Thermodynamic Functions of Kerr Black Holes Using Hamiltonian Equation

Najat Abdel-Rahman Hussien[1], Dalia Ahmed Eisa[2], Zeyad Ali[2*]

[1] Department of Mathematics, Faculty of Science, Assiut University, Assiut, Egypt

[2] Department of Mathematics, Faculty of Science, New Valley University, Al-Kharga, Al-Wadi Al-Gadid, 72511 Egypt.

Doi: 10.21608/nujbas.2024.295034.1025

Abstract

This paper focuses on calculating key thermodynamic functions—temperature, internal energy, enthalpy, and Helmholtz free energy—of uncharged rotating black holes using the Hamiltonian equation. Analytical mechanics methods, paired with general relativity and thermodynamics, are employed to predict the behavior of these functions by studying the motion of charged particles near a Kerr black hole. The calculations were conducted using Mathematica, providing precise results along with graphical representations of the thermodynamic functions. Planck units are used throughout the analysis to simplify the equations. The findings align with the behavior predicted by Hawking radiation, as both follow similar patterns, suggesting a deeper connection between black hole thermodynamics and radiation. This paper offers a theoretical framework to understand how thermodynamic functions behave within black holes, contributing valuable insights into the role of thermodynamic principles in describing the dynamic properties of black hole systems and enhancing our understanding of black hole physics. shedding light on the variability of this method.

Keywords: Kerr Black hole; Thermodynamics; Hamiltonian; Hawking radiation; Temperature; Internal energy; Enthalpy; Helmholtz Free energy.

Introduction

Supergravity cosmological systems and black holes have always been topics that fascinated researchers. These are two fundamental predictions of General relativity theory; those topics were confirmed by numerous observations. Since the discovery of the first black hole, (Kbh) hold the attributes such as mass besides angular momentum but are not electrically charged [1], [2], [3].

The Kerr black hole (Kbh) possesses an Ergosphere, which is a region that surrounds the event horizon where the rotational velocity of spacetime exceeds the speed of light. Objects within the Ergosphere are forced to rotate in conjunction with the rotation of the (bh) [4]. One of the most interesting distinctive attributes of the (Kbh) is the effect called frame-dragging. This effect causes the spacetime around a rotating body to be dragged along with the body's rotation. Near a Kerr black hole, this effect is significant and can have observable consequences for nearby objects [5]. In the vicinity of a rotating body, spacetime itself is dragged along with the rotation of the body. This means that any object or observer in the vicinity will experience a dragging or twisting of its local frame of reference due to the rotation of the massive object [6]. Near a Kerr black hole, this effect is particularly pronounced, especially close to the black hole's event horizon. Hence, the frame-dragging effect can have observable consequences for nearby objects, such as a precession in the orbits of nearby particles or a twisting of light rays passing near the (bh) [5], [6]. The frame-dragging effect is a key prediction of general relativity and has been confirmed to some extent by experimental observations [7], [8]. The event horizon of a (Kbh) is more complex than that of a mass only with no angular momentum (bh) due to its rotation. The event horizon is an oblate spheroid, with the poles being closer to the black hole than the equator [9].

Many scientists contributed to the study of black hole thermodynamics, starting with Steven Hawking [10] who was the first to introduce the concept for the first-time proving that (bh) indeed have entropy and abide by laws of thermodynamics of their own followed by numerous contributions of scientists till this day.

Black whole parameters which are distance from the observer, inclination angle, and angular momentum are determined using a new method proposed by Hioki and Miyamoto [11]. This method uses the size and shape of the (Kbh). The method aims to treat the distance as an independent variable, extend the shadow

* Corresponding authors: zeyad22@scinv.au.edu.eg

outline, and construct principal components from the Fourier coefficients. A one-to-one mapping between observable principal components provided previously had been obtained. This method can be applied to different types of (bh) including (bh) that have accretion disks.

Black hole motion had been studied under the circumstances of a homogeneous, scalar, static, massless field. The generalized case is applied, gradient constant vector behaves time-like, space-like, or null. A comparison had been established between all the cases mentioned previously this influenced the authors Frolov and Koek [12] to conclude that as a result of black hole interaction with the scalar field, its spin, relative velocity, and mass with the scalar field can differ, through these equations which are describing these parameters evolution had been obtained. Furthermore, the authors presented solutions of equations for simple cases.

