{(m-1)\Omega_1} = \frac{\sum_{i=1}^m(\mathbf{R}_i+1)(U_i^{\mathbf{R}}-U_1^{\mathbf{R}})}{n(m-1)U_{1:m:n:k}^{\mathbf{R}}}$$
$$= \frac{\sum_{i=1}^m(\mathbf{R}_i+1)\left[\log(1-x_{i:m:n:k}^{\alpha})-\log(1-x_{1:m:n:k}^{\alpha})\right]}{n(m-1)\left(\log(1-x_{1:m:n:k}^{\alpha})\right)}, \quad (26)$$

and

$$\eta_2(\lambda,\theta) = \Omega_1+\Omega_2 = 2\sum_{i=1}^{m}(\mathbf{R}_i+1)U_i^{\mathbf{R}}. \quad (27)$$

We can show that $\eta_1(\theta) \sim F(2m-2,2)$, and $\eta_2(\lambda,\theta) \sim \chi^2(2m)$. Moreover, $\eta_1(\alpha)$ and $\eta_2(\lambda,\theta)$ are independent.

We require this lemma in order to get ECI for θ and EJCR for λ and θ.
Lemma. Assume that $0 < x_1 < x_2 < ... < x_m$,

$$\eta_1(\theta) = \frac{\sum_{i=1}^m(\mathbf{R}_i+1)\left[\log(1-x_i^{\theta})-\log(1-x_1^{\theta})\right]}{n(m-1)\left(\log(1-x_1^{\theta})\right)}$$

Then, $\eta_1(\theta)$ is strictly increasing for any $\theta > 0$. Also, if $t > 0$, the equation $\eta_1(\theta) = t$ has a unique solution for any $\theta > 0$.
Proof. $\eta_1(\alpha)$ writing it as

$$\eta_1(\theta) = \frac{1}{n(m-1)}\sum_{i=1}^{m}(\mathbf{R}_i+1)\left(\frac{\log(1-x_i^{\theta})}{\log(1-x_1^{\theta})}-1\right).$$

Hanieh [17] showed that $\log(1-x_i^{\theta})/\log(1-x_1^{\theta})$ is increasing of θ. Thus, obvious that $\eta_1(\theta)$ is a strictly increasing in θ. (ECI) for θ can be got using the following theorem.

Theorem 1. Let $x = \{x_1^{\mathbf{R_1}}, x_2^{\mathbf{R_2}}, x_3^{\mathbf{R_3}} ..., x_m^{\mathbf{R_m}}\}$ are PFFCS from KD(λ,θ) and with censored scheme $\mathbf{R}$. The $100(1-\alpha)\%$ ECI for θ equal

$$G(X^{\mathbf{R}}, F_{1-\frac{\alpha}{2}}(2m-2,2)) < \theta < G(X^{\mathbf{R}}, F_{\frac{\alpha}{2}}(2m-2,2)), \quad (28)$$

hence

$$\frac{\sum_{i=1}^m(\mathbf{R}_i+1)\left[\log(1-x_i^{\theta})-\log(1-x_1^{\theta})\right]}{n(m-1)\left(\log(1-x_1^{\theta})\right)} = t,$$

the solution for θ of this equation is $G(X^R,t)$.

Proof.
From (26), we have

$$\eta_1(\theta) = \frac{\sum_{i=1}^m(\mathbf{R}_i+1)\left[\log(1-x_i^{\theta})-\log(1-x_1^{\theta})\right]}{n(m-1)\left(\log(1-x_1^{\theta})\right)},$$

where $\eta_1(\theta) \sim F(2m-2,2)$. By Lemma, we can observe that $\eta_1(\theta) = t$ has a unique solution with any $\theta > 0$. Then we have

$$F_1 < \frac{\sum_{i=1}^m(\mathbf{R}_i+1)\left[\log(1-x_{i:m:n:k}^{\theta})-\log(1-x_{1:m:n:k}^{\theta})\right]}{n(m-1)\left(\log(1-x_{1:m:n:k}^{\theta})\right)} < F_2,$$

where $F_1 = F_{1-\frac{\alpha}{2}}(2m-2,2)$, and $F_2 = F_{\frac{\alpha}{2}}(2m-2,2)$, then obtain

$$G(X^{\mathbf{R}}, F_{1-\frac{\alpha}{2}}(2m-2,2)) < \theta < G(X^{\mathbf{R}}, F_{\frac{\alpha}{2}}(2m-2,2)).$$

This completes the proof.
We observe that the ECI for θ does not depend on k. So, the ECI for θ is similar to progressive type II censored (Hanieh [17]) and PFFCS. We can get EJCR for the parameters λ and θ by the following theorem.

Theorem 2.
Assume $x = \{x_1^{\mathbf{R}}, x_2^{\mathbf{R}}, x_3^{\mathbf{R}} ..., x_m^{\mathbf{R}}\}$ are PFFCS from KD(λ,θ) with censored scheme $\mathbf{R}$. The $100(1-\alpha)\%$ EJCR for λ and θ becomes

$$\begin{cases} \dfrac{\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m)}{\eta_3} < \lambda < \dfrac{\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m)}{\eta_3} \\ G(X^{\mathbf{R}},F_3) < \theta < G(X^{\mathbf{R}},F_4). \end{cases} \quad (29)$$

where $F_3 = F_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2,2)$, $F_4 = F_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2,2)$, $\eta_3(\theta) = 2k\sum_{i=1}^m(\mathbf{R}_i+1)\log(1-x_i^{\theta})^{-1}$ and $G(X^R,t)$ is defined in Theorem 1.

