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Bayesian Estimation and Prediction for Marshall-Olkin Weibull-Exponential Distribution Based on Progressive Type-II Censoring Scheme

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Abstract

Lifetime distributions under progressive Type-II censored scheme have been attracting great interest due to their wide application in the fields of science, engineering, social sciences and medicine. Also, prediction of future events on the basis of the past and present knowledge without any doubt is one of the most important problems in statistics. In this paper, the Bayes estimators for the parameters of the Marshall-Olkin Weibull-exponential distribution are derived based on progressive Type-II censored scheme. The estimators are considered under two different loss functions, the balanced squared error loss function; as a symmetric loss function and the balanced linear exponential loss function; as an asymmetric loss function. Also, the two-sample prediction method is applied to obtain the Bayesian prediction (point and interval) for future order statistics. A numerical example is provided to illustrate the theoretical results and an application using real data set is used to demonstrate how the results can be used in practice.

Keywords: *Marshall-Olkin Weibull-exponential distribution; progressive Type-II censored samples; Bayesian estimation and prediction.*

1. Introduction

In recent years, various generalized distributions have been proposed and their flexibilities over their baseline distributions when applied to real life data have been established. Interest have been increased among statisticians by adding one parameter or more to a baseline distribution; to provide great flexibility in modeling data in several applied areas such as reliability, engineering, economics, environmental sciences, finance and medical.

Life-testing and reliability experiments contain many situations where units are removed or lost from the test before failure. For example, units may break accidently in an industrial experiment, individuals may drop out of the study in a clinical trial, or they have to be terminated early due to lack of funds. In many scenarios, the removal of units before failure is very often procedure due to limitations of time and cost associated with the experiment. The data of such tests or experiments are called censored data.

There are several types of censoring schemes but the two most common censoring schemes are Type-I and Type-II censoring. The conventional Type-I and Type-II censoring schemes do not have flexibility of allowing removal of units at point other than the terminal point of the experiment. For this reason, more general censoring scheme called progressive censoring scheme is considered. The progressive censoring possesses such flexibility and thus allows in between removals of units as well. Different progressive censoring schemes have been introduced in the literature. The most popular one is known as the progressive Type-II censoring scheme and it can be briefly described as follows:

Considering n identical units are put to test and the lifetime distribution of the n units are denoted by $X_1, X_2, \dots, X_n$.The integer $m(< n)$ is fixed at the beginning of the experiment and $R_1, R_2, \dots, R_m$ are m pre-fixed integers satisfying $R_1 + R_2 + \cdots + R_m + m = n$. At the time of the first failure $X_{1:m:n}$, R_1 units are chosen randomly from the remaining $n - 1$ units and they are removed from the experiment. Similarly at the time of the second failure $X_{2:m:n}$, R_2 of the remaining $n - R_1 - 2$ units are removed from the test and so on. Finally, when the m-th failure is observed the experiment is terminated and the remaining surviving units R_m with $R_m = n - R_1 - R_2 - \cdots - R_{m-1} - m$ are removed. Here $(R_1, R_2, \ldots, R_m)$ is known as the censoring scheme and it is prefixed before the experiment starts.

Lifetime distributions under progressive Type-II censored scheme have been attracting great interest due to their wide application in the fields of science, engineering, social sciences and medicine [see, Almetwally and Almongy (2018), Karakoca and Pekgör (2019), Li and Gui (2020), Salah (2020), Alshenawy *et al*. (2020), AL-Sayed *et al*. (2022) and others].

Prediction of future events on the basis of the past and present information is a fundamental problem of statistics, arising in many contexts and producing varied solutions. As in estimation, a predictor can be either a point or an interval predictor. Prediction has been applied in a variety of disciplines such as medicine, engineering, business, economic and other areas as well. Prediction for order statistics of future observables from certain distributions has been studied by several authors, such as, Valiollahi *et al.* (2017), Faizan and Sana (2018), Arshad and Jamal (2019), Okasha *et al.* (2020), Ahmad (2021) and AL-Dayian *et al.* (2021).

The *Marshall-Olkin Weibull-exponential* (MOW-E) distribution is introduced by Klakattawi *et al*. (2022), as a special case of the Marshall-Olkin Weibull generated family. The MOW-E distribution is a flexible distribution which provides symmetrical, right-skewed, left-skewed, J shaped and reversed-J shaped densities. Its *hazard rate function* (hrf) can provide increasing, decreasing, bathtub, upside-down bathtub, J shaped and reversed-J shaped hrfs. It is noted that, the MOW-E distribution is a suitable distribution for fitting positively skewed data which may not be adequately modelled by many other distributions. Thus, it can be used to fit data related to public health, biomedical studies, industrial reliability, survival analysis and several other areas.

The *probability density function* (pdf) and *cumulative distribution function* (cdf) of the MOW-E distribution are, respectively, given by

$$f(x;\underline{\Phi}) = \frac{\alpha\theta\lambda^{\theta}}{\beta^{\theta}} x^{\theta-1}\, e^{-\left(\frac{\lambda x}{\beta}\right)^{\theta}} \left[1-\ \bar{\alpha}\, e^{-\left(\frac{\lambda x}{\beta}\right)^{\theta}}\right]^{-2}, \qquad x>0;(\underline{\Phi}\ >\underline{0}), \qquad (1)$$

and

$$F(x;\underline{\Phi}) = \frac{1-e^{-\left(\frac{\lambda x}{\beta}\right)^{\theta}}}{1-\bar{\alpha}\, e^{-\left(\frac{\lambda x}{\beta}\right)^{\theta}}}, \qquad x>0;(\underline{\Phi}\ >\underline{0}), \qquad (2)$$

where $\underline{\Phi} = (\lambda,\alpha,\beta,\theta)'$; α,θ are shape parameters, λ,β are scale parameters and $\bar{\alpha} = 1-\alpha$.

The *reliability function* (rf), hrf and *reversed hazard rate function* (rhrf) of the MOW-E distribution are, respectively, given by

$$R(x;\underline{\Phi}) = \frac{\alpha\, e^{-\left(\frac{\lambda x}{\beta}\right)^{\theta}}}{1-\bar{\alpha}\, e^{-\left(\frac{\lambda x}{\beta}\right)^{\theta}}}, \qquad x>0;(\underline{\Phi}>\underline{0}), \qquad (3)$$

$$h(x;\underline{\Phi}) = \frac{\theta\lambda^{\theta}}{\beta^{\theta}} x^{\theta-1}\left[1-\bar{\alpha}\, e^{-\left(\frac{\lambda x}{\beta}\right)^{\theta}}\right]^{-1}, \qquad x>0;(\underline{\Phi}\ >\underline{0}), \qquad (4)$$

and

$$rh(x;\underline{\Phi}) = \frac{\alpha\theta\lambda^{\theta}}{\beta^{\theta}} x^{\theta-1}\, e^{-\left(\frac{\lambda x}{\beta}\right)^{\theta}} \left[\left(1-\ e^{-\left(\frac{\lambda x}{\beta}\right)^{\theta}}\right)\left(1-\ \bar{\alpha} e^{-\left(\frac{\lambda x}{\beta}\right)^{\theta}}\right)\right]^{-1}, \quad x>0;(\underline{\Phi}\ >\underline{0}). \qquad (5)$$

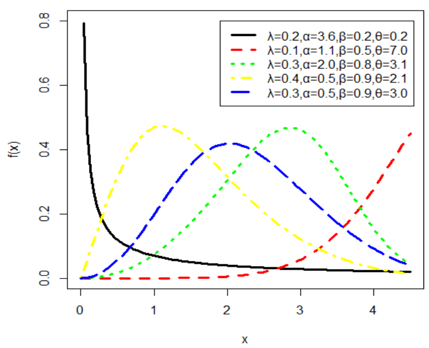

Figure 1. Different shapes for the pdf

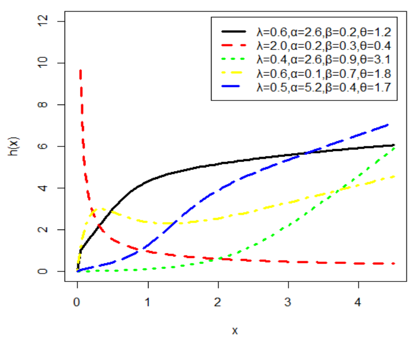

Figure 2. Different shapes for the hrf

The rest of this paper is organized as follows: in Section 2, the Bayes estimators for the parameters of the MOW-E distribution based on progressive Type-II censoring scheme under the *balanced squared error loss* (BSEL) and *balanced linear exponential loss* (BLL) functions are obtained. Also, the *credible intervals* (CIs) for the parameters are introduced. In Section 3, the Bayesian prediction (point and interval) for a future observation of the MOW-E distribution based on two-sample prediction method are considered. A numerical example is given to illustrate the theoretical results and an application using real data set is used to demonstrate how the results can be used in practice in Section 4. Finally, general conclusion is presented in Section 5.

2. Bayesian Estimation

In this section, the Bayes estimators and CIs for the parameters of the MOW-E distribution based on progressive Type-II censored sample are derived.