The parametrized Kerr spacetimes are useful for testing the nature of (bh) in a model-independent manner. These spacetimes involve numerous arbitrary typically require a Taylor expansion around infinity, truncated to finite orders. However, this truncation can lead to unphysical divergences in the metric. To solve this issue, the authors redefine the arbitrary functions to eliminate these divergences for various known black hole solutions that fit the parametrized Kerr spacetime framework. Yagi et al. [13] present a couple refined classification parametrized Kerr spacetimes, each containing only a single or couple of function that are of the arbitrary type. The original parametrized Kerr spacetime is classified as Petrov type I, while the restricted classification with a single arbitrary function remains Petrov type D. Additionally, the paper calculates the ringdown frequencies and black hole shadow shapes for these spacetimes, demonstrating their deviations from the Kerr solution.

Sharif and Ama-Tul-Mughani did a study that analyzes the critical phenomena and phase transitions of Kerr-Sen-anti-de Sitter black holes using Maxwell's equal-area law, taking in mind the (temperature-entropy) diagram. Thermodynamic quantities such as angular momentum, entropy, and Hawking temperature are reviewed, conforming to the thermodynamics' first law and the Smarr–Gibbs–Duhem relation. Critical behavior is examined using Maxwell's equal-area law and a van der Waals-like equation of state, with the former proving more effective. The diagram is constructed, showing lower than the critical pressure, black holes undergo phase transitions similar to liquid-gas transitions in van der Waals fluids [14].

Also, Hristov [15] has a study that reformulates the thermodynamics of thermal Kerr-Newman black holes in 4D flat space using left- and right-moving variables for regions surrounding, simplifying the expressions. It extends this approach to four deformations of the 4D Einstein-Maxwell theory within supergravity: higher derivative corrections, additional scalar and vector couplings, higher dimensions (5D), and a cosmological constant in 4D minimal gauged supergravity with an AdS vacuum. Each extension offers new insights, particularly regarding BPS limits, showcasing the broad applicability of natural variables.

Yang *et al*. [16] did a study that explores the thermodynamics and weak cosmic censorship conjecture (WCCC) for in a certain type of Kerr Newman black holes. Using the electric charge defined as a Komar integral over the event horizon, the thermodynamics' first law is established via Euclidean action. (WCCC) is then examined for the black hole with a complex scalar field. Results show that an extremal (bh) is indestructible by either

The paper discusses the development of new explicit symplectic integrators for the Kerr spacetime, which is challenging due to the non-separability of variables. By introducing the function to the Hamiltonian of geometry of Kerr, the authors create a type of Hamiltonian that can be split into parts with explicit solutions in the new coordinate time. These new algorithms, which adapt time steps and split the Hamiltonian effectively, show superior long-term conservation of Hamiltonian quantities and better computational efficiency compared to existing methods. The approach can also be applied to other relativistic problems beyond the Kerr metric [17].

The organization of this paper is as follows. In section 2, the mathematical model is proposed. In section 3, the Hamilton equation formula of the (Kbh) is given. In section 4, the temperature of the (Kbh) is Explored. In section 5, the thermodynamic functions of (Kbh) are derived.

Formulation and Physical Model

In general relativity, the motion of a particle in curved spacetime is described by the geodesic equation, which is derived from the principle of stationary action applied to the proper time of the particle. A vacuum solution of the Einstein equation for rotating the black is the Kerr metric given as follows [2], [3]:

$$ds^2 = -\left(1-\frac{2r}{\Sigma}\right)dt^2 - \frac{4arsin^2\theta}{\Sigma}dtd\phi + \frac{Asin^2\theta}{\Sigma}d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2. \quad (1)$$

(Kbh) has a gravitational field which is described by the spacetime metric:

$$-d\tau^2 = ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \quad (2)$$

In the standard Boyer-Lindquist coordinates (t, r, θ, φ), this metric is stationary symmetric axially and has covariant components [1], [3], [17]:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2r}{\Sigma}\right) & 0 & 0 & \frac{2arsin^2\theta}{\Sigma} \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ \frac{2arsin^2\theta}{\Sigma} & 0 & 0 & \left(\rho^2 + \frac{2ra^2}{\Sigma}sin^2\theta\right)sin^2\theta \end{pmatrix}, \quad (3)$$

and it has the following contravariant components:

$$g^{\mu\nu} = \begin{pmatrix} -\frac{A}{\Delta\Sigma} & 0 & 0 & -\frac{2ar}{\Delta\Sigma} \\ 0 & \frac{\Delta}{\Sigma} & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ -\frac{2ar}{\Delta\Sigma} & 0 & 0 & \frac{\Sigma-2r}{\Delta\Sigma sin^2\theta} \end{pmatrix}, \quad (4)$$

where:

$$\left.\begin{aligned} \Sigma &= r^2 + a^2cos^2\theta, \\ \Delta &= \rho^2 - 2r, \\ \rho^2 &= r^2 + a^2, \\ A &= \rho^4 - \Delta a^2 sin^2\theta. \end{aligned}\right\} \quad (5)$$

