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On the Location of The Lagrangian Collinear Points In The Photo-Gravitational Relativistic Restricted 3-Body Problem

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ABSTRACT **: we outlined the history of the restricted three body problem, beginning from the early works due to the brilliant scientist Lagrange, Euler, Jacobi, Poincare etc. we also continued to the up to date references. We formulated the basic scientific materials relevant to our work, e.g., the restricted three body problem, the Equation of motion in the rotating frame. We addressed the Lagrangian point, computations of their locations. We explained the curves of zero velocity and the permissible motions. In the field of the restricted three body problem. We obtained the locations of the three collinear points of the photo-gravitation relativistic restricted three body problem.**

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I. INTRODUCTION

The history of the restricted problem began with Euler and Lagrange continues with Jacobi (1836), Hill (1878) , Poincare' (1892-1899), and Birkhoff (1915). In 1772 , Euler first introduced a synodic (rotating) coordinate system , the use of which led to an integral of the equations of motion, known today as the Jacobian integral. Euler himself did not discover the Jacobian integral which was first given by Jacobi(1836)who, as Wintner remarks ,"rediscovered "the synodic system. The actual situation is somewhat complex since Jacobi published his integral in a sideral (fixed) system in which its significance is definitely less than I the synodic system. Hill (1878) used this integral to show that the Earth -Moon distance remains bounded from above for all time (assuming his model for the Sun-Earth-Moon system is valid), and Brown (1896) gave the most precise lunar theory of his time. Poincare' (1892-1899), published his monumental (Mēthodes Nouvelles). Emphasizing qualitative aspects of celestial mechanics, including modern concepts such at phase space surfaces of section . Birkhoff (1915) further developed these qualitative methods. The important problem of regularization was considered by Levi-Civita (1903) ; Burrrau (1906), Sündman (1912) and Birkhoff (1915) proved that all singularities are collisions for n = 3. Sündman (1912) found a uniformly convergent infinite series involving known function that "solves" the restricted three-body problem in the whole plane (once singularities are removed through the process of regularization). Since such global regularization's are available for this problem ,the restricted problem of three bodies can be considered to be complete "solved" . However, this "solution" does not address issues of stability, allowed regions of motion, and so on, and so is of limited practical utility (Szebehely 1967, p. 42). Furthermore, an unreasonably large number of terms (of order 108,000,000) of Sundman's series are required to attain anything like the accuracy required for astronomical observations. Lagrange (1867-1892) showed that the three- body problem has five relative equilibrium configurations. The circular restricted three problems is reviewed. The restricted three body problem equations of motion in the synodic frame of reference are derived. The Jacobi integral is obtained. The Lagrange points are highlighted. The curves of zero relative velocity are shown. The locations of the collinear points are computed. We consider the collinear equilibrium points of the relativistic (RTBP). We determine approximate positions of the collinear points by series expansions in μ and $1/c^2$

1.2 The Circular Restricted Three- Body Problem

In an effort to obtain insight into the possible types of motions Poincaré, Hill, and others coined the so-called circular restricted three-body problem (RTBP). Suppose two massive bodies move in circular orbits about their common center of mass called the primaries and attract (but are not attracted by) a third particle of infinitesimal mass. The problem of motion of the third body is called the circular restricted three-body problem, henceforth referred to as the CRTBP. If we further restrict the motion of the third body to be in the orbital plane of the other two bodies, the problem is called the planar circular restricted three-body problem, or the PCRTBP, the problem is to determine the possible types of motions of the third particle given the coordinates and velocities of the system at some epoch. The (RTBP) is a classical problem of celestial mechanics. Attempts for its solution led to the foundation for dynamical systems theory and alerted Poincare' to the existence of deterministic chaos within Newtonian mechanics (Poincare' [1892- 1899] .

Given an isolated system of two bodies with initial conditions, the equations of motion allow one to predict the position and velocity of either body at any later time. However, the difficulties that arise when a third body is introduced to a two-body system change the dynamical equations so much that there is no closed form solution. For nearly a century now, astrodynamists, physicists and mathematicians are developing methods of approximation to best predict the motion, yet they perform these calculations under basic assumptions that otherwise would make them impossible to solve. One of the most promising solutions is that produced by the (RTBP) where an infinitesimal body moves about 3-dimensional space under the gravitational influence of two finite bodies whose rotation with respect to one another defines a plane.