Proof.
From (27), We have

$$\eta_2(\lambda,\theta)=2\sum_{i=1}^{m}(\mathbf{R}_i+1)U_i^R=2k\lambda\sum_{i=1}^{m}(R_i+1)\log(1-x_i^{\theta})^{-1},$$

where $\eta_2(\lambda,\theta)\sim\chi^2(2m)$, $\eta_1(\theta)$ and η_2 are independent. Next, we have

$$P(F_3<\eta_1(\theta)<F_4)=\sqrt{1-\alpha},$$

and

$$P\left(\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m)<\eta_2(\lambda,\theta)<\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m)\right)=\sqrt{1-\alpha},$$

where $F_3=F_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2,2)$, $F_4=F_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2,2)$, then we can obtain

$$P\left\{\begin{array}{c}F_3<\eta_1(\theta)<F_4,\\ \chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m)<\eta_2(\lambda,\theta)<\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m)\end{array}\right\}=1-\alpha,$$

then we have

$$P\left\{\begin{array}{c}G(X^{\mathbf{R}},F_3)<\theta<G(X^{\mathbf{R}},F_4),\\ \dfrac{\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m)}{\eta_3}<\lambda<\dfrac{\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m)}{\eta_3}\end{array}\right\}=1-\alpha.$$

where $\eta_3(\theta)=2k\sum_{i=1}^{m}(\mathbf{R}_i+1)\log(1-x_i^{\theta})^{-1}$. This concludes the evidence.

4.2 Approximate interval estimation

We must determine the variance of $S(t)$ and $\mathbf{H}(t)$ in order to create the asymptotic confidence interval. To get approximations of the variances $S(t)$ and $\mathbf{H}(t)$, we use the delta method, see Greene [18]. Let $\acute{G}_1=\left(\frac{\partial \mathbf{S}(t)}{\partial\theta},\frac{\partial \mathbf{S}(t)}{\partial\lambda}\right)$ and $\acute{G}_2=\left(\frac{\partial \mathbf{H}(t)}{\partial\theta},\frac{\partial \mathbf{H}(t)}{\partial\lambda}\right)$, where

$$\frac{\partial \mathbf{S}(t)}{\partial\theta}=-\lambda t^{\theta}\log(t)(1-t^{\theta})^{\lambda-1}, \quad (30)$$

$$\frac{\partial \mathbf{S}(t)}{\partial\lambda}=(1-t^{\theta})^{\lambda}\log(1-t^{\theta}), \quad (31)$$

$$\frac{\partial \mathbf{H}(t)}{\partial\theta}=\frac{\lambda t^{\theta-1}}{1-t^{\theta}}+\frac{\theta\lambda t^{\alpha-1}\log(t)}{1-t^{\theta}}[1+\frac{t^{\theta}}{1-t^{\theta}}], \quad (32)$$

$$\frac{\partial \mathbf{H}(t)}{\partial\lambda}=\frac{\theta t^{\theta-1}}{1-t^{\theta}}. \quad (33)$$

Then

$$\widehat{Var}(\hat{\mathbf{S}})\simeq\left[\acute{G}_1\mathbf{I}^{-1}G_1\right]_{(\alpha,\beta)=(\hat{\alpha}_{ML},\hat{\beta}_{ML})}, \quad (34)$$

and

$$\widehat{Var}(\hat{\mathbf{H}})\simeq\left[\acute{G}_2\mathbf{I}^{-1}G_2\right]_{(\alpha,\beta)=(\hat{\alpha}_{ML},\hat{\beta}_{ML})}. \quad (35)$$

Thus, both $\frac{(\hat{\mathbf{S}}(t)-\mathbf{S}(t))}{\sqrt{\widehat{Var}(\hat{\mathbf{S}})}}$ and $\frac{(\hat{\mathbf{H}}(t)-\mathbf{H}(t))}{\sqrt{\widehat{Var}(\hat{\mathbf{H}})}}\sim N(0,1)$. Then yields the approximate confidence intervals (ACI_s)for S(t) and $\mathbf{H}$(t) become:

$$\hat{\mathbf{S}}(t)\pm Z_{\frac{\gamma}{2}}\sqrt{\widehat{Var}(\hat{\mathbf{S}})} \text{ and } \hat{\mathbf{H}}(t)\pm Z_{\frac{\gamma}{2}}\sqrt{\widehat{Var}(\hat{\mathbf{H}})}. \quad (36)$$

5 Real Data Analysis

In this section, real-life data are presented for illustrative purposes. These data were presented by Dasgupta [19]. Dey et al. [20] proved that the Kumaraswamy distribution is a better model for this data. The analytical methods are presented in the previous parts. Based on complete sample (k=1, n=50, m=50, $\mathbf{R}=(R_1,R_2,R_3,\ldots\ldots,R_m)=(0^m)$), say x$^{(0)}$.Using the ML and MCMC estimates, of λ,θ, S(t=0.2) and H(t=0.2) presented in Table 2. In order to demonstrate the strategies discussed in this example, with $n=25$, $m=20$ $k=2$ and $R_1=5$, $R_2=\cdots=R_{20}=0$, a progressively first-failure scheme (PFFCS) has been generated from data in Table 1. Progressive first-failure censoring data is:
x$^{(1)}$ = {0.02, 0.02, 0.04, 0.06, 0.06, 0.08, 0.08, 0.08, 0.08, 0.12, 0.12, 0.14, 0.14, 0.14, 0.14, 0.16, 0.16, 0.18, 0.24, 0.28}. The ML and Bayes estimates under ($BSEL$) *and* ($BLINEXL$) functions, of α, β, $S(t=0.2)$ and $H(t=0.2)$ by $\underline{x}^{(1)}$ are also listed in Table2. The 95% (ACI_s) and CI_s of (of λ,θ, $S(0.2)$ and $H(0.2)$. are computed in Table3.
To find a 95% exact confidence interval for α.
By Theorem 1, 95% ECI of θ under x$^{(0)}$ is (1.73519, 4.52735) and 95% ECI of θ under x$^{(1)}$ is (1.04382, 3.34865).
Moreover, using Theorem 2, the 95% EJCR of λ,θ for x$^{(0)}$ is:

$$\left\{\begin{array}{c}1.63381<\theta<4.89233\\ \dfrac{71.0694}{2\sum_{i=1}^{50}(\mathbf{R}_i+1)\log(1-x_i^{\theta})^{-1}}<\lambda<\dfrac{134.257}{2\sum_{i=1}^{50}(\mathbf{R}_i+1)\log(1-x_i^{\theta})^{-1}},\end{array}\right.$$

and the 95% EJCR of λ,θ for x$^{(1)}$ is determined by the following inequalities:

$$\left\{\begin{array}{c}0.957383<\theta<3.64148\\ \dfrac{22.7065}{4\sum_{i=1}^{20}(\mathbf{R}_i+1)\log(1-x_i^{\theta})^{-1}}<\lambda<\dfrac{62.6056}{4\sum_{i=1}^{20}(\mathbf{R}_i+1)\log(1-x_i^{\theta})^{-1}}.\end{array}\right.$$

Table 1:The real data set presented in Dasgupta [19].

0.04	0.02	0.06	0.12	0.14	0.08	0.22	0.12	0.08
0.26	0.24	0.04	0.14	0.16	0.08	0.26	0.32	0.28
0.14	0.16	0.24	0.22	0.12	0.18	0.24	0.32	0.16
0.14	0.08	0.16	0.24	0.16	0.32	0.18	0.24	0.22
0.16	0.12	0.24	0.06	0.02	0.18	0.22	0.14	0.06
0.04	0.14	0.26	0.18	0.16				

Table 3: The 95% (ACI_s) and CI_s of (of λ, θ, $S(0.2)$ and $H(0.2)$.

	Parameter	θ	λ
$x^{(1)}$	MCMC	$(1.1958, 2.4495)$	$(5.2601, 61.9888)$
$x^{(0)}$		$(1.5379, 2.5977)$	$(13.1508, 73.1512)$
		$S(t)$	$H(t)$
$x^{(1)}$	MLE	$(0.1771, 0.5113)$	$(3.7832, 16.2598)$
$x^{(0)}$		$(0.2023, 0.4053)$	$(8.4189, 16.7819)$
$x^{(1)}$	MCMC	$(0.2047, 0.5353)$	$(4.8193, 16.9574)$
$x^{(0)}$		$(0.2116, 0.4156)$	$(8.5512, 16.983)$

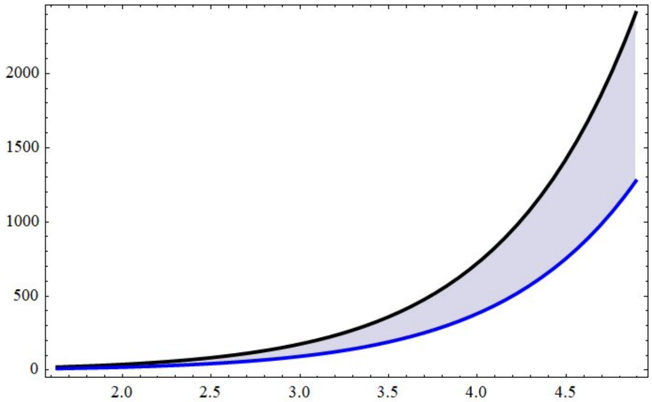

Figure 1: Joint confidence region based on progressive first failure ($x^{(0)}$) for λ *and* θ.

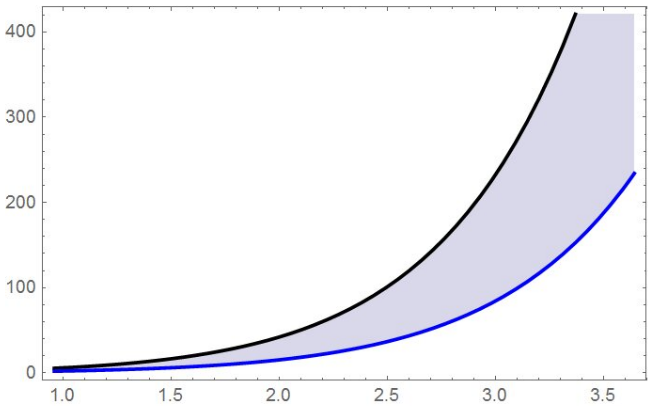

Figure 2: EJCR of based on progressive first failure (PFFC) ($x^{(1)}$) for λ *and* θ.

Figures 1 and 2 show EJCR *for* λ *and* θ. When θ is large, it is simple to observe that the confidence region is large.
Moreover, area of EJCR of λ *and* θ can be obtain by computing the integration:

$$\int_{1.63381}^{4.89233} \frac{134.2557 - 71.0694}{2\sum_{i=1}^{50}(\mathbf{R}_i + 1)\log(1 - x_{i:50:50:1}^{\alpha})^{-1}} d\alpha,$$

and

$$\int_{0.957383}^{3.64148} \frac{62.6056 - 22.7065}{4\sum_{i=1}^{20}(\mathbf{R}_i + 1)\log(1 - x_{i:20:25:2}^{\alpha})^{-1}} d\alpha.$$

Then, area of EJCR of λ *and* θ by $x^{(0)}$ *and* $x^{(1)}$, respectively, equals 822.016 and 250.398.

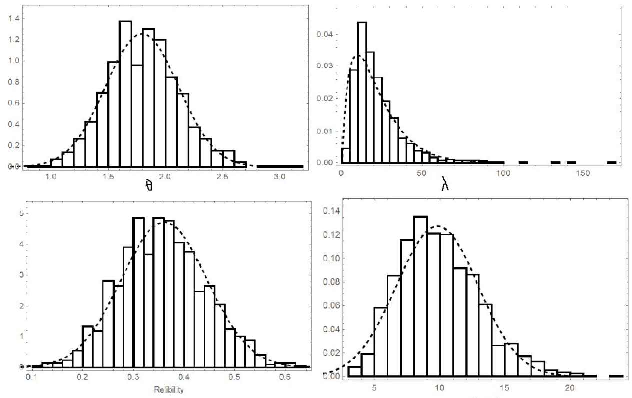

Figure 3: Histogram for λ, θ , $S(t)$ *and* $H(t)$ for $x^{(1)}$, generated by MCMC.