Planck units were used, making $c = M = G = 1$ where c is the speed of light, M is the mass of the black hole and G is the gravitational constant, while:

$$a = \frac{J}{M}, \quad (6)$$

Where J is the angular momentum of the black hole. Based on the space on the (Kbh) spacetime metric, the Lagrangian system could be expressed as:

$$\mathcal{L} = \frac{1}{2}\left(\frac{ds}{d\tau}\right)^2 = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \quad (7)$$

Four-velocity $\dot{x}^\mu = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$ satisfies the relation.

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1, \quad (8)$$

According to classical mechanics, a covariant generalized four-momentum is defined as

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\mu\nu}\dot{x}^\nu, \tag{9}$$

The coordinate ϕ and t don't appear in all of the components therefore the components don't depend on t nor ϕ explicitly which results in two constant momentums quantities, substituting from equation (2) into equation (8) we get:

$$p_t = g_{tt}\dot{t} + g_{t\varphi}\dot{\varphi} = -\left(1 - \frac{2r}{\Sigma}\right)\dot{t} - \frac{2ra\sin^2\theta}{\Sigma}\dot{\varphi} = -E, \tag{10}$$

$$p_\phi = g_{\varphi\varphi}\dot{\varphi} + g_{t\varphi}\dot{t} = \left(\rho^2 + \frac{2ra^2}{\Sigma}\sin^2\theta\right)\sin^2\theta\,\dot{\varphi} - \frac{2ra\sin^2\theta}{\Sigma}\dot{t} = L, \tag{11}$$

E represents the energy of a test particle moving around the rotating body, and L is the angular momentum of the test particle.

There exists a direct relation between this Lagrangian and the Hamiltonian as follows.

$$\mathcal{H} = \boldsymbol{\mathcal{U}}\,.\boldsymbol{p} - \mathcal{L} = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu = \mathbb{F} + \frac{1}{2}\frac{\Delta}{\Sigma}p_r^2 + \frac{1}{2}\frac{p_\theta^2}{\Sigma}, \tag{12}$$

where $\mathbb{F}$ is a function of r and θ and reads.

$$\mathbb{F} = \frac{1}{2}(g^{tt}E^2 + g^{\varphi\varphi}L^2) - g^{t\varphi}EL = -\frac{AE^2}{2\Delta\Sigma} + \frac{L^2(\Sigma - 2r)}{2\Delta\Sigma\sin^2\theta} + \frac{2ar}{\Delta\Sigma}EL. \tag{13}$$

Analytical mechanics methods can be applied to derive the thermodynamic functions of black holes. This paper focuses on using the Hamiltonian function$\mathcal{H}$. If the Hamiltonian function does not depend on time, it can represent the total energy of a system, and in the context of a black holes the Hamiltonian describes the total mass of the black hole. Also, The Hamiltonian equation describes the motion of a charged particle near a black hole. Using methods of differentiation one can derive the thermodynamic functions using the Hamiltonian equation. One can derive this method to derive the temperature of (KBH), than other thermodynamic function could be derived.

Hamilton equation of a (Kbh)

Hamiltonian equation is given by [17]:

$$\mathcal{H} = \mathbb{F} + \frac{1}{2\Sigma}\left(\Delta\, p_r^2 + p_\theta^2\right), \tag{14}$$

$\mathcal{H}$ is the Hamilton, $p_i(i = r, \theta, \varphi, t)$ is momentum, r is the radial coordinate ($0 < r < r_+$; r_+ is the outer event horizon), L is the angular momentum, and E is the energy of a test particle moving in the gravitational field. When a black hole is rotating, there exists a preferable direction in space, i.e. the axis of rotation. In such a gravitational field the generic orbit of particle or light is no longer planned. The motion of the equatorial plane is a special case. Because of the reflection symmetry $\theta \to \pi - \theta$, this plane is a geodesic submanifold. This means that the motion of a particle with the initial data $\theta = \frac{\pi}{2}, \dot{\theta} = 0$ is restricted to the equatorial plane. This means that $\dot{p}_\theta = 0$. There exists a critical (Kbh) metric solution with $M = a$ radial normalized momentum that could be expressed as [3]:

$$p_r = r^{-3/2}\sqrt{\mathcal{P}}, \tag{15}$$

where

$$\mathcal{P} = E^2(r^3 + a^2 r + 2a^2) - 4aEL - (2 - r)L^2 - r\Delta \tag{16}$$

The entropy s is given by:

$$S = \frac{A}{4} = \frac{r}{4}(2 + r + r^3), \tag{17}$$

where A is the area of the event horizon.