Let $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$ denote a progressive Type-II censored sample obtained from the MOW-E distribution. The *likelihood function* (LF) is given by

$$L(\underline{\Phi};\underline{x}) = C(n,m-1)\prod_{i=1}^{m} f\left(x_{(i)};\underline{\Phi}\right)\left[1-F\left(x_{(i)};\underline{\Phi}\right)\right]^{R_i}, \tag{6}$$

where $\underline{\Phi} = (\lambda,\alpha,\beta,\theta)'$, $\underline{x} = (x_{1:m:n}, x_{2:m:n}, \dots, x_{m:m:n})$ denotes an observed value of $\underline{X} = (X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n})$ and $C(n,m-1) = n(n-R_1-1)(n-R_1-R_2-2)\dots(n-R_1-\dots-R_{m-1}-m+1)$, with $C(n,0)=n$. Then substituting (1) and (2) in (6) yields

$$L(\underline{\Phi};\underline{x}) = C(n,m-1)\prod_{i=1}^{m}\frac{\alpha\theta\lambda^{\theta}}{\beta^{\theta}}x_{(i)}^{\theta-1}\; e^{-\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\left[1-\bar{\alpha}\, e^{-\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\right]^{-2}$$

$$\times\prod_{i=1}^{m}\left[1-\frac{1-e^{-\left(\frac{\lambda x}{\beta}\right)^{\theta}}}{1-\bar{\alpha}\, e^{-\left(\frac{\lambda x}{\beta}\right)^{\theta}}}\right]^{R_i}$$

$$= C(n,m-1)\prod_{i=1}^{m}\frac{\alpha\theta\lambda^{\theta}}{\beta^{\theta}}\, x_{(i)}^{\theta-1}\,\alpha^{R_i}\, e^{-(R_i+1)\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\left[1-\bar{\alpha}\, e^{-\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\right]^{-(R_i+2)}. \tag{7}$$

The natural logarithm of $L(\underline{\Phi};\underline{x})$ is given by

$$\ell \propto m\ln\alpha + m\ln\theta + \theta m\ln\lambda - \theta m\ln\beta + \sum_{i=1}^{m} R_i \ln\alpha$$
$$+(\theta-1)\sum_{i=1}^{m}\ln x_{(i)} - \sum_{i=1}^{m}(R_i+1)\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}$$
$$-\sum_{i=1}^{m}(R_i+2)\ln\left[1-\ \bar{\alpha}\, e^{-\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\right]. \qquad (8)$$

The *maximum likelihood* (ML) estimators of the parameters $\underline{\Phi} = (\lambda,\alpha,\beta,\theta)'$ can be obtained by differentiating (8) with respect to λ,α,β and θ and then setting to zeros. Hence

$$\frac{\partial\ell}{\partial\lambda} = \frac{\theta m}{\lambda} - \frac{\theta}{\lambda}\sum_{i=1}^{m}(R_i+1)\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta} + \frac{\theta\bar{\alpha}}{\lambda}\sum_{i=1}^{m}(R_i+2)\frac{\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}{\left[\bar{\alpha}-e^{\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\right]}, \qquad (9)$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m}\frac{R_i}{\alpha} - \sum_{i=1}^{m}\frac{(R_i+2)}{\left[e^{\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}-\bar{\alpha}\right]}, \qquad (10)$$

$$\frac{\partial\ell}{\partial\beta} = -\frac{\theta m}{\beta} + \frac{\theta}{\beta}\sum_{i=1}^{m}(R_i+1)\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta} + \frac{\theta\,\bar{\alpha}}{\beta}\sum_{i=1}^{m}(R_i+2)\frac{\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}{\left[e^{\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}-\bar{\alpha}\right]}, \qquad (11)$$

and

$$\frac{\partial\ell}{\partial\theta} = \frac{m}{\theta} + m\ln\lambda - m\ln\beta + \sum_{i=1}^{m}\ln x_{(i)} - \sum_{i=1}^{m}(R_i+1)\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}\ln\left(\frac{\lambda x_{(i)}}{\beta}\right)$$
$$+\ \bar{\alpha}\sum_{i=1}^{m}(R_i+2)\frac{\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}\ln\left(\frac{\lambda x_{(i)}}{\beta}\right)}{\left[\bar{\alpha}-e^{\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\right]}. \qquad (12)$$

The ML estimators are obtained by equating the derivatives (9)-(12) to zeros. The system of non-linear equations can be solved numerically using

Newton-Raphson method, to obtain the ML estimates of the parameters λ, α, β and θ .

Considering the prior knowledge of the vector of parameters, $\underline{\Phi} = (\lambda, \alpha, \beta, \theta)'$, is adequately represented by informative prior which is gamma distribution with parameters a_j and b_j and pdf as follows:

$$\pi(\Phi_j; a_j, b_j) = \frac{{b_j}^{a_j}}{\Gamma(a_j)} \Phi_j^{a_j-1} \exp(-b_j \Phi_j), \quad \Phi_j > 0;\ (a_j, b_j > 0),\ j = 1,2,3,4 \qquad (13)$$

where $\Phi_1 = \lambda, \Phi_2 = \alpha, \Phi_3 = \beta$ and $\Phi_4 = \theta$, a_j and b_j are the hyper-parameters of the prior distribution.

Assuming that the parameters, $\underline{\Phi} = (\lambda, \alpha, \beta, \theta)'$, are unknown and independent. Then the joint prior distribution of the unknown parameters has a joint pdf given by

$$\pi(\underline{\Phi};\ \underline{a}, \underline{b}) \propto \lambda^{a_1-1}\, \alpha^{a_2-1} \beta^{a_3-1}\, \theta^{a_4-1} \exp[-(b_1\lambda + b_2\alpha + b_3\beta + b_4\theta)],$$

$$\underline{\Phi} > \underline{0};\ (\underline{a}, \underline{b} > \underline{0}). \qquad (14)$$

Combining the LF in (7) and the joint prior distribution given by (14), then the joint posterior distribution of the parameters, $\underline{\Phi} = (\lambda, \alpha, \beta, \theta)'$, can be obtained as follows:

$$\pi(\underline{\Phi}|\underline{x}) = A\, L(\underline{\Phi}|\underline{x})\, \pi(\underline{\Phi}; \underline{a}, \underline{b})$$

$$= A\, \lambda^{\theta m + a_1 - 1}\, \alpha^{m + a_2 - 1} \beta^{a_3 - \theta m - 1} \theta^{m + a_4 - 1} \exp[-(b_1\lambda + b_2\alpha + b_3\beta + b_4\theta)]$$

$$\times \prod_{i=1}^{m} \alpha^{R_i}\, x_{(i)}^{\theta-1}\;\; e^{-(R_i+1)\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}} \left[1 - \bar{\alpha}\, e^{-\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\right]^{-(R_i+2)}, \qquad (15)$$

where A is a normalizing constant,

$$A^{-1} = \int_{\underline{\Phi}} \lambda^{\theta m + a_1 - 1}\, \alpha^{m + a_2 - 1} \beta^{a_3 - \theta m - 1} \theta^{m + a_4 - 1} \exp[-(b_1\lambda + b_2\alpha + b_3\beta + b_4\theta)]$$

$$\times \prod_{i=1}^{m} \alpha^{R_i}\, x_{(i)}^{\theta-1}\;\; e^{-(R_i+1)\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}} \left[1 - \bar{\alpha}\, e^{-\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\right]^{-(R_i+2)} d\underline{\Phi}, \qquad (16)$$

where $\int_{\underline{\Phi}} = \int_{\lambda} \int_{\alpha} \int_{\beta} \int_{\theta}$ and $\underline{\Phi} = d\theta\, d\beta\, d\alpha\, d\lambda$. (17)

The Bayes estimators for the parameters of the MOW-E distribution are considered under the *balanced loss function* (BLF). The estimator of a function using BLF is a mixture of the ML estimator, least squares estimators or any other estimator and the Bayes estimator using any loss function.

2.1 Bayesian estimation under balanced loss functions

Ahmadi *et al.* (2009) suggested the use of the BLF, which was originated by Zellner (1994), to be of the form

$$L^*(\Phi,\tilde{\Phi}) = \omega\, l\left(\Phi,\ \hat{\Phi}\right) + (1-\omega)\, L\left(\Phi,\tilde{\Phi}\right), \tag{18}$$

where $L\left(\Phi,\tilde{\Phi}\right)$ is an arbitrary loss function, $\hat{\Phi}$ is a chosen target estimator of Φ and the weight $\omega\epsilon[0,1]$.

The BLF specializes to various choices of loss functions such as the absolute error loss, entropy, *linear exponential* (LINEX) and *squared error loss* (SEL) functions.