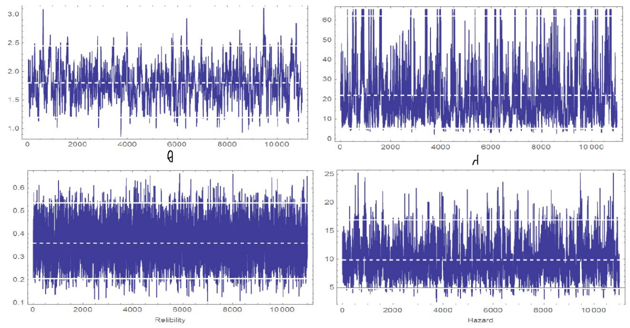

Figure 4: Trace plots for λ, θ, $S(t)$ and $H(t)$ for $x^{(1)}$, generated by MCMC.

For Bayesian estimation, we run the Metropolis-Hastings algorithm within the Gibbs sampling 11,000 with the first 1000 value as brun-in. Using trace plots, the convergence is observed, from Figs. 3 and 4. For ($a_1 = a_2 = b_1 = b_2 = 0$), which occurs when the hyperparameters are 0, non-informative priors *for* λ *and* θ. Outcomes of MCMC estimates with respect *BSEL and BLINEXL* functions are shown in Table2, These estimates used various values of (c) of the *BLINEXL* function, and varied of ω.

6 Simulation study

To evaluate the performance of the suggested methods, some computations in accordance with Monte Carlo simulation experiments are performed utilizing (MATHEMATICA ver.12.0). Applying the algorithm offered by [21], with distribution $1-(1-F(x))^k$, *we* generated PFFC samples from KD(λ =1.5 ,θ=2.1) distribution. For the informative prior, we choose $a_1=b_1=1$, $a_2=b_2=2$. We compute the MLE of parameters by Eqs (8)-(11). Furthermore, the MCMC method is calculated by constructing a Markov chain that converges to the stationary distribution and free with free from the effect of the values initially selected, it will not appear as abnormal parameter estimates. By using 11,000 MCMC samples, we obtained Bayes estimates of parameters for four distinct values of $\omega = 0, 0.3, 0.6, 0.9$, $c = -2$, 2, with $k = 2$, and with various schemes. We calculate the average mean (Mean) values and mean square error (MSE) from all estimations, which results in Tables 4-7. Furthermore, Table 8 gives 95% CIs of λ, *and* θ based on MCMC, ECI by (Theorem 1) *and* (EJCR) by (Theorem 2), also the 95% (CIs) for *S*(t) and H(t) *under* asymptotic normality (AN) *and* MCMC procedures are summarized in Table 8.
According to the simulation results, We observe the following:

1. As sample size rises, the mean squared error of MLE estimators and Bayes estimates decrease, as shown in Tables 4 through 7.
2. The mean squared errors for the Bayes estimate for λ, θ, $S(t)$, and $H(t)$ are the lowest.
3. In terms of λ, θ, in addition to the survival function and hazard rate function, the symmetric Bayes estimates (under BSEL) function and asymmetric Bayes estimates (under BLINEXL) function are generally better than MLE estimates according to the MSE, which makes them more attractive for use in practical problems.

4. Also, it is clear from the results, in all cases, as *c* tends to zero, MSEs under the (BLINEXL) are the same using the (BSEL) function.
5. Finally, from Table 8, the recommended (CIs), (ECI) *and* (EJCR) coverage probability is close to the required level (0.95).

7 Conclusions

In this paper, we devised a variety of techniques to discuss estimating the parameters of Kumaraswamy distribution based on PFFCS. We obtained MLEs and Bayes estimates for parameters. Furthermore, we created ECI and EJCR of parameters. To get Bayes estimates for parameters, the Metropolis-Hastings algorithm within the Gibbs Technique is used. Under the BSEL and BLINEXL functions, the Bayes estimates are obtained. Numerical data analysis and Monte Carlo simulation study are studied to illustrate the use of suggested point and interval estimators. Finally, the numerical results have shown that estimation results under an informative prior distribution in the Bayes approach are superior to likelihood estimates.

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Table 2: The ML *and* MCMC estimates.