The temperature formula of (Kbh)

It is well known that temperature can be calculated by differentiating the Hamiltonian with respect to entropy, which is given by.

$$T=\frac{2}{r^6(-1+r)^3(1+r+2r^3)}(-2(-1+r)^6r+E^2r^4(-1+3r+r^2+r^3)+L^2(5-25r+51r^2-54r^3+30r^4-6r^5)+E((-1+r)^4(5-r-r^2+r^3)-2L(5-23r+42r^2-38r^3+16r^4))) \qquad (18)$$

The thermodynamic functions of (Kbh)

The volume of a black hole had been calculated as a function of entropy [18]:

$$V=\frac{4\pi}{3}\left(\frac{S}{\pi}\right)^{\frac{3}{2}}=\frac{(r(2+r+r^3))^{3/2}}{6\sqrt{\pi}}. \qquad (19)$$

The pressure of a (Kbh) can be expressed in the following form.

$$P=\frac{T}{2\,v^{\frac{2}{3}}}-\frac{1}{8\,\pi v^{\frac{2}{3}}}+\frac{J^2}{4\,\pi v}=\frac{1}{2}\left(\frac{4\sqrt{\pi}}{(r(2+r+r^3))^{3/2}}-\frac{1}{r(2+r+r^3)}+(4\sqrt{\pi}(-2(-1+r)^6r+E^2r^4(-1+3r+r^2+r^3)+L^2(5-25r+51r^2-54r^3+30r^4-6r^5)+E((-1+r)^4(5-r-r^2+r^3)-2L(5-23r+42r^2-38r^3+16r^4))))/((-1+r)^3r^6(r(2+r+r^3))^{\frac{1}{2}}(1+r+2r^3))\right), \qquad (20)$$

Where $v=\frac{3V}{4\,\pi}=\frac{(r(2+r+r^3))^{3/2}}{8\pi^{3/2}}$.

The internal energy U of a black hole is the total energy to create a black hole and place it in a cosmological environment [19]. The internal energy U is given by:

$$U=TS-PV, \qquad (21)$$

One obtains the internal energy U form by substituting equations (18), (19), (20), and (21) in equation (22).

$$U=\frac{1}{2}\left(\frac{(2+r+r^3)(-2(-1+r)^3}{r^4(1+r+2r^3)}+E^2r^4(-1+3r+r^2+r^3)+L^2(5-25r+51r^2-54r^3+30r^4-6r^5)+E((-1+r)^4(5-r-r^2+r^3)-2L(5-23r+42r^2-38r^3+16r^4))\right)+\frac{1}{\sqrt{\pi}}(r(2+r+r^3))^{3/2}\left(-\frac{2\sqrt{\pi}}{(r(2+r+r^3))^{3/2}}+\frac{1}{2(r(2+r+r^3)}-(2\sqrt{\pi}(-2(-1+r)^6r+E^2r^4(-1+3r+r^2+r^3)+L^2(5-25r+51r^2-54r^3+30r^4-6r^5)+E((-1+r)^4(5-r-r^2+r^3)-2L(5-23r+42r^2-38r^3+16r^4))))/((-1+r)^3r^6(r(2+r+r^3))^{\frac{1}{2}}(1+r+2r^3))\right)). \qquad (22)$$

The enthalpy of the (Kbh) is given by:

$$H=U+PV, \qquad (23)$$

Enthalpy is obtained by substituting equations (20), (21) and (23) in equation (24). Enthalpy is given by:

$$H=\frac{1}{2(-1+r)^3r^5(1+r+2r^3)}(2+r+r^3)(-2(-1+r)^6r$$
$$+E^2r^4(-1+3r+r^2+r^3)$$
$$+L^2(5-25r+51r^2-54r^3+30r^4-6r^5)$$
$$+E((-1+r)^4(5-r-r^2+r^3)-2L(5-23r+42r^2-38r^3+16r^4))). \qquad (24)$$