The Bayes estimator of Φ, using the BSEL function is as follows:

$$\tilde{\Phi}_{BSE} = \omega\, \hat{\Phi}_{ML} + (1-\omega)\, \tilde{\Phi}_{SE}, \tag{19}$$

where $\hat{\Phi}_{\text{ML}}$ is the ML estimator of Φ and $\tilde{\Phi}_{\text{SE}}$ is its Bayes estimator using SEL function.
Also, the Bayes estimator using the BLL function of Φ is obtained as follows:

$$\tilde{\Phi}_{BL} = \frac{-1}{v} \ln\left\{\omega\, exp\left(-v\hat{\Phi}_{ML}\right) + (1-\omega)\, E\left(exp(-v\Phi)\,\middle|\underline{x}\right)\right\}, \tag{20}$$

where $v \neq 0$ is the shape parameter of BLL function.

The Bayes estimators are considered under the BSEL function; as a symmetric loss function and BLL function; as an asymmetric loss function.

2.1.1 Bayes estimators under balanced squared error loss function

The Bayes estimators of the parameters under BSEL function can be obtained from (15) and (19) as given below

$$\tilde{\Phi}_{jBSE} = \omega\hat{\Phi}_{jML} + (1-\omega)E(\Phi_j|\underline{x})$$

$$= \omega\hat{\Phi}_{jML} + (1-\omega)\int_{\underline{\Phi}} \Phi_j\, \pi(\underline{\Phi}|\underline{x})\, d\underline{\Phi}$$

$$= \omega\hat{\Phi}_{jML} + (1-\omega)A\int_{\underline{\Phi}} \Phi_j\ \lambda^{\theta m+a_1-1}\, \alpha^{m+a_2-1}\beta^{a_3-\theta m-1}\theta^{m+a_4-1}$$

$$\times \prod_{i=1}^{m} \alpha^{R_i}\, x_{(i)}^{\theta-1}\ \ e^{-(R_i+1)\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}} \left[1-\ \bar{\alpha}\, e^{-\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\right]^{-(R_i+2)}$$

$$\times \exp[-(b_1\lambda + b_2\alpha + b_3\beta + b_4\theta)]d\underline{\Phi},\ \Phi_j > 0, j = 1,2,3,4, \quad (21)$$

where $\Phi_1 = \lambda, \Phi_2 = \alpha, \Phi_3 = \beta$ and $\Phi_4 = \theta, \hat{\Phi}_{jML}$ is the estimator of Φ_j using the ML method based on (8), A is defined in (16), $\int_{\underline{\Phi}}$ and $d\underline{\Phi}$ are given by (17).

2.1.2 Bayes estimators under balanced linear exponential loss function

The Bayes estimators of the parameters under BLL function can be obtained from (15) and (20) as follows:

$$\tilde{\Phi}_{jBL} = \frac{-1}{v}\ln\left\{\omega\, exp(-v\hat{\Phi}_{ML}) + (1-\omega)\int_{\underline{\Phi}} exp(-v\Phi_j)\ \pi(\underline{\Phi}|\underline{x})\, d\underline{\Phi}\right\}$$

$$= \frac{-1}{v}\ln\{\omega\, exp(-v\hat{\Phi}_{ML}) + (1-\omega)A\int_{\underline{\Phi}} exp(-v\Phi_j)\ \ \lambda^{\theta m+a_1-1}\, \alpha^{m+a_2-1}$$

$$\times \beta^{a_3-\theta m-1}\theta^{m+a_4-1}\exp[-(b_1\lambda + b_2\alpha + b_3\beta + b_4\theta)]$$

$$\times \prod_{i=1}^{m} \alpha^{R_i}\, x_{(i)}^{\theta-1}\ \ e^{-(R_i+1)\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}} \left[1-\ \bar{\alpha}\, e^{-\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\right]^{-(R_i+2)} d\underline{\Phi}\},$$

$$\Phi_j > 0, j = 1,2,3,4, \qquad (22)$$

where $\Phi_1 = \lambda, \Phi_2 = \alpha, \Phi_3 = \beta$ and $\Phi_4 = \theta, \hat{\Phi}_{jML}$ is the estimator of Φ_j using the ML method based on (8), A is defined in (16), $\int_{\underline{\Phi}}$ and $d\underline{\Phi}$ are given by (17).

Remarks:

- If $R_i = 0, i = 1,2,\ldots,m$ and n equals m, all the results obtained for progressive Type-II censored sample reduced to the complete sample case.

- If $R_i = 0, i = 1,2, \dots, m-1$ and $R_m = n - m,$ then progressive Type-II censored sampling reduces to traditional Type-II censoring.

2.2 Credible intervals

In general, a two-sided 100(1-τ) % CIs of Φ_j are given by

$$P\left[L_{\Phi_j}(\underline{x}) < \Phi_j < U_{\Phi_j}(\underline{x}) \middle| \underline{x}\right] = \int_{L_{\Phi_j}(\underline{x})}^{U_{\Phi_j}(\underline{x})} \pi(\Phi_j|\underline{x})\, d\Phi_j = 1 - \tau,$$
$$j = 1, 2, 3,4, \qquad (23)$$

where L_{Φ_j} and U_{Φ_j} are the *lower limit* (LL) and *upper limit* (UL). The LL and UL can be obtained by evaluating

$$P(\Phi_j > L_{\Phi_j}(\underline{x})|\underline{x}) = 1 - \frac{\tau}{2} \quad \text{and} \quad P(\Phi_j > U_{\Phi_j}(\underline{x})|\underline{x}) = \frac{\tau}{2}. \qquad (24)$$

3. Bayesian Prediction

In this section, the Bayesian prediction (point and interval) for a future observation $Y_{(s)}$, of the MOW-E distribution based on two-sample prediction method are derived.

Considering that $X_{(1)} \le X_{(2)} \le \cdots \le X_{(r)}$ are the first r ordered life times in a random sample of n components (progressive Type-II censoring) whose failure times are identically distributed as a random variable X having the MOW-E $(\lambda, \alpha, \beta, \theta)'$ distribution; informative sample, and $Y_{(1)}, Y_{(2)}, \dots, Y_{(\jmath)}$ is a future independent random sample (of size $\jmath$) from the same distribution. Our aim is to predict a statistic in the future sample based on the informative sample.

For the future sample of size $\jmath$, let $Y_{(s)}$ denotes the s^{th} order statistic, $1 \le s \le \jmath$. The conditional density function of $Y_{(s)}$, given the vector of the parameters $\underline{\Phi}$, is given by

$$h(y_{(s)}|\underline{\Phi}) = D(s) f(y_{(s)}|\underline{\Phi}) [F(y_{(s)}|\underline{\Phi})]^{s-1} [1 - F(y_{(s)}|\underline{\Phi})]^{\jmath - s}, \; y_{(s)} > 0, \qquad (25)$$

where s is the order statistic of the predicted future observation in the future sample,

$$D(s) = s\binom{j}{s} = \frac{j!}{(s-1)!(j-s)!} = \frac{1}{B(s,j-s+1)} \quad \text{and} \quad s = 1,2,3,\dots,j. \qquad (26)$$

Using the binomial expansion theorem for $\left[1 - F\left(y_{(s)}|\underline{\Phi}\right)\right]^{j-s}$, yields

$$h\left(y_{(s)}|\underline{\Phi}\right) = D(s) f\left(y_{(s)}|\underline{\Phi}\right) \sum_{l_1=0}^{j-s} (-1)^{l_1} \binom{j-s}{l_1} \left[F\left(y_{(s)}|\underline{\Phi}\right)\right]^{s+l_1-1}$$

$$= D(s)\frac{\alpha\theta\lambda^{\theta}}{\beta^{\theta}} \sum_{l_1=0}^{j-s} (-1)^{l_1} \binom{j-s}{l_1} y_{(s)}^{\theta-1}\, e^{-\left(\frac{\lambda y_{(s)}}{\beta}\right)^{\theta}}$$

$$\times \left[1 - \bar{\alpha}\, e^{-\left(\frac{\lambda y_{(s)}}{\beta}\right)^{\theta}}\right]^{-(s+l_1+1)} \left[1 - e^{-\left(\frac{\lambda y_{(s)}}{\beta}\right)^{\theta}}\right]^{s+l_1-1}. \qquad (27)$$

For real value of k, using the binomial expansion

$$(1+x)^{k} = \sum_{j=0}^{\infty} \binom{k}{j} x^{j} = \sum_{j=0}^{\infty} \frac{\Gamma(k+1)}{j!\,\Gamma(k-j+1)} x^{j}, \qquad (28)$$

then, the $h\left(y_{(s)}|\underline{\Phi}\right)$ in (27) is expressed as an infinite sum which is given by

$$h\left(y_{(s)}|\underline{\Phi}\right) = D(s)\frac{\alpha\theta\lambda^{\theta}}{\beta^{\theta}} \sum_{l_1=0}^{j-s} \sum_{l_2=0}^{s+l_1-1} \sum_{l_3=0}^{\infty} (\bar{\alpha})^{l_3} (-1)^{(l_1+l_2+l_3)}$$

$$\times \binom{j-s}{l_1}\binom{s+l_1-1}{l_2}\binom{-(s+l_1+1)}{l_3} y_{(s)}^{\theta-1}\, e^{-(l_2+l_3+1)\left(\frac{\lambda y_{(s)}}{\beta}\right)^{\theta}}. \qquad (29)$$