1.2.1 The Equations of Motion in Synodic Frame

In this section, we familiarize the reader with some of the terminology of the (RTBP) and the all important concepts of viewing the motion in the rotating frame. We consider the motion of a small particle of negligible mass moving under the gravitational influence of two masses m_1 and m_2. We assume that the two masses have circular orbits about their common center of mass and that they exert a force on the particle although the particles cannot affect the two masses. The system is made non dimensional by the following choice of units: the unit of mass is taken to be $m_1 + m_2$; the unit of length is chosen to be the constant separation between m_1 and m_2 (e.g., the distance between the centers of the sun and planet); the unit of time is chosen such that the orbital period of m_1 and m_2 about their center of mass is 2π. The universal constant of gravitation then becomes $G = 1$. It then follows that the common mean motion, n, of the primaries is also unity. We will refer to this system of units as non dimensional or normalized units throughout the thesis. We will use the normalized units for nearly all the discussions in this thesis. When appropriate, we can convert to dimensional units (e.g., km, km/s, s) to scale a problem. The conversion from units of distance, velocity, and time in the unprimed, normalized system to the primed, dimensionalized system is where L is the distance between the centers of m_1 and m_2 while V is the orbital velocity of m_1 but T is the orbital period of m_1 and m_2.

$$\text{Distance} = Ld ,$$
$$\text{Velocity} = Vs ,$$
$$\text{Time} = \frac{T}{2\pi}t$$

The only parameter of the system is the mass parameter,

$$\mu = \frac{m_2}{m_1 + m_2}$$

If we assume that $m_1 > m_2$, then the masses of m_1 and m_2 in this system of units are, respectively,

$$m_1 = 1 - \mu,$$
$$m_2 = \mu.$$

Now the mean angular velocity (or mean motion) of the two bodies is the unity, i.e.,

$$n^2 a^3 = G\,(m_1 + m_2) \,.$$

Consider a set of axes $(X,\, Y,\, Z)$ in the inertial frame (non-rotating frame) referred to the center of mass of the two finite bodies, see Fig. 1. If the coordinates of the masses $1-\mu$ and μ are $P_1(x_1,\, y_1,\, z_1)$ and $P_2(x_2, y_2, z_2)$ respectively, and the coordinates of the test particle are $P(x, y, z)$. The equations of motion of this particle are

$$\ddot{x} = (1-\mu)\frac{x_1 - x}{r_1^3} + \mu\frac{x_2 - x}{r_2^3}$$

$$\ddot{y} = (1-\mu)\frac{y_1 - y}{r_1^3} + \mu\frac{y_2 - y}{r_2^3}, \qquad (1.2) \qquad (1.1)$$

$$\ddot{z} = (1-\mu)\frac{z_1 - z}{r_1^3} + \mu\frac{z_2 - z}{r_2^3}$$

(1.3)

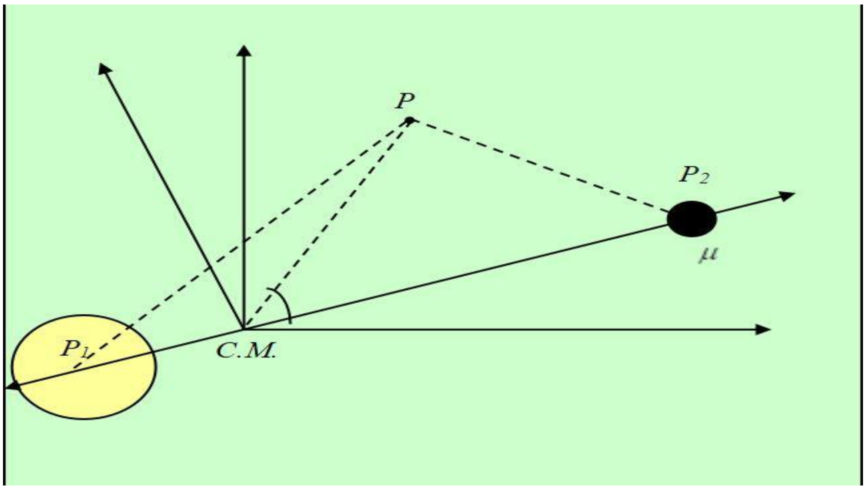

Figure 1.1 Inertial and rotating frames.