At a constant temperature, Helmholtz's free energy is the maximum amount of work a system can perform in a process. Helmholtz free energy is obtained by

$$F=U-TS \qquad (25)$$

F is obtained by substituting equations (18), (19) and (24) in equation (25). One can get:

$$F = \frac{1}{12\sqrt{\pi}}\left(r(2+r+r^3)\right)^{3/2}\left(-\frac{4\sqrt{\pi}}{\left(r(2+r+r^3)\right)^{3/2}} + \frac{1}{(r(2+r+r^3)} - (4\sqrt{\pi}(-2(-1+r)^6 r + E^2 r^4(-1+3r+r^2+r^3) + L^2(5-25r+51r^2-54r^3+30r^4-6r^5) + E((-1+r)^4(5-r-r^2+r^3) - 2L(5-23r+42r^2-38r^3+16r^4))))/((-1+r)^3 r^6((r(2+r+r^3))^{3/2})^{1/3}(1+r+2r^3))\right). \quad (26)$$

Results and discussion

Thermodynamic functions of (Kbh) are obtained using the Hamiltonian equation, also we calculated the Helmholtz free energy. From **Figure 1** one can note that temperature T is at its highest approaching infinity at the Ergosphere, $r \to r_+$ and $r > r_+$ near the outer part of the event horizon where the Hawking radiation effect takes place. From **Figure 1**, one can note that T approaches $-\infty$ when r approaches 0 near the singularity and r approaches 1 near the interior part of the event horizon in addition to a stagnant interval where T approaches zero, however near the exterior part of the event horizon temperature rises exponentially approaching ∞ , the farther away from the event horizon the lower the temperature gets reaching zero.

The peak in the temperature graph near $r = 1$ corresponds to the event horizon of the black hole. This indicates that the temperature of the black hole is highest at the event horizon, consistent with the idea that the event horizon is a region of intense gravitational and thermal activity. The rapid decrease in temperature as r moves away from the black hole reflects the diminishing gravitational effects on the surrounding spacetime. This behavior demonstrates how the gravitational field of the black hole influences its thermodynamic properties. The behavior of the temperature near $r = 0$ and $r = 1$ may reveal insights into quantum effects near the black hole. Near $r = 0$, where the singularity may be present, the temperature behavior could indicate the breakdown of classical physics and the need for a quantum theory of gravity.

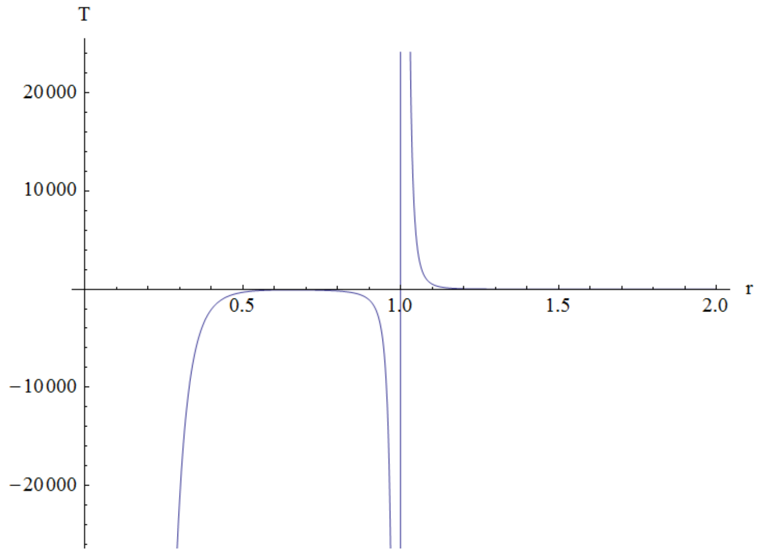

Figure 1. The behavior of (Kbh) temperature T with respect to r, where $a = 1$, $E = 0.995$, $p_t = -E$, $L = 4.6$, and $r_+ = 1$.

The overall shape of the temperature graph reflects the thermodynamic stability of the (Kbh). A well-behaved temperature profile, without abrupt changes or singularities, suggests that the black hole is in a stable thermodynamic state. The temperature graph is related to the concept of Hawking radiation, which predicts that black holes emit thermal radiation due to quantum effects near the event horizon (**Figure 2**). The temperature graph provides insights into the properties of this radiation and its dependence on the parameters of the black hole. Overall, the temperature graph of a (Kbh) provides a window into the complex interplay between gravity, thermodynamics, and quantum mechanics in the extreme conditions near black holes.