Applying the power series for the exponential function, hence

$$exp\left[-(l_2 + l_3 + 1)\left(\frac{\lambda y_{(s)}}{\beta}\right)^{\theta}\right] = \sum_{l_4=0}^{\infty} \frac{(-1)^{l_4}\,[l_2+l_3+1]^{l_4}\left(\frac{\lambda y_{(s)}}{\beta}\right)^{\theta l_4}}{l_4!}, \qquad (30)$$

substituting (30) in (29) yields

$$h\left(y_{(s)}|\underline{\Phi}\right) = D(s) \sum_{l_1=0}^{j-s} \sum_{l_2=0}^{s+l_1-1} \sum_{(l_3,l_4)=0}^{\infty} \delta_{(l_1+l_2+l_3+l_4)}(\lambda,\alpha,\beta,\theta)\; y_{(s)}^{\theta(l_4+1)-1}, \qquad (31)$$

where

$$\delta_{(l_1+l_2+l_3+l_4)}(\lambda,\alpha,\beta,\theta) = \alpha\theta\,(\bar{\alpha})^{l_3}(-1)^{(l_1+l_2+l_3+l_4)}\,\frac{[l_2+l_3+1]^{l_4}}{l_4!}$$

$$\times \binom{j-s}{l_1}\binom{s+l_1-1}{l_2}\binom{-(s+l_1+1)}{l_3}\left(\frac{\lambda}{\beta}\right)^{\theta(l_4+1)}. \qquad (32)$$

Assuming that the parameters $\underline{\Phi}$ are unknown and independent, then the conditional *ML predictive density* (MLPD) of $Y_{(s)}$ given $\underline{\hat{\Phi}}_{ML}$ can be obtained using the conditional pdf of the s^{th} order statistic which is given by (31) after replacing the vector of parameters $\underline{\Phi}$ by their ML estimators $\underline{\hat{\Phi}}_{ML}$ as follows:

$$h_1\left(y_{(s)}\middle|\underline{\hat{\Phi}}_{ML}\right) = D(s)\sum_{l_1=0}^{j-s}\sum_{l_2=0}^{s+l_1-1}\sum_{(l_3,l_4)=0}^{\infty}\delta_{(l_1+l_2+l_3+l_4)}(\hat{\lambda},\hat{\alpha},\hat{\beta},\hat{\theta})$$

$$\times\ y_{(s)}^{\hat{\theta}(l_4+1)-1}, \qquad y_{(s)} > 0;\ (\underline{\hat{\Phi}}_{ML} > \underline{0}). \qquad (33)$$

The conditional *ML predictor* (MLP) for the future observation $Y_{(s)}$, based on progressive Type-II censoring can be derived using (33) as follows:

$$\hat{y}_{(s)(ML)} = E\left(y_{(s)}\middle|\underline{\hat{\Phi}}_{ML}\right)$$

$$= \int_{y_{(s)}} y_{(s)}\, h_1\left(y_{(s)}\middle|\underline{\hat{\Phi}}_{ML}\right) dy_{(s)}$$

$$= D(s)\sum_{l_1=0}^{j-s}\sum_{l_2=0}^{s+l_1-1}\sum_{(l_3,l_4)=0}^{\infty}\delta_{(l_1+l_2+l_3+l_4)}(\hat{\lambda},\hat{\alpha},\hat{\beta},\hat{\theta})\int_0^{\infty} y_{(s)}^{\hat{\theta}(l_4+1)}\, dy_{(s)}, \qquad (34)$$

where $D(s)$ is defined in (26) and $\delta_{(l_1+l_2+l_3+l_4)}(\lambda,\alpha,\beta,\theta)$ is given by (32).

Assuming that the parameters $(\lambda,\alpha,\beta,\theta)'$ are unknown and independent, then the *Bayesian predictive density* (BPD) of $Y_{(s)}$ given $\underline{x}$ based on informative prior can be obtained as follows:

$$h_2\left(y_{(s)}\middle|\underline{x}\right) = \int_{\underline{\Phi}} h\left(y_{(s)}\middle|\underline{\Phi}\right)\pi\left(\underline{\Phi}\middle|\underline{x}\right) d\underline{\Phi}, \quad y_{(s)} > 0;\ (\underline{\Phi} > \underline{0}), \qquad (35)$$

where $\pi\left(\underline{\Phi}\middle|\underline{x}\right)$ is given by (15), $\int_{\underline{\Phi}}$ and $d\underline{\Phi}$ are given by (17) and $h\left(y_{(s)}\middle|\underline{\Phi}\right)$ is defined in (31).

Substituting (15) and (31) into (35), then the BPD of $Y_{(s)}$ given $\underline{x}$ is

$$h_2\left(y_{(s)}\middle|\underline{x}\right) = D(s)A\sum_{l_1=0}^{j-s}\sum_{l_2=0}^{s+l_1-1}\sum_{(l_3,l_4)=0}^{\infty}\int_{\underline{\Phi}}\delta^{*}_{(l_1+l_2+l_3+l_4)}(\lambda,\alpha,\beta,\theta)$$

$$\times\ y_{(s)}^{\theta(l_4+1)-1}\exp[-(b_1\lambda + b_2\alpha + b_3\beta + b_4\theta)]$$

$$\times\prod_{i=1}^{m}\alpha^{R_i}\, x_{(i)}^{\theta-1}\ e^{-(R_i+1)\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\left[1-\ \bar{\alpha}\, e^{-\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\right]^{-(R_i+2)} d\underline{\Phi}, \qquad (36)$$

where

$$\delta^{*}_{(l_1+l_2+l_3+l_4)}(\lambda,\alpha,\beta,\theta) = (\bar{\alpha})^{l_3}\,\alpha^{m+a_2}\,\theta^{m+a_4}\,\lambda^{[\theta(m+l_4+1)+a_1-1]}\,\beta^{[-\theta(m+l_4+1)+a_3-1]}$$

$$\times(-1)^{(l_1+l_2+l_3+l_4)}\,\frac{[l_2+l_3+1]^{l_4}}{l_4!}\binom{j-s}{l_1}\binom{s+l_1-1}{l_2}\binom{-(s+l_1+1)}{l_3}. \quad (37)$$

3.1 Point prediction

Based on progressive Type-II censoring, the Bayesian prediction is considered under two types of loss functions, the BSEL function; as a symmetric loss function and BLL function; as an asymmetric loss function.

I. Balanced squared error loss function

The *Bayes predictor* (BP) for the future observation $Y_{(s)}$, under BSEL function can be derived using (19) and (36) as given below

$$\hat{y}_{(s)BSE} = \omega\hat{y}_{(s)ML} + (1-\omega)\int_{y_{(s)}} y_{(s)}\,h_2\left(y_{(s)}\middle|\underline{x}\right)\,dy_{(s)}$$

$$= \omega\hat{y}_{(s)ML} + (1-\omega)D(s)A\sum_{l_1=0}^{j-s}\sum_{l_2=0}^{s+l_1-1}\sum_{(l_3,l_4)=0}^{\infty}\int_{\underline{\Phi_*}} y_{(s)}^{\theta(l_4+1)}$$

$$\times\ \delta^{*}_{(l_1+l_2+l_3+l_4)}(\lambda,\alpha,\beta,\theta)\exp[-(b_1\lambda+b_2\alpha+b_3\beta+b_4\theta)]$$

$$\times\prod_{i=1}^{m}\alpha^{R_i}\,x_{(i)}^{\theta-1}\ e^{-(R_i+1)\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\left[1-\bar{\alpha}\,e^{-\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\right]^{-(R_i+2)} d\underline{\Phi_*}, \quad (38)$$

where $\hat{y}_{(s)ML}$ is the conditional MLP for the future observation of $y_{(s)}$, A^{-1} is defined in (16), $D(s)$ is given by (26), $\delta^{*}_{(l_1+l_2+l_3+l_4)}(\lambda,\alpha,\beta,\theta)$ is defined in (37), $\int_{\underline{\Phi_*}} = \int_{y_{(s)}}\int_{\lambda}\int_{\alpha}\int_{\beta}\int_{\theta}$ and $d\underline{\Phi_*} = d\theta\,d\beta\,d\alpha\,d\lambda\,dy_{(s)}$. (39)

II. Balanced linear exponential loss function

The BP for the future observation $Y_{(s)}$, under BLL function can be derived using (20) and (36) as follows:

$$\hat{y}_{(s)BLL} = \frac{-1}{v}\ln\left\{\omega\, exp\left(-v\, \hat{y}_{(s)ML}\right) + (1-\omega)\int_{y_{(s)}} exp\left(-v\, y_{(s)}\right) h_2\left(y_{(s)}\middle|\underline{x}\right) dy_{(s)}\right\}$$

$$= \frac{-1}{v}\ln\{\omega\, exp\left(-v\, \hat{y}_{(s)ML}\right) + (1-\omega) D(s) A \sum_{l_1=0}^{j-s}\sum_{l_2=0}^{s+l_1-1}\sum_{(l_3,l_4)=0}^{\infty}\int_{\underline{\Phi_*}} y_{(s)}^{\theta(l_4+1)-1}$$

$$\times\ \delta^*_{(l_1+l_2+l_3+l_4)}(\lambda,\alpha,\beta,\theta)\exp\left[-\left(v\, y_{(s)} + b_1\lambda + b_2\alpha + b_3\beta + b_4\theta\right)\right]$$

$$\times \prod_{i=1}^{m}\alpha^{R_i} x_{(i)}^{\theta-1}\ e^{-(R_i+1)\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\left[1-\bar{\alpha}\, e^{-\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\right]^{-(R_i+2)} d\underline{\Phi_*}\}, \qquad (40)$$

where $\int_{\underline{\Phi_*}}$ and $d\underline{\Phi_*}$ are given by (39).