Parameter		*MLE*	ω	*BSEL*	BLINEXL			
					c =-0.05	c =0.05	c =-0.02	c =0.02
λ	$x^{(1)}$	19.7012	0	21.6695	71.8511	17.8358	25.8535	19.7271
	$x^{(0)}$	33.1374		34.1566	43.5957	29.6331	36.9502	32.0468
	$x^{(1)}$		0.3	21.079	65.3397	18.3774	24.0859	19.7193
	$x^{(0)}$			33.8508	40.9898	30.6215	35.8366	32.3715
	$x^{(1)}$		0.6	20.4885	55.6229	18.9341	22.2537	19.7115
	$x^{(0)}$			33.5451	37.9928	31.6612	34.6976	32.6983
	$x^{(1)}$		0.9	19.898	35.9775	19.5067	20.3516	19.7038
	$x^{(0)}$			33.2393	34.4662	32.758	33.5321	33.0273
θ	$x^{(1)}$	1.8287	0	1.794	1.7965	1.7916	1.795	1.793
	$x^{(0)}$	2.0774		2.0465	2.0482	2.0448	2.0472	2.0458
	$x^{(1)}$		0.3	1.8044	1.8062	1.8027	1.8051	1.8037
	$x^{(0)}$			2.0558	2.057	2.0545	2.0563	2.0553
	$x^{(1)}$		0.6	1.8148	1.8158	1.8138	1.8152	1.8144
	$x^{(0)}$			2.065	2.0657	2.0643	2.0653	2.0648
	$x^{(1)}$		0.9	1.8252	1.8254	1.8249	1.8253	1.8251
	$x^{(0)}$			2.0743	2.0745	2.0741	2.0744	2.0742
$S(t)$	$x^{(1)}$	0.3442	0	0.3603	0.3604	0.3601	0.3603	0.3602
	$x^{(0)}$	0.3038		0.3087	0.3087	0.3086	0.3087	0.3086
	$x^{(1)}$		0.3	0.3554	0.3556	0.3553	0.3555	0.3554
	$x^{(0)}$			0.3072	0.3073	0.3072	0.3072	0.3072
	$x^{(1)}$		0.6	0.3506	0.3507	0.3505	0.3506	0.3506
	$x^{(0)}$			0.3057	0.3058	0.3057	0.3058	0.3057
	$x^{(1)}$		0.9	0.3458	0.3458	0.3458	0.3458	0.3458
	$x^{(0)}$			0.3043	0.3043	0.3043	0.3043	0.3043
$H(t)$	$x^{(1)}$	10.0215	0	9.8005	10.051	9.5683	9.8984	9.7056
	$x^{(0)}$	12.6004		12.387	12.5045	12.2726	12.4336	12.3409
	$x^{(1)}$		0.3	9.8668	10.0422	9.7032	9.9354	9.8001
	$x^{(0)}$			12.451	12.5333	12.3704	12.4837	12.4186
	$x^{(1)}$		0.6	9.9331	10.0333	9.839	9.9723	9.8949
	$x^{(0)}$			12.5151	12.5621	12.4686	12.5338	12.4965
	$x^{(1)}$		0.9	9.9994	10.0245	9.9757	10.0092	9.9898
	$x^{(0)}$			12.5791	12.5908	12.5674	12.5838	12.5744

Table 4: Mean and MSE of λ and θ, with $\omega = 0, 0.3$.

n	m	*Scheme*	Parameter	*MLE*	MCMC (ω =0)			MCMC (ω=0.3)		
					BSEL	BLINEXL		*BSEL*	BLINEXL	
						c= -2	c= 2		c= -2	c = 2
25	20	$(5,19^{0})$	λ	1.8797 0.5583	1.4109 0.12003	1.7302 0.37933	1.2385 0.13518	1.5516 0.27582	1.8111 0.80832	1.3361 0.11275
			θ	2.2852 0.2859	1.9795 0.11458	2.1615 0.12826	1.8223 0.16443	2.0712 0.15344	2.211 0.20749	1.9155 0.13939
		$(9^{0},5,10^{0})$	λ	1.9277 0.569	1.4019 0.12184	1.7543 0.39923	1.2203 0.14038	1.5596 0.27985	1.8541 0.88379	1.3228 0.11745
			θ	2.2924 0.29918	1.9694 0.11534	2.1431 0.12996	1.815 0.15734	2.0663 0.13769	2.2029 0.18885	1.9104 0.12807
		$(19^{0},5)$	λ	1.9325 0.6517	1.4181 0.12908	1.7556 0.41534	1.2411 0.1358	1.5725 0.28596	1.8523 0.81172	1.3417 0.11939
			θ	2.3156 0.38558	1.9724 0.14139	2.1692 0.17267	1.8023 0.18308	2.0754 0.17642	2.2295 0.24952	1.9011 0.15349
40	35	$(5,34^{0})$	λ	1.6823 0.30997	1.4535 0.10120	1.6391 0.20351	1.3294 0.09182	1.5222 0.14016	1.6559 0.24451	1.4029 0.09158
			θ	2.2141 0.14035	2.0364 0.08617	2.1541 0.09183	1.9288 0.10249	2.0897 0.10826	2.1747 0.10890	1.9963 0.09358
		$(17^{0},5,17^{0})$	λ	1.7157 0.31711	1.4652 0.09891	1.6664 0.21008	1.3329 0.09224	1.5403 0.14136	1.6842 0.24587	1.4115 0.09224
			θ	2.1948 0.14831	2.0107 0.08755	2.1205 0.09374	1.9087 0.10460	2.0659 0.096438	2.1457 0.10988	1.9765 0.094242
		$(34^{0},5)$	λ	1.6689 0.31987	1.4386 0.097504	1.6224 0.20376	1.3157 0.1011	1.5077 0.14696	1.6401 0.24597	1.3888 0.09944
			θ	2.1947 0.16010	2.0126 0.094366	2.1326 0.10414	1.903 0.11216	2.0673 0.10468	2.154 0.12116	1.9718 0.10148
50	45	$(5,44^{0})$	λ	1.6623 0.18858	1.4784 0.08169	1.6283 0.13184	1.3711 0.08796	1.5336 0.10334	1.6399 0.15339	1.4362 0.07714
			θ	2.1977 0.10055	2.0502 0.074399	2.1469 0.078964	1.9601 0.086626	2.0944 0.079695	2.1637 0.11132	2.0188 0.079547
		$(22^{0},5,22^{0})$	λ	1.6406 0.19750	1.4652 0.08245	1.6145 0.14473	1.3576 0.07749	1.5178 0.10623	1.6235 0.16187	1.4222 0.07848
			θ	2.1581 0.10796	2.02 0.075726	2.1092 0.07906	1.9364 0.08768	2.0614 0.080375	2.1251 0.087358	1.9921 0.080572
		$(44^{0},5)$	λ	1.6623 0.24858	1.4784 0.10169	1.6283 0.18184	1.3711 0.08796	1.5336 0.13334	1.6399 0.20339	1.4362 0.09414
			θ	2.1977 0.14055	2.0502 0.08640	2.1469 0.098964	1.9601 0.09263	2.0944 0.096695	2.1637 0.11132	2.0188 0.089847
60	40	$(20,39^{0})$	λ	1.6567 0.18293	1.4673 0.07924	1.6256 0.12837	1.3555 0.07919	1.5241 0.09198	1.6375 0.14547	1.4211 0.06965
			θ	2.1733 0.09598	2.0333 0.066154	2.1282 0.07029	1.946 0.07535	2.0753 0.075075	2.1435 0.085454	2.0025 0.07143
		$(19^{0},20,20^{0})$	λ	1.7111 0.20873	1.4722 0.11621	1.6648 0.24736	1.3441 0.09945	1.5439 0.17360	1.6839 0.30959	1.418 0.10351
			θ	2.1957 0.11839	2.0412 0.067562	2.1274 0.073323	1.9595 0.076419	2.0876 0.076018	2.1504 0.086461	2.0178 0.07228
		$(39^{0},20)$	λ	1.7389 0.21697	1.4426 0.10607	1.6888 0.26094	1.2972 0.10509	1.5315 0.17229	1.7128 0.34844	1.38 0.10013
			θ	2.2028 0.13645	2.0114 0.07498	2.1102 0.076813	1.9188 0.09153	2.0688 0.085544	2.1427 0.09782	1.9857 0.08297