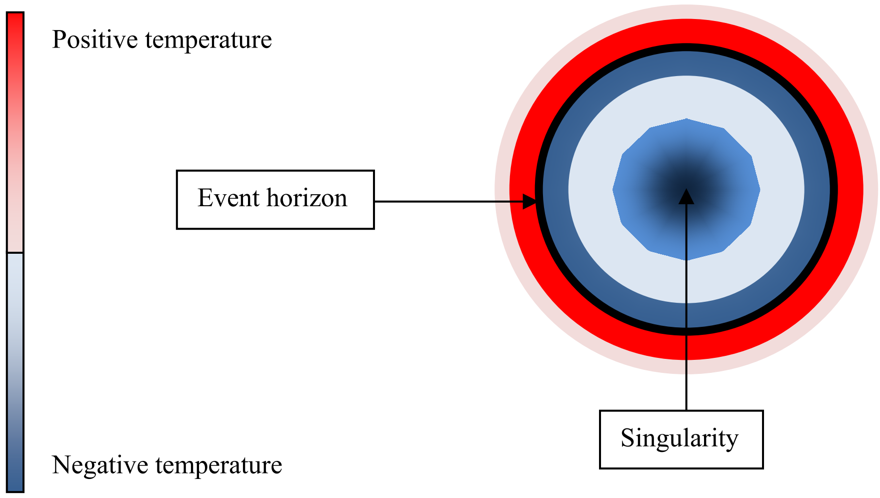

Figure 2. Diagram of distribution of (Kbh) temperature concerning r.

Figure 3 shows that internal energy U is at its highest near the singularity and the exterior part of the event horizon approaching infinity and U is at its lowest near the interior part of the event horizon, meanwhile it reaches 0 everywhere else. The behavior of U reflects the stability of the (Kbh). A well-behaved internal energy function indicates that the black hole is in a stable thermodynamic state. The peak or rapid increase in U near the event horizon ($r \approx 1$) signifies the intense gravitational and thermodynamic activity at this boundary. This behavior consists of high internal energy and temperature near the event horizon. The decrease in U for large r reflects the diminishing gravitational effects of the black hole as r increases. This behavior demonstrates how the gravitational field of the black hole influences its internal energy. Any discontinuities or singular behavior in U at small r ($r \approx 0$) could indicate the presence of quantum effects near the black hole. Understanding these effects is crucial for developing a gravity quantum theory.

The behavior of U may also be related to the concept of Hawking radiation, which predicts that black holes emit thermal radiation. The internal energy function could provide insights into the properties of this radiation and its dependence on the parameters of the black hole. The graph of internal energy U of a (Kbh) as a function of r reveals valuable information about the thermodynamic and gravitational properties of the black hole system, shedding light on its stability, interaction with its surroundings, and the nature of spacetime near the event horizon.

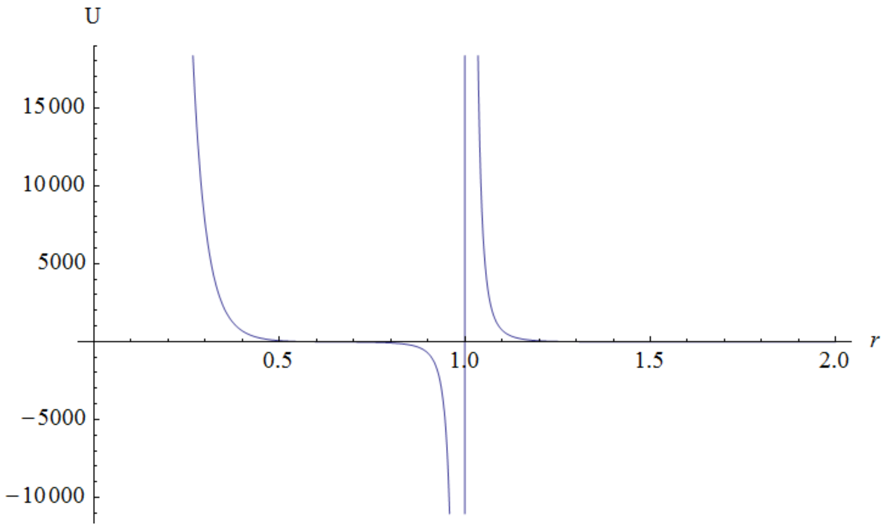

Figure 3. The behavior of (Kbh) Internal energy U with respect to r, where $a = 1$, $E = 0.995$, $p_t = -E$, $L = 4.6$, and $r_+ = 1$.