3.2 Interval prediction

A 100(1-τ)% *Bayesian predictive interval* (BPI) for the future observation $Y_{(s)}$, such that $P\left(L_{(s)}(\underline{x}) < Y_{(s)} < U_{(s)}(\underline{x})\middle|\underline{x}\right) = 1-\tau$, can be obtained from (36) as given below

$$P(Y_{(s)} > L_{(s)}(\underline{x})|\underline{x}) = \int_{L_{(s)}(\underline{x})}^{\infty} h_2\left(y_{(s)}\middle|\underline{x}\right) dy_{(s)} = 1-\frac{\tau}{2}, \qquad (41)$$

and

$$P(\ Y_{(s)} > U_{(s)}(\underline{x})|\underline{x}) = \int_{U_{(s)}(\underline{x})}^{\infty} h_2\left(y_{(s)}\middle|\underline{x}\right) dy_{(s)} = \frac{\tau}{2}. \qquad (42)$$

Substituting (36) in (41) and (42), then the BPI are obtained as follows:

$$P(Y_{(s)} > L_{(s)}(\underline{x})|\underline{x}) = D(s) A \sum_{l_1=0}^{j-s}\sum_{l_2=0}^{s+l_1-1}\sum_{(l_3,l_4)=0}^{\infty}\int_{L_{(s)}(\underline{x})}^{\infty}\int_{\underline{\Phi}}\delta^*_{(l_1+l_2+l_3+l_4)}(\lambda,\alpha,\beta,\theta)$$

$$\times\ y_{(s)}^{\theta(l_4+1)-1}\exp[-(b_1\lambda + b_2\alpha + b_3\beta + b_4\theta)]\prod_{i=1}^{m}\alpha^{R_i} x_{(i)}^{\theta-1}$$

$$\times \prod_{i=1}^{m} e^{-(R_i+1)\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\left[1-\bar{\alpha}\, e^{-\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\right]^{-(R_i+2)} d\underline{\Phi}\, dy_{(s)} = 1-\frac{\tau}{2}, \qquad (43)$$

and

$$P(Y_{(s)} > U_{(s)}(\underline{x})|\underline{x}) = D(s) A \sum_{l_1=0}^{j-s}\sum_{l_2=0}^{s+l_1-1}\sum_{(l_3,l_4)=0}^{\infty}\int_{U_{(s)}(\underline{x})}^{\infty}\int_{\underline{\Phi}}\delta^*_{(l_1+l_2+l_3+l_4)}(\lambda,\alpha,\beta,\theta)$$

$$\times\ y_{(s)}^{\theta(l_4+1)-1}\exp[-(b_1\lambda + b_2\alpha + b_3\beta + b_4\theta)]\prod_{i=1}^{m}\alpha^{R_i} x_{(i)}^{\theta-1}$$

$$\times \prod_{i=1}^{m} e^{-(R_i+1)\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\left[1-\bar{\alpha}\, e^{-\left(\frac{\lambda x_{(i)}}{\beta}\right)^{\theta}}\right]^{-(R_i+2)} d\underline{\Phi}\, dy_{(s)} = \frac{\tau}{2}. \qquad (44)$$

Remark:

- If $s = 1$, in (38) and (40), one can predict the minimum observable, $Y_{(1)}$, which represents the first failure time in a future sample of size $\jmath$.
- If $s = \jmath$, in (38) and (40), one can predict the maximum observable, $Y_{(\jmath)}$, which represents the largest failure time in a future sample of size $\jmath$.
- If $s = \frac{\jmath+1}{2}$, in (38) and (40), one can predict the median observable if $\jmath$ is odd, $Y_{\left(\frac{\jmath+1}{2}\right)}$, which represents the median failure time in a future sample of size $\jmath$.

4. Numerical Illustration

This section aims to investigate the precision of the theoretical results of Bayesian estimation and prediction on the basis of simulated and real data set.

4.1 Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the presented Bayes estimates on the basis of generated data from the MOW-E distribution. The Bayes estimates of the parameters based on progressive Type-II censoring scheme are computed. Moreover, CIs for the parameters are calculated. Also, the Bayesian prediction (point and interval) for a future observation from the MOW-E distribution based on progressive Type-II censored data are computed. All simulation studies are performed using R programming language.

4.2 Simulation algorithm

Applying the algorithm given by Balakrishnan and Sandhu (1995), the following steps are used to generate a progressive Type-II censored sample from the MOW-E distribution as follows:

Step 1: Generate m independent $U(0,1)$ random variables $U_1, U_2, \dots, U_m$.
Step 2: For given values of the progressive censoring scheme $R_1, R_2, \dots, R_m$, set $Y_i = U_i^{1/(i+\sum_{j=m-i+1}^{m} R_j)}$, for $i = 1,2, \dots, m$.

Step 3: Set $UP_i = 1 - (Y_m Y_{m-1} Y_{m-2} \dots Y_{m-i+1}), i = 1,2,\dots,m$. Hence, $UP_1, UP_2, \dots, UP_m$ are progressive Type-II censored sample of size m from $U(0,1)$ distribution.

Step 4: For given values of the parameters λ, α, β and θ , the inverse cdf method, can be used to generate m progressive Type-II censored sample from the MOW-E distribution whose cdf is given by (2). Thus, by solving the nonlinear equation

$$x_i = \left(\frac{\beta}{\lambda}\right)\left[-\ln\left(\frac{1-(UP_i)}{1-\bar{\alpha}(UP_i)}\right)\right]^{1/\theta}, \quad i = 1,2,\dots,m,$$

the resulting set, $(x_1, x_2, \dots, x_m)$, is the required progressive Type-II censored sample of size m from the MOW-E distribution and this obtained sample is ordered.

Step 5: For given values of a_j and b_j, generate λ, α, β and θ from the gamma prior distributions.

Step 6: Calculate the Bayes estimates of the parameters under BSEL and BLL functions.

Step 7: Repeat all the previous steps N=10000 times for the samples of size (n =30, 60, 100).

Step 8: The BPD of the future observation $Y_{(s)}$, can be obtained.

Step 9: The BPs are calculated under BSEL and BLL functions. Also, the BPI is evaluated.

For each sample size, different ratio of effective sample sizes, $r = \frac{m}{n} = 0.2, 0.5$ and 0.8 and set of different samples schemes, where

Scheme I: $R_1 = n - m$ and $R_2 = R_3 = \cdots = R_m = 0$.

Scheme II: $R_1 = R_2 = \cdots = R_{m-1} = 1$ and $R_m = n - 2m + 1$, for $r = 0.2$ and 0.5. If $r = 0.8$, then $R_{i+1} = 1$ and $R_j = 0$; $i = 0, 4, 8, 12, \dots, m,$ $i \neq j$.

Scheme III: $0, 0, \dots, 0, \frac{n-m}{3}, 0, 0, \dots, 0, \frac{n-m}{3}, 0, 0, \dots, 0, \frac{n-m}{3}$.

Evaluating the performance of the estimates is considered through measurement of accuracy. In order to study the precision and variation of the estimates, it is convenient to use the *estimated risk* (ER) $= \frac{\sum_{i=1}^{N}(\text{estimated value} - \text{true value})^2}{N}$.

Tables 1-6 display the Bayes averages, ERs and CIs of the unknown parameters under BSEL and BLL functions based on progressive Type-II censoring under different samples schemes. Also, BPs and BPI of the future observations under BSEL and BLL functions based on progressive Type-II censoring under two-sample prediction method and different sample schemes are presented in Table 8.

4.3 Some applications

The main aim of this subsection is to demonstrate how the proposed methods can be used in practice. A real lifetime data set is used for this purpose. The MOW-E distribution is fitted to the real data using *Kolmogorov-Smirnov* (K–S) goodness of fit test through R programming language.

In lung transplant recipients, respiratory tract infections are associated with faster progression through stages of bronchiolitis obliterans syndrome and mortality. Common causative pathogens for *respiratory tract infections* (RTIs) include *non-fermenting gram-negative bacilli* (NFGNB). Data to guide optimal treatment durations for limited NFGNB RTIs in this population. This was a single-center, retrospective, cohort study of adult lung transplant recipients who received systemic antibiotic treatment for RTIs caused by NFNGB and had at least 28 days of post-treatment follow-up.