Table 5: Mean and MSE of λ and θ, with $\omega = 0.6, 0.9$.

n	m	Scheme	Parameter	MCMC ($\omega = 0.6$)			MCMC ($\omega = 0.9$)		
				BSEL	BLINEXL		BSEL	BLINEXL	
					$c=-2$	$c=2$		$c=-2$	$c=2$
25	20	$(5,19^0)$	λ	1.6922 0.54049	1.8464 0.64419	1.4665 0.134483	1.8328 0.60404	1.8723 0.64458	1.6815 0.32196
			θ	2.1629 0.18595	2.2472 0.22661	2.0339 0.12168	2.2546 0.25411	2.2765 0.27156	2.2024 0.20241
		$(9^0,5,10^0)$	λ	1.7174 0.57887	1.8927 0.64442	1.4607 0.13599	1.8751 0.61889	1.9201 0.66278	1.6919 0.33324
			θ	2.1632 0.18815	2.2463 0.23902	2.0319 0.12568	2.2601 0.26674	2.2818 0.28469	2.2058 0.20437
		$(19^0,5)$	λ	1.7268 0.56708	1.8942 0.97696	1.4768 0.14631	1.8811 0.97244	1.9241 0.98963	1.7039 0.35879
			θ	2.1783 0.24422	2.2719 0.31244	2.0286 0.15486	2.2813 0.34478	2.3057 0.36815	2.2166 0.25679
40	35	$(5,34^0)$	λ	1.5908 0.20655	1.6685 0.28254	1.4938 0.11856	1.6594 0.28838	1.6791 0.31109	1.6184 0.20863
			θ	2.143 0.13912	2.1928 0.15749	2.0759 0.11567	2.1963 0.16875	2.209 0.17469	2.1746 0.15441
		$(17^0,5,17^0)$	λ	1.6155 0.20354	1.699 0.27802	1.5099 0.11966	1.6906 0.28543	1.7118 0.30767	1.6478 0.22324
			θ	2.1211 0.11333	2.1681 0.12631	2.0567 0.09758	2.1764 0.13823	2.1884 0.14281	2.1555 0.12742
		$(34^0,5)$	λ	1.5768 0.20820	1.6539 0.28026	1.4793 0.12215	1.6458 0.28874	1.6654 0.31045	1.6045 0.21533
			θ	2.1219 0.123056	2.1727 0.13802	2.0532 0.10512	2.1765 0.14949	2.1895 0.15462	2.1542 0.13720
50	45	$(5,44^0)$	λ	1.5654 0.13463	1.6654 0.17346	1.5451 0.09566	1.6654 0.17857	1.6457 0.18245	1.6231 0.14410
			θ	2.2456 0.07219	2.1654 0.08481	2.0654 0.07298	2.1654 0.09458	2.1451 0.10126	2.1432 0.09321
		$(22^0,5,22^0)$	λ	1.5704 0.139207	1.6314 0.17771	1.4994 0.09688	1.623 0.18139	1.6384 0.19266	1.5983 0.15459
			θ	2.1028 0.08933	2.1399 0.09602	2.0557 0.08209	2.1443 0.10258	2.1537 0.10494	2.13 0.09782
		$(44^0,5)$	λ	1.5887 0.17563	1.6502 0.22346	1.5146 0.12066	1.6439 0.22857	1.6594 0.24245	1.6172 0.19410
			θ	2.1387 0.11209	2.179 0.12381	2.0864 0.09798	2.1829 0.13258	2.1932 0.13636	2.1669 0.12482
60	40	$(20,39^0)$	λ	1.581 0.13033	1.6467 0.16219	1.5004 0.08895	1.6378 0.16826	1.6544 0.17562	1.6057 0.13087
			θ	2.1173 0.07075	2.1571 0.08137	2.0673 0.0693308	2.1593 0.09386	2.1695 0.09810	2.144 0.08673
		$(19^0,20,20^0)$	λ	1.6155 0.13544	1.6971 0.17184	1.5098 0.09507	1.6872 0.17875	1.7079 0.18771	1.6391 0.14890
			θ	2.1339 0.08030	2.171 0.09305	2.0849 0.0786215	2.1802 0.10040	2.1898 0.10080	2.1649 0.09331
		$(39^0,20)$	λ	1.6204 0.16455	1.7265 0.20561	1.4855 0.106102	1.7092 0.21286	1.7362 0.22267	1.6415 0.20495
			θ	2.1262 0.10543	2.1706 0.11882	2.0649 0.0885318	2.1837 0.11464	2.1952 0.11959	2.1632 0.10247

Table 6: Mean and MSE of $S(0.2)$=0.94936 and $H(0.2)$=0.55525, with $\omega = 0, 0.3$.