From **Figure 4**, enthalpy has very similar behavior due to its connection to mass, negative mass particles are denser near the event horizon and the singularity and positive mass particles are denser at the outer part of the event horizon. The high value of H near the singularity indicates that the system has a large total energy content, including both internal energy and the energy required to maintain the system at constant pressure and volume. The behavior of H reflects the strong gravitational effects near the singularity and event horizon. These effects influence the distribution of energy and the balance between internal energy and pressure-volume work. The fluctuations and changes in H suggest that the system is not in a thermodynamic equilibrium state at all points. Near the singularity and event horizon, where gravitational effects are significant, the system may be far from equilibrium. The rise in H near the event horizon indicates that significant energy changes occur in this region. This could be related to the release of energy due to interaction of (bh) with its surroundings, such as the emission of Hawking radiation.

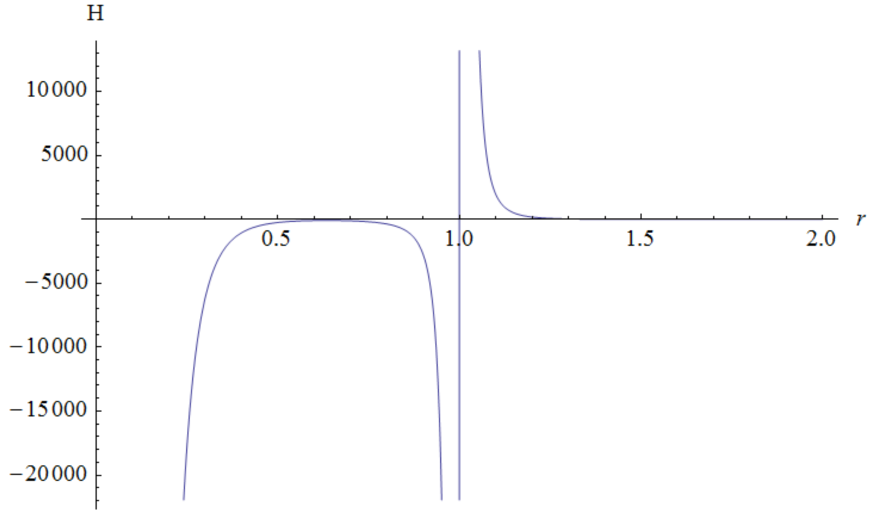

Figure 4. The behavior of (Kbh) enthalpy H with respect to r, where $a = 1$, $E = 0.995$, $p_t = -E$, $L = 4.6$, and $r_+ = 1$.

Form **Figure 5**, as r increases from 0 to 0.6 Helmholtz Free energy decreases from $F \to \infty$ to $F \to 0$ while r increases from 0.6 to 0.7 Helmholtz Free energy at $F \to 0$, then as r increases from $r = 0.7$ to $r \to 1$ near the event horizon, Helmholtz Free energy increases exponentially approaching ∞. The Helmholtz free energy F represents the energy available to do work in the (Kbh) system at constant temperature and volume. It reflects the balance between internal energy, temperature effects, and the influence of angular momentum. The behavior of F provides insights into the stability of the (Kbh). Regions where F exhibit unusual behavior may indicate points of instability or phase transitions within the black hole system.

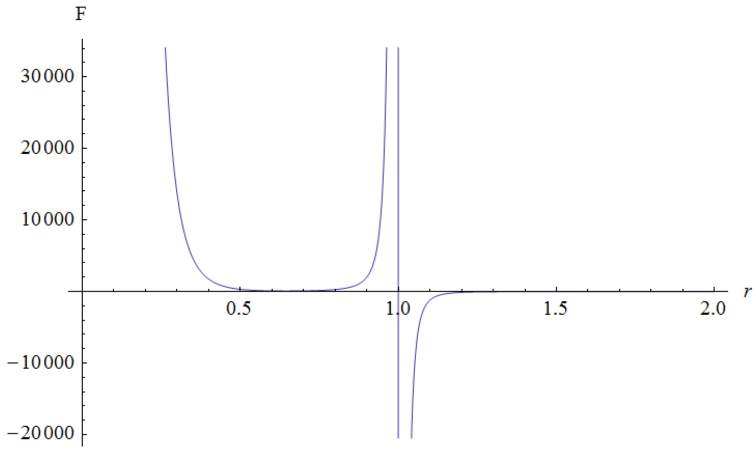

Figure 5. The behavior of (Kbh) Helmholtz Free energy concerning r, where $a = 1$, $E = 0.995$, $p_t = -E$, $L = 4.6$, and $r_+ = 1$.

Conclusion

Since the Hamiltonian function predicts the motion of a test particle near a black hole, this could be informative about the interior part and the exterior part of the black hole past the event horizon. Furthermore, Once the temperature formula is calculated, Other thermodynamic functions can easily be calculated.