The data set was taken from web portal of Indiana University Health (2021), http://dx.doi.org/10.17632/5xfkn6kk82.1 and it refers to the duration of therapy (days) for NFGNB RTI treatment (DOT) for 60 lung transplant recipients. The data set is:

9, 10, 14, 29, 16, 18, 16, 10, 7, 15, 11, 10, 21, 14, 13, 10, 7, 12, 18, 18, 19, 13, 12, 10, 13, 13, 17, 15, 10, 10, 18, 14, 15, 7, 14, 14, 12, 12, 19, 26, 15, 5, 9, 10, 10, 18, 17, 12, 10, 7, 10, 14, 14, 15, 5, 19, 13, 9, 7 and 7.

The K–S goodness of fit test is applied to check the validity of the introduced fitted model. The p-value is 0.6604. The p-value showed that the proposed model fits the data very well. The three progressive censoring schemes considered in the simulation study were applied for the real data set. The results are displayed in Table 7. Table 7 presents the Bayes estimates and *standard errors* (SEs) of the unknown parameters for the real data set under BSEL and BLL functions based on progressive Type-II censoring under different samples schemes. Also, the BPs and BPI of the future observations for the real data set under BSEL and BLL functions based on progressive Type-II censoring under different sample schemes are presented in Table 9.

4.4 Concluding remarks

- It is observed that from Tables 1-6, the accuracy of the ER gets better when the sample size increases.
- It is noticed, from Tables 1-6, that the ERs are decreasing as the ratio of effective samples size (r) increases.
- The ERs and the lengths of the CIs of the Bayes estimates under BLL function are in most cases less than the ERs and the lengths of the CIs of the Bayes estimates under BSEL function, so the Bayes estimators under BLL function perform better than the Bayes estimators under BSEL function.
- Scheme I is the best censoring scheme where in most cases it has the lowest ERs and the narrower lengths of the CIs.

- The results in Tables 8 and 9 indicate that the length of the interval of the first future order statistic is smaller than the length of the interval of the last future order statistic.
- The Bayes intervals include the predictive values (between the LL and UL).
- In all cases, the BPs and the lengths of the intervals under BLL function are less than the BPs and the lengths of the intervals under BSEL function.
- The lengths of the intervals of the Bayes predictors increase when s increases.

5. General Conclusion

In this research, the Bayes estimators are obtained for the parameters of the MOW-E distribution based on progressive Type-II censored samples. The estimators are considered under two different loss functions, the BSEL function; as a symmetric loss function and BLL function; as an asymmetric loss function. The BLF is a mixture of Bayes and non-Bayes estimators. The estimators are obtained based on informative gamma prior distributions. Also, the two-sample prediction method is applied to obtain the Bayesian prediction (point and interval) for future order statistics of the MOW-E distribution based on progressive Type-II censored samples. The predictors are considered under two different loss functions, the BSEL function; as a symmetric loss function and BLL function; as an asymmetric loss function. A numerical example is given to illustrate the theoretical results and an application using real data set is used to demonstrate how the results can be used in practice. In general, numerical computations showed that when the sample size n and the ratio of effective samples of size (r) increases, the ERs and the lengths of the CIs decreases. The ERs of the Bayes estimates under BLL function are in most cases less than the ERs of the Bayes estimates under BSEL function, so the Bayes estimators under BLL function perform better than the Bayes estimators under BSEL function. Also, the Bayes intervals include the predictive values, the length of the interval of the first future order statistic is smaller than the length of the interval of the last future order statistic and in all cases, the BPs and the lengths of the intervals under BLL function are less than the BPs and the lengths of the intervals under BSEL function. The E-Bayesian estimation and prediction methods under different type of loss functions such as general entropy and Precautionary loss functions for estimating the parameters of MOW-E distribution would be useful as a basis for further researches in distribution theory.

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Appendix

Table 1

Bayes averages, estimated risks and 95% credible intervals of the parameters under balanced squared error loss function based on progressive Type-II censoring under Scheme I

(N=10000, $r = 0.2, 0.5$ and 0.8, $\lambda = 1.2, \alpha = 1.3$, $\beta = 2.2, \theta = 1.3$ and $\omega = 0.3$)

n	m	Φ	Average	ER	UL	LL	Length
30	6	λ	1.1606	0.1075	1.9387	0.9936	0.9451
		α	1.2990	0.1328	1.8435	1.1129	0.7306
		β	0.9434	0.8764	1.9079	0.6316	1.2764
		θ	1.1942	0.1190	1.8217	0.7180	1.1037
	15	λ	0.8129	0.0599	1.1583	0.6665	0.4918
		α	1.2522	0.0967	1.9040	1.1753	0.7287
		β	1.7218	0.2189	3.0355	1.8206	1.2149
		θ	1.1819	0.0910	1.7480	0.7898	0.9581
	24	λ	0.8706	0.0291	1.1730	0.8390	0.3341
		α	1.1924	0.0423	1.6997	1.2089	0.4908
		β	1.6647	0.1017	2.7731	1.9451	0.8279
		θ	1.1765	0.0463	1.5936	0.9535	0.6400
60	12	λ	0.8046	0.0623	1.1516	0.6627	0.4889
		α	1.2577	0.1019	1.9135	1.1886	0.7249
		β	1.7201	0.2177	3.0329	1.8149	1.2180
		θ	1.1885	0.0903	1.7519	0.8075	0.9444

Table 1 (continued)

60	30	λ	0.8830	0.0277	1.1767	0.8441	0.3326
		α	1.1971	0.0434	1.7049	1.2143	0.4907
		β	1.6629	0.1008	2.7679	1.9447	0.8232
		θ	1.1739	0.0462	1.5976	0.9464	0.6511
	48	λ	1.0969	0.0208	1.4207	1.0983	0.3223
		α	1.1037	0.0072	1.4310	1.1139	0.3171
		β	1.5522	0.0057	2.3149	2.0442	0.2707
		θ	1.2745	0.0209	1.5665	1.2875	0.2789
100	20	λ	1.0165	0.0090	1.4137	1.1170	0.2967
		α	0.9772	0.0227	1.2712	1.0135	0.2577
		β	1.6218	0.0320	2.5350	2.1171	0.4180
		θ	1.3571	0.0656	1.7017	1.1838	0.5179
	50	λ	1.0272	0.0011	1.2603	1.1337	0.1266
		α	1.0930	0.0022	1.3661	1.2251	0.1410
		β	1.4470	0.0224	2.1817	1.9510	0.2217
		θ	1.2118	0.0016	1.3851	1.2769	0.1081
	80	λ	1.0215	0.0001	1.2149	1.1585	0.0564
		α	1.1107	0.0013	1.3628	1.2825	0.0803
		β	1.5574	0.0016	2.2684	2.1414	0.1270
		θ	1.1763	0.0006	1.3289	1.2346	0.0942

Table 2

Bayes averages, estimated risks and 95% credible intervals of the parameters under balanced linear exponential loss function based on progressive Type-II censoring under Scheme I

(N=10000, $r = 0.2, 0.5$ and 0.8, $\lambda = 1.2, \alpha = 1.3$, $\beta = 2.2, \theta = 1.3, \omega = 0.3$ and $v = -2$)

n	m	Φ	Average	ER	UL	LL	Length
30	6	λ	0.9490	0.0292	1.3862	0.7761	0.6102
		α	0.8562	0.1227	1.2533	0.6534	0.5999
		β	1.8539	0.3770	3.3709	2.0172	1.3537
		θ	1.5081	0.1874	1.9374	1.3102	0.6273
	15	λ	0.9277	0.0077	1.2629	0.9991	0.2637
		α	1.0153	0.0285	1.4165	0.8663	0.5502
		β	1.6857	0.0578	2.6502	2.1567	0.4934
		θ	1.3331	0.0284	1.5575	1.2364	0.3211
	24	λ	0.9582	0.0018	1.2345	1.1007	0.1338
		α	1.0441	0.0073	1.3562	1.0829	0.2733
		β	1.6125	0.0144	2.4254	2.1780	0.2474
		θ	1.2598	0.0062	1.4229	1.2627	0.1601
60	12	λ	1.0057	0.0159	1.4355	1.0091	0.4264
		α	1.1378	0.0191	1.5386	1.1233	0.4153
		β	1.3775	0.0899	2.2383	1.5684	0.6699
		θ	0.9350	0.1842	1.1907	0.6373	0.5534

Table 2 (continued)