n	m	Scheme	Parameter	MLE	MCMC ($\omega = 0$)			MCMC ($\omega = 0.3$)		
					BSEL	BLINEXL		BSEL	BLINEXL	
						$c=-2$	$c=2$		$c=-2$	$c=2$
25	20	$(5,19^{0})$	$S(t)$	0.9505	0.9356	0.9364	0.9348	0.9401	0.9407	0.9394
				0.00082	0.00075	0.00072	0.00079	0.00066	0.00064	0.00069
			$H(t)$	0.5361	0.5992	0.6363	0.5665	0.5803	0.6087	0.5567
				0.03660	0.02809	0.03593	0.02364	0.02936	0.033615	0.02696
		$(9^{0},5,10^{0})$	$S(t)$	0.9509	0.9363	0.937	0.9355	0.9406	0.9412	0.94
				0.00072	0.00065	0.00062	0.00068	0.00056	0.00054	0.00058
			$H(t)$	0.5402	0.5968	0.6299	0.5674	0.5798	0.605	0.5586
				0.03325	0.02565	0.03223	0.02180	0.02687	0.03047	0.02475
		$(19^{0},5)$	$S(t)$	0.9505	0.9336	0.9345	0.9326	0.9387	0.9394	0.9379
				0.00087	0.00086	0.00081	0.00091	0.00074	0.00071	0.00077
			$H(t)$	0.5355	0.6103	0.6508	0.575	0.5879	0.6194	0.5621
				0.04162	0.03133	0.04090	0.02580	0.03238	0.0375	0.02951
40	35	$(5,34^{0})$	$S(t)$	0.9501	0.9408	0.9413	0.9404	0.9436	0.944	0.9433
				0.00049	0.00046	0.00045	0.00048	0.00043	0.00042	0.00044
			$H(t)$	0.5413	0.5808	0.6027	0.5605	0.569	0.5852	0.5545
				0.02525	0.02126	0.02424	0.01938	0.02201	0.02362	0.02097
		$(17^{0},5,17^{0})$	$S(t)$	0.9486	0.9393	0.9398	0.9389	0.9421	0.9424	0.9418
				0.00047	0.00045	0.00044	0.00047	0.00041	0.0004	0.00042
			$H(t)$	0.5585	0.5953	0.6161	0.5761	0.5843	0.5996	0.5706
				0.02261	0.01901	0.02218	0.01687	0.01940	0.02121	0.01813
		$(34^{0},5)$	$S(t)$	0.9497	0.9397	0.9402	0.9392	0.9427	0.9431	0.9423
				0.00051	0.00048	0.00047	0.0005	0.00044	0.00043	0.00045
			$H(t)$	0.5437	0.5867	0.6095	0.5657	0.5738	0.5908	0.5588
				0.02281	0.01933	0.02245	0.01739	0.01985	0.02151	0.01880
50	45	$(5,44^{0})$	$S(t)$	0.9501	0.9422	0.9426	0.9419	0.9446	0.9449	0.9443
				0.00044	0.00040	0.00039	0.00042	0.00038	0.00038	0.00039
			$H(t)$	0.5421	0.5772	0.5949	0.5605	0.5667	0.5797	0.5548
				0.02053	0.01743	0.01935	0.01620	0.01802	0.01902	0.01737
		$(22^{0},5,22^{0})$	$S(t)$	0.948	0.9407	0.941	0.9403	0.9429	0.9431	0.9426
				0.00039	0.00038	0.00037	0.00039	0.00035	0.00034	0.00035
			$H(t)$	0.562	0.591	0.6075	0.5754	0.5823	0.5943	0.5713
				0.01975	0.01626	0.01840	0.01474	0.01647	0.01772	0.01555
		$(44^{0},5)$	$S(t)$	0.9501	0.9422	0.9426	0.9419	0.9446	0.9449	0.9443
				0.00043	0.00040	0.00039	0.00042	0.00038	0.00038	0.00039
			$H(t)$	0.5421	0.5772	0.5949	0.5605	0.5667	0.5797	0.5548
				0.02053	0.01743	0.01935	0.01620	0.01802	0.01902	0.01737
60	40	$(20,39^{0})$	$S(t)$	0.949	0.9416	0.942	0.9412	0.9438	0.9441	0.9436
				0.00039	0.00037	0.00036	0.00038	0.00035	0.00034	0.00035
			$H(t)$	0.5535	0.5827	0.601	0.5655	0.5739	0.5873	0.5618
				0.02003	0.01681	0.01909	0.01531	0.01722	0.01852	0.01633
		$(19^{0},20,20^{0})$	$S(t)$	0.9502	0.9436	0.9439	0.9433	0.9456	0.9458	0.9454
				0.00029	0.00027	0.00026	0.00028	0.00026	0.00025	0.00026
			$H(t)$	0.5487	0.571	0.5852	0.5575	0.5643	0.5746	0.5548
				0.01548	0.01338	0.01469	0.01553	0.01384	0.01458	0.01334
		$(39^{0},20)$	$S(t)$	0.9502	0.9425	0.9427	0.9422	0.9448	0.945	0.9445
				0.00029	0.00028	0.00028	0.00029	0.00027	0.00026	0.00027
			$H(t)$	0.5475	0.5728	0.5863	0.5601	0.5652	0.575	0.5562
				0.01465	0.01176	0.01291	0.01103	0.01240	0.01299	0.01200

Table 7: Mean and MSEs of $S(0.2)$=0.94936 and $H(0.2)=0.55525$ with $\omega=0.6,0.9$.