Figure 1 could be easily explained by the Hawking radiation effect, a great number of particles come into existence through a quantum process both of negative masses and positive masses; while positive masses are being emitted outside the black hole while negative mass particles fall in towards the singularity. This study represents an understanding of the action of negative mass particles falling toward the singularity as they come to existence near the interior part of the event horizon temperature reaches $-\infty$ where they come to existence, then on their path to the singularity time gets faster so the density of these particles decreases, finally after negative mass particles reach a certain distance traveling towards the singularity, they become much denser because as they gather in a smaller space (**Figure 2**). **Figure 3** shows that particles approaching the singularity of a Kerr black hole consume energy as they travel within the inner Ergosphere. For **Figure 4**, enthalpy has very similar behavior due to its connection to mass, negative mass particles are denser near the event horizon and the singularity and positive mass particles are denser at the outer part of the event horizon.

Conflict of Interest

None.

References

[1] R. P. Kerr, "Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics," *Physical Review Letters,* vol. 11, no. 5, p. 237–238, 1963.

[2] J. M. Bardeen, "Kerr Metric Black Holes," *Nature,* vol. 226, no. 5240, p. 64–65, 1970.

[3] V. P. Frolov and A. Zelnikov, Introduction to Black Hole Physics, New York: Oxford University Press, 2011.

[4] G. Contopoulos, "Orbits through the ergosphere of a Kerr black hole," *General Relativity and Gravitation,* vol. 16, no. 1, p. 43–70, 1984.

[5] L. F. Costa and J. Natário, "Frame-Dragging: Meaning, Myths, and Misconceptions," *Universe,* vol. 7, no. 10, p. 388, 2021.

[6] H. Thirring and J. Lense, "On the influence of the proper rotation of a central body on the motion of the planets and the moon, according to Einstein's theory of gravitation," *Zeitschrift für Physik,* vol. 19, no. 156, p. 47, 1918.

[7] C. W. F. Everitt et al, "Gravity Probe B: Final Results of a Space Experiment to Test General Relativity," *Physical Review Letters,* vol. 106, no. 22, 2011.

[8] V. Bozza, "Optical caustics of Kerr spacetime: The full structure," *Physical Review D,* vol. 78, no. 6, 2008.

[9] S. A. Teukolsky and W. H. Press, "Perturbations of a rotating black hole. III - Interaction of the hole with gravitational and electromagnetic radiation," *The Astrophysical Journal,* vol. 193, p. 443, 1974.

[10] H. Stephen, "Particle creation by black holes," *Communications in Mathematical Physics,* vol. 43, no. 3, p. 199-220, 1975.

[11] K. Hioki and U. Miyamoto, "Determining parameters of Kerr black holes at finite distance by shadow observation," *Physical Review D,* vol. 109, no. 4, p. 044030, 2024.

[12] V. P. Frolov and A. Koek, "Motion of rotating black holes in homogeneous scalar fields: A general case," *Physical Review D,* vol. 109, no. 8, p. 044017, 2024.

[13] K. Yagi, S. Lomuscio, T. Lowrey and Z. Carson, "Regularizing parametrized black hole spacetimes with Kerr symmetries," *Physical Review D,* vol. 109, no. 4, p. 044017, 2024.

[14] M. Sharif and Q. Ama-Tul-Mughani, "$$P-V$$ criticality and phase transition of the Kerr-Sen-AdS Black Hole," *The European Physical Journal Plus,* vol. 136, no. 3, 2021.

[15] K. Hristov, "Black hole thermodynamics in natural variables: quadrophenia," *Journal of High Energy Physics,* vol. 2024, no. 2, 2024.

[16] S.-J. Yang, W.-D. Guo, S.-W. Wei, and Y.-X. Liu, "First law of black hole thermodynamics and the weak cosmic censorship conjecture for Kerr–Newman Taub–NUT black holes," *The European Physical Journal C,* vol. 83, no. 12, 2023.

[17] X. Wu, Y. Wang, W. Sun and F. Liu, "Construction of Explicit Symplectic Integrators in General Relativity. IV. Kerr Black Holes," *The Astrophysical Journal,* vol. 914, no. 1, p. 63, 2021.

[18] P. D. Brain, "Pressure and volume in the first law of black hole thermodynamics," *Classical and Quantum Gravity,* vol. 28, no. 23, p. 235017, 2011.

[19] D. Kubizňák, R. B. Mann and M. Teo, "Black hole chemistry: thermodynamics with Lambda," *Classical and Quantum Gravity,* vol. 34, no. 6, p. 063001, 2017.