60	30	λ	1.0120	0.0044	1.3244	1.1096	0.2148
		α	1.0399	0.0048	1.3139	1.1286	0.1853
		β	1.4812	0.0112	2.2300	1.9612	0.2688
		θ	1.1559	0.0065	1.3059	1.0789	0.2270
	48	λ	1.0305	0.0012	1.2653	1.1594	0.1059
		α	1.1032	0.0017	1.3684	1.2389	0.1294
		β	1.5335	0.0009	2.2462	2.1331	0.1131
		θ	1.1732	0.0011	1.3290	1.2152	0.1138
100	20	λ	0.9191	0.0096	1.2104	0.9448	0.2655
		α	1.0343	0.0092	1.3796	1.1133	0.2662
		β	1.5678	0.0046	2.3277	2.1148	0.2129
		θ	1.3487	0.0426	1.5862	1.2692	0.3170
	50	λ	1.0027	0.0010	1.2071	1.1172	0.0899
		α	1.1334	0.0048	1.4208	1.2967	0.1241
		β	1.5612	0.0037	2.3117	2.1374	0.1744
		θ	1.1624	0.0020	1.3041	1.1862	0.1178
	80	λ	1.0233	0.0003	1.2262	1.1568	0.0694
		α	1.1128	0.0016	1.3655	1.2908	0.0747
		β	1.5496	0.0007	2.2440	2.1577	0.0863
		θ	1.1782	0.0003	1.3302	1.255	0.0755

Table 3

Bayes averages, estimated risks and 95% credible intervals of the parameters under balanced squared error loss function based on progressive Type-II censoring under Scheme II

(N=10000, $r = 0.2, 0.5$ and 0.8, $\lambda = 1.2, \alpha = 1.3, \beta = 2.2, \theta = 1.3$ and $\omega = 0.3$)

n	m	Φ	Average	ER	UL	LL	Length
30	6	λ	1.5209	0.9046	2.7471	0.8108	1.9363
		α	1.3532	0.2232	2.1694	1.1356	1.0339
		β	2.0921	0.8441	3.7511	2.0371	1.7140
		θ	1.5448	0.6419	2.8278	0.9030	1.9247
	15	λ	1.2586	0.2678	2.1166	0.8312	1.2854
		α	1.2452	0.1073	1.9139	1.0761	0.8378
		β	1.5810	0.1918	2.8203	1.5065	1.3138
		θ	0.8800	0.2840	1.1887	0.2868	0.9019
	24	λ	1.1872	0.1266	1.7742	1.0457	0.7285
		α	1.0508	0.0265	1.5187	0.8725	0.6462
		β	1.6008	0.1035	2.7929	1.7350	1.0579
		θ	1.3468	0.0878	1.8116	1.0782	0.7333
60	12	λ	1.3159	0.4039	2.1881	0.7492	1.4388
		α	1.3416	0.1990	2.1044	1.2005	0.9039
		β	1.5642	0.1218	2.6912	1.4219	1.2692
		θ	1.3689	0.1408	1.954	0.9404	1.0138

Table 3 (continued)

60	30	λ	1.1715	0.1418	1.8254	0.9129	0.9125
		α	1.1881	0.0530	1.8184	1.1105	0.7079
		β	1.3287	0.1084	2.1300	1.5627	0.5673
		θ	1.3002	0.0727	1.7716	0.9936	0.7780
	48	λ	1.1402	0.0550	1.5843	1.0647	0.5196
		α	1.0228	0.0169	1.3650	1.0655	0.2994
		β	1.4245	0.0328	2.1670	1.8428	0.3242
		θ	1.2483	0.0234	1.5670	1.1262	0.4408
100	20	λ	1.1527	0.1804	1.8878	0.7906	1.0972
		α	1.0147	0.0334	1.5771	0.9053	0.6718
		β	1.6466	0.0444	2.6336	2.0220	0.6116
		θ	1.2340	0.0121	1.4814	0.9877	0.4937
	50	λ	0.8399	0.1131	1.2932	0.6033	0.6899
		α	1.0587	0.0064	1.3768	1.1119	0.2649
		β	1.6166	0.0234	2.4746	2.0977	0.3769
		θ	1.1428	0.0088	1.3271	1.1037	0.2233
	80	λ	1.0697	0.0064	1.3442	1.1786	0.1656
		α	1.1106	0.0015	1.3806	1.2813	0.0993
		β	1.5149	0.0024	2.2201	2.1046	0.1155
		θ	1.1335	0.0066	1.3090	1.1734	0.1356

Table 4

Bayes averages, estimated risks and 95% credible intervals of the parameters under balanced linear exponential loss function based on progressive Type-II censoring under Scheme II

(N=10000, $r = 0.2, 0.5$ and 0.8, $\lambda = 1.2, \alpha = 1.3$, $\beta = 2.2, \theta = 1.3, \omega = 0.3$ and $v = -2$)

n	m	Φ	Average	ER	UL	LL	Length
30	6	λ	1.0893	0.1343	1.8077	0.5152	1.2926
		α	1.1402	0.2628	2.1425	0.5606	1.5819
		β	1.1930	0.4034	2.3250	0.9400	1.3849
		θ	1.6029	0.3702	2.2242	1.1568	1.0674
	15	λ	1.0237	0.0763	1.6437	0.6471	0.9966
		α	0.9044	0.0965	1.4254	0.5707	0.8547
		β	1.4384	0.0740	2.3500	1.3572	0.9928
		θ	1.4236	0.1317	1.9282	1.1907	0.7375
	24	λ	0.9464	0.0187	1.3349	0.8650	0.4698
		α	0.9517	0.0622	1.3669	0.7666	0.6003
		β	1.6117	0.0380	2.6318	1.9516	0.6801
		θ	1.2520	0.0105	1.4745	1.0927	0.3817
60	12	λ	1.0294	0.0457	1.5956	0.8492	0.7464
		α	1.0331	0.0513	1.6704	0.8272	0.8433
		β	1.8130	0.2196	2.9372	2.1056	0.8316
		θ	1.5814	0.3390	2.2486	1.1843	1.0643

Table 4 (continued)

60	30	λ	1.0445	0.0415	1.5466	0.8312	0.7154
		α	1.1511	0.0478	1.6448	0.9302	0.7147
		β	1.7070	0.0720	2.5901	2.0909	0.4993
		θ	1.1223	0.0209	1.3212	0.8547	0.4665
	48	λ	1.0592	0.0062	1.3364	1.1140	0.2224
		α	1.0109	0.0247	1.3252	0.9316	0.3936
		β	1.6097	0.0121	2.3597	2.1664	0.1933
		θ	1.1920	0.0028	1.3981	1.1967	0.2015
100	20	λ	1.0186	0.0387	1.5286	0.8133	0.7154
		α	1.0267	0.0299	1.4857	0.8585	0.6272
		β	1.3155	0.1247	2.1578	1.5465	0.6113
		θ	1.5473	0.2633	2.0102	1.2229	0.7872
	50	λ	0.9953	0.0019	1.2075	1.1035	0.1040
		α	1.1159	0.0019	1.3747	1.2821	0.0926
		β	1.4708	0.0123	2.1717	2.0028	0.1689
		θ	1.1377	0.0094	1.3072	1.1256	0.1816
	80	λ	1.0459	0.0014	1.2733	1.1879	0.0854
		α	1.1052	0.0006	1.3402	1.2809	0.0593
		β	1.5336	0.0005	2.2214	2.1405	0.0809
		θ	1.1627	0.0013	1.3060	1.2141	0.0918

Table 5

Bayes averages, estimated risks and 95% credible intervals of the parameters under balanced squared error loss function based on progressive Type-II censoring under Scheme III

(N=10000, $r = 0.2, 0.5$ and 0.8, $\lambda = 1.2, \alpha = 1.3$, $\beta = 2.2, \theta = 1.3$ and $\omega = 0.3$)

n	m	Φ	Average	ER	UL	LL	Length
30	6	λ	1.2436	0.2353	2.2798	0.9375	1.3423
		α	1.3685	0.2599	2.2801	1.0790	1.2011
		β	1.7603	0.2286	3.1101	1.9167	1.1934
		θ	1.2540	0.1915	2.1021	0.6088	1.4934
	15	λ	1.1877	0.1889	2.1699	0.9518	1.2181
		α	1.3680	0.2278	2.0068	1.0792	0.9276
		β	1.7539	0.2015	3.0470	1.9520	1.0950
		θ	1.2188	0.1634	2.0101	0.6477	1.3623
	24	λ	1.1490	0.1128	1.9485	1.0030	0.9455
		α	1.2753	0.1259	1.9957	1.1370	0.8587
		β	1.7056	0.1230	2.8621	1.9954	0.8667
		θ	1.1703	0.1190	1.8206	0.7260	1.0946
60	12	λ	1.2306	0.2019	1.9664	1.0517	0.9147
		α	1.3362	0.2124	2.1876	1.1134	1.0742
		β	1.2649	0.1839	2.1169	1.3642	0.7527
		θ	1.2268	0.1641	2.0199	0.6742	1.3457

Table 5 (continued)