n	m	*Scheme*	Parameter	MCMC (ω =0.6)			MCMC (ω =0.9)		
				BSEL	BLINEXL		*BSEL*	BLINEXL	
					$c=-2$	$c=2$		$c=-2$	$c=2$
25	20	$(5,19^0)$	$S(t)$	0.9445 0.00065	0.9449 0.00061	0.9441 0.00063	0.949 0.00061	0.9491 0.00061	0.9489 0.00062
			$H(t)$	0.5613 0.03573	0.5791 0.03320	0.5475 0.03080	0.5424 0.03520	0.5473 0.03524	0.5388 0.03508
		$(9^0,5,10^0)$	$S(t)$	0.945 0.00055	0.9454 0.00051	0.9446 0.00052	0.9494 0.00051	0.9495 0.00051	0.9493 0.00051
			$H(t)$	0.5629 0.032	0.5785 0.03030	0.5504 0.02814	0.5459 0.03204	0.5501 0.03211	0.5427 0.03192
		$(19^0,5)$	$S(t)$	0.9437 0.00069	0.9442 0.00066	0.9433 0.00068	0.9488 0.00066	0.9489 0.00066	0.9487 0.00066
			$H(t)$	0.5654 0.03992	0.5855 0.03661	0.5502 0.03391	0.5429 0.03894	0.5486 0.03893	0.539 0.03886
40	35	$(5,34^0)$	$S(t)$	0.9464 0.00044	0.9466 0.00040	0.9462 0.00041	0.9492 0.00040	0.9492 0.00040	0.9491 0.00040
			$H(t)$	0.5571 0.02515	0.5669 0.02374	0.5487 0.02272	0.5453 0.02466	0.5479 0.02472	0.5431 0.02460
		$(17^0,5,17^0)$	$S(t)$	0.9449 0.00042	0.9451 0.00038	0.9447 0.00039	0.9477 0.00037	0.9477 0.00037	0.9476 0.00038
			$H(t)$	0.5732 0.02312	0.5825 0.02088	0.5653 0.01954	0.5622 0.02118	0.5646 0.02129	0.5602 0.02107
		$(34^0,5)$	$S(t)$	0.9457 0.00045	0.9459 0.00041	0.9454 0.00042	0.9487 0.00041	0.9487 0.00041	0.9486 0.00041
			$H(t)$	0.5609 0.02282	0.5712 0.02140	0.5522 0.02041	0.548 0.02223	0.5508 0.02227	0.5458 0.02218
50	45	$(5,44^0)$	$S(t)$	0.947 0.00039	0.9471 0.00037	0.9468 0.00037	0.9493 0.00037	0.9494 0.00037	0.9493 0.00037
			$H(t)$	0.5562 0.0249	0.564 0.01924	0.5493 0.01865	0.5456 0.02007	0.5477 0.02009	0.5439 0.02004
		$(22^0,5,22^0)$	$S(t)$	0.9451 0.00033	0.9452 0.00032	0.9449 0.00033	0.9473 0.00035	0.9473 0.00032	0.9472 0.00032
			$H(t)$	0.5736 0.01989	0.5808 0.01744	0.5672 0.01645	0.5649 0.01750	0.5668 0.01760	0.5633 0.01742
		$(44^0,5)$	$S(t)$	0.947 0.00040	0.9471 0.00037	0.9468 0.00037	0.9493 0.00037	0.9494 0.00037	0.9493 0.00037
			$H(t)$	0.5562 0.0209	0.564 0.01924	0.5493 0.01865	0.5456 0.02007	0.5477 0.02009	0.5439 0.02004
60	40	$(20,39^0)$	$S(t)$	0.946 0.00033	0.9462 0.00033	0.9459 0.00033	0.9482 0.00035	0.9483 0.00033	0.9482 0.00033
			$H(t)$	0.5652 0.01985	0.5731 0.01840	0.5582 0.01744	0.5564 0.0187	0.5585 0.01878	0.5546 0.01862
		$(19^0,20,20^0)$	$S(t)$	0.9476 0.00029	0.9477 0.00025	0.9475 0.00025	0.9496 0.00025	0.9496 0.00025	0.9495 0.00025
			$H(t)$	0.5576 0.01645	0.5637 0.01475	0.5521 0.01422	0.5509 0.01520	0.5525 0.01524	0.5496 0.01515
		$(39^0,20)$	$S(t)$	0.9471 0.00030	0.9472 0.00026	0.9469 0.00026	0.9494 0.00027	0.9494 0.00027	0.9493 0.00027
			$H(t)$	0.5576 0.01523	0.5635 0.01343	0.5524 0.01307	0.55 0.01426	0.5516 0.01427	0.5487 0.01424

Table8: 95% *coverage probabilities* of θ, λ, S(0.2)=0.94936 and H(0.2) =0.55525.

n	m	Scheme	λ	θ		(θ,λ)	S(0.2)		H(0.2)	
			MCMC	MCMC	By Theorem 1	By Theorem 2	AN	MCMC	AN	MCMC
25	20	$(5,19^0)$	0.959	0.962	0.978	0.959	0.871	0.967	0.871	0.976
		$(9^0,5,10^0)$	0.974	0.97	0.974	0.953	0.886	0.969	0.886	0.973
		$(19^0,5)$	0.969	0.967	0.984	0.944	0.852	0.966	0.852	0.973
40	35	$(5,34^0)$	0.958	0.946	0.972	0.948	0.884	0.94	0.884	0.948
		$(17^0,5,17^0)$	0.962	0.958	0.984	0.87	0.912	0.954	0.912	0.968
		$(34^0,5)$	0.952	0.95	0.976	0.95	0.906	0.952	0.906	0.962
50	45	$(5,44^0)$	0.956	0.96	0.966	0.954	0.916	0.96	0.916	0.966
		$(22^0,5,22^0)$	0.972	0.95	0.986	0.956	0.934	0.946	0.934	0.952
		$(44^0,5)$	0.96	0.936	0.962	0.926	0.896	0.93	0.896	0.936
60	40	$(20,39^0)$	0.948	0.956	0.958	0.934	0.918	0.956	0.918	0.95
		$(19^0,20,20^0)$	0.97	0.962	0.976	0.944	0.92	0.962	0.92	0.962
		$(39^0,20)$	0.97	0.966	0.988	0.958	0.92	0.958	0.92	0.96