60	30	λ	1.1818	0.1106	1.7752	1.0652	0.7100
		α	1.2817	0.1292	2.0034	1.1464	0.8570
		β	1.3287	0.1084	2.1300	1.5627	0.5673
		θ	1.1691	0.1187	1.8260	0.7126	1.1133
	48	λ	1.1747	0.0810	1.6385	0.9967	0.6418
		α	1.1208	0.0277	1.5577	0.9348	0.6230
		β	1.5580	0.0227	2.4223	1.8758	0.5465
		θ	1.3678	0.0923	1.8543	1.2751	0.5793
100	20	λ	1.0778	0.0504	1.7072	1.0091	0.6981
		α	1.2483	0.1041	1.9515	1.1808	0.7707
		β	1.7879	0.1682	2.9150	2.1920	0.7230
		θ	1.3563	0.1072	1.8504	0.8057	1.0446
	50	λ	1.0140	0.0030	1.2727	1.0670	0.2057
		α	1.1587	0.0129	1.4821	1.2758	0.2063
		β	1.7291	0.0872	2.6098	2.1827	0.4271
		θ	1.3296	0.0500	1.6315	1.2627	0.3688
	80	λ	1.0302	0.0015	1.2579	1.1142	0.1437
		α	1.1006	0.0013	1.3588	1.2371	0.1217
		β	1.5700	0.0026	2.2895	2.1956	0.0939
		θ	1.2377	0.0104	1.4682	1.2880	0.1802

Table 6

Bayes averages, estimated risks and 95% credible intervals of the parameters under balanced linear exponential loss function based on progressive Type-II censoring under Scheme III

(N=10000, $r = 0.2, 0.5$ and 0.8, $\lambda = 1.2, \alpha = 1.3$, $\beta = 2.2, \theta = 1.3, \omega = 0.3$ and $v = -2$)

n	m	Φ	Average	ER	UL	LL	Length
30	6	λ	1.0690	0.1192	1.7750	0.5058	1.2692
		α	0.8670	0.1305	1.3036	0.4536	0.8501
		β	1.3015	0.1940	2.3530	1.1810	1.1720
		θ	1.4210	0.1332	1.9735	1.0673	0.9062
	15	λ	1.0994	0.0933	1.7620	0.8571	0.9048
		α	1.1467	0.0383	1.6613	0.9914	0.6699
		β	1.3926	0.1203	2.4757	1.5387	0.9370
		θ	1.3658	0.0837	1.8325	1.1331	0.6994
	24	λ	0.9259	0.0122	1.2792	0.9480	0.3311
		α	1.1340	0.0230	1.5715	1.0572	0.5142
		β	1.7213	0.0898	2.7639	2.1446	0.6192
		θ	1.3418	0.0423	1.6146	1.2249	0.3897
60	12	λ	1.1203	0.0983	1.7723	0.9855	0.7868
		α	1.1489	0.0394	1.6654	0.9940	0.6714
		β	1.7876	0.1547	2.7681	2.0706	0.6975
		θ	1.3386	0.0986	1.8813	1.0199	0.8614

Table 6 (continued)

60	30	λ	1.0406	0.0250	1.4952	0.9583	0.5369
		α	1.1319	0.0217	1.5621	1.0557	0.5063
		β	1.7241	0.1040	2.7137	2.1225	0.5912
		θ	1.1223	0.0209	1.3212	0.8547	0.4665
	48	λ	1.0449	0.0027	1.2909	1.1426	0.1482
		α	1.0367	0.0110	1.3168	1.0544	0.2624
		β	1.6135	0.0166	2.4058	2.1680	0.2378
		θ	1.2073	0.0051	1.4326	1.2092	0.2234
100	20	λ	0.9336	0.2717	1.4558	0.8251	0.6307
		α	1.0267	0.0299	1.4857	0.8585	0.6272
		β	1.6282	0.0429	2.5800	1.9606	0.6194
		θ	1.2560	0.0437	1.7178	0.9501	0.7677
	50	λ	1.0004	0.0013	1.2072	1.1080	0.0992
		α	1.0712	0.0010	1.3054	1.2294	0.0760
		β	1.6138	0.0127	2.3522	2.2019	0.1502
		θ	1.1396	0.0070	1.2951	1.1470	0.1481
	80	λ	1.0111	0.0008	1.2155	1.1323	0.0832
		α	1.1068	0.0009	1.3531	1.2790	0.0741
		β	1.5294	0.0008	2.2149	2.1305	0.0844
		θ	1.1805	0.0007	1.3508	1.2475	0.1033

Table 7

Bayes estimates and standard errors of the parameters for the real data set under balanced loss functions based on progressive Type-II censoring under different samples schemes

	n	m	Φ	Scheme I		Scheme II		Scheme III	
				Estimate	SE	Estimate	SE	Estimate	SE
BSE	60	12	λ	0.4959	0.1236	0.7527	0.5585	0.7490	0.4186
			α	2.9718	0.3037	1.6074	0.9125	3.3438	0.9786
			β	2.7219	0.0657	3.2294	0.7017	2.8867	0.2027
			θ	3.8460	0.0708	3.5095	0.4695	3.6755	0.2596
		30	λ	0.3189	0.0321	0.2858	0.1602	0.6859	0.3120
			α	2.7056	0.0676	2.9664	0.3161	2.9752	0.2994
			β	2.4452	0.0080	2.7500	0.1389	2.2518	0.1519
			θ	4.1259	0.0242	4.1348	0.1015	3.5962	0.2186
		48	λ	0.1251	0.0068	0.2812	0.0355	0.3747	0.0435
			α	2.4824	0.0017	2.4310	0.0218	2.7832	0.1094
			β	2.5159	0.0019	2.2508	0.1351	2.6364	0.0247
			θ	3.9325	0.0075	4.1954	0.0784	3.9124	0.0246
		12	λ	0.2157	0.0092	0.2848	0.0718	0.2440	0.0525
			α	2.4517	0.0080	2.7891	0.1600	2.3826	0.0449
			β	2.7119	0.0536	2.1312	0.1613	2.1312	0.1613
			θ	4.0922	0.0423	3.6402	0.1891	4.1817	0.2247

Table 7 (continued)

BLL	**60**	**30**	λ	0.2101	0.0014	0.1234	0.0231	0.1484	0.0477
			α	2.4450	0.0038	2.7812	0.1210	2.4330	0.0148
			β	2.5749	0.0101	2.8046	0.1129	2.6422	0.0311
			θ	3.9435	0.0044	4.1141	0.0327	4.2870	0.1788
			λ	0.2051	0.0008	0.2266	0.01120	0.2059	0.0010
		48	α	2.4717	0.0011	2.5755	0.01995	2.4839	0.0009
			β	2.4724	0.0014	2.3904	0.0306	2.5723	0.0062
			θ	4.0249	0.0011	4.0376	0.0045	4.0326	0.0047

Table 8

Bayes predictors and intervals of the future observation under balanced loss functions based on progressive Type-II censoring under different samples schemes

(N=10000, $n = 60, m = 30, j = 30, \lambda = 1.2, \alpha = 1.3, \beta = 2.2, \theta = 1.3$ and $\omega = 0.3$)

		BSEL				BLL			
Scheme	s	$\hat{y}_{(s)(BES)}$	**UL**	**LL**	**Length**	$\hat{y}_{(s)(BLL)}$	**UL**	**LL**	**Length**
Scheme I	1	0.3351	0.3674	0.2975	0.0699	0.3079	0.3166	0.2972	0.0194
	15	0.6236	0.9153	0.3575	0.5578	0.5600	0.6461	0.4426	0.2035
	30	1.6447	2.2342	0.7550	1.4792	0.8263	1.1959	0.3009	0.8949
Scheme II	1	0.4311	0.4699	0.3992	0.0707	0.4308	0.4483	0.3977	0.0506
	15	1.1400	1.5885	0.6917	0.8968	1.1363	1.3962	0.6691	0.7271
	30	2.1494	3.0330	0.3820	2.6510	1.1909	1.5232	0.6944	0.8288
Scheme III	1	0.4382	0.4678	0.3899	0.0779	0.4053	0.4151	0.3840	0.0311
	15	1.2102	1.5206	0.6360	0.8846	0.8534	1.0360	0.5609	0.4751
	30	2.3461	3.5861	0.8586	2.7275	1.9546	2.4784	0.8228	1.6556

Table 9

Bayes predictors and intervals of the future observation for the real data set under balanced loss functions based on progressive Type-II censoring under different samples schemes

	BSEL					BLL				
Scheme	s	$\hat{y}_{(s)(BES)}$	UL	LL	Length	s	$\hat{y}_{(s)(BES)}$	UL	LL	Length
Scheme I	1	1.3710	1.9463	0.6405	1.3058	1	0.9978	1.3187	0.5893	0.7295
	12	2.5721	3.3703	1.6527	1.7176	12	1.4242	1.8537	0.9302	0.9235
	22	3.2796	4.7404	0.9331	3.8073	22	3.2415	3.7365	2.6541	1.0824
Scheme II	1	1.6709	2.7674	0.4804	2.2870	1	1.2284	1.5119	1.2283	0.7757
	12	2.7691	4.3531	0.6535	3.6995	12	2.0481	3.4878	0.7800	2.7078
	22	3.3657	5.5702	1.1341	4.4361	22	2.8345	4.5803	0.5593	4.0210
Scheme III	1	2.5788	3.4635	1.5939	1.8696	1	2.4101	3.0102	1.4093	1.6009
	12	3.4827	4.5805	2.2180	2.3624	12	1.7680	2.6153	0.5964	2.0189
	22	5.1707	6.9598	2.6438	4.3160	22	4.0533	5.2446	3.0518	2.1928