Where, from Fig. 1.1,

$$r_1^2 = (x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2 , \qquad (1.4)$$

$$r_2^2 = (x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2 . \qquad (1.5)$$

Where r_1 and r_2 are the distances of the infinitesimal mass and the two massive primaries respectively, and stands for $\frac{d^2}{dt}$.

Note that these equations are also valid in the general three-body problem since they do not require any assumptions about the paths of the two masses.

If the $Z-$axis perpendicular to the plane of rotation of the two massive particles, $z_1 = z_2 = 0$.

Now we introduce a coordinates system $(\xi,\, \eta,\, \zeta)$ rotating about Z axis with constant angular velocity (unity), and having the same origin as before. The direction of ξ-axis can be chosen such that the two massive particles $1-\mu$ and μ always lie on it. This is usually called synodic frame of reference.

The coordinates of the masses $(1-\mu)$ and μ are $P_1(\xi_1,0,0)$ and $P_2(\xi_2,0,0)$ respectively, such that $\xi_2-\xi_1=1$.

In addition, in the units chosen,

$$\xi_1=-\mu, \quad \xi_2=1-\mu. \qquad (1.6)$$

Hence

$$\left.\begin{aligned} r_1^2&=(\xi_1-\xi)^2+\eta^2+\zeta^2=(\mu+\xi)^2+\eta^2+\zeta^2,\\ r_2^2&=(\xi_2-\xi)^2+\eta^2+\zeta^2=(1-\mu-\xi)^2+\eta^2+\zeta^2.\end{aligned}\right\} \qquad (1.7)$$

The coordinates of the test particle $(x,\ y,\ z)$ in terms of the rotating frame is given by

$$\begin{aligned} x&=\xi\cos t-\eta\sin t,\\ y&=\xi\sin t+\eta\cos t,\\ z&=\zeta.\end{aligned} \qquad (1.8)$$

Differentiating (1.8) twice, and substituting the resulting equations into equation (1.1)-(1.3), we have

$$\begin{aligned} \ddot{\xi}-2n\dot{\eta}&=\frac{\partial U}{\partial \xi},\\ \ddot{\eta}+2n\dot{\xi}&=\frac{\partial U}{\partial \eta},\\ \ddot{\zeta}&=\frac{\partial U}{\partial \zeta}\end{aligned} \qquad (1.9)$$

where

$$U=\frac{1}{2}(\xi^2+\eta^2)+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}. \qquad (1.10)$$

In this equation the term $\frac{1}{2}(\xi^2+\eta^2)$ is the centrifugal potential while the terms $\frac{1-\mu}{r_1}$ and $\frac{\mu}{r_2}$ are the gravitational potentials of the massive primaries.

Note also that U is not true potential and it is best referred to as a scalar function from which some (but not all) of the accelerations experienced by the particle in the rotating frame can be derived. U is called a (Pseudo-Potential)

1.2.2 Jacobi's Integral

Equations (1.9) can be specifically solved in closed algebraic form if the 1st equation of (1.9) is multiplied by $\dot{\xi}$, the 2nd equation is multiplied by $\dot{\eta}$ and the 3rd equation is multiplied by $\dot{\zeta}$. Then adding together yields

$$\ddot{\xi}\dot{\xi}+\ddot{\eta}\dot{\eta}+\ddot{\zeta}\dot{\zeta}=\frac{\partial U}{\partial \xi}\dot{\xi}+\frac{\partial U}{\partial \eta}\dot{\eta}+\frac{\partial U}{\partial \zeta}\dot{\zeta} \qquad (1.11)$$

The R.H.S. represents the total differentiation of U , since it does not depend implicitly on the time , but is a function of $(\xi,\ \eta,\ \zeta)$ only.

Integrating (1.11) , we therefore obtain

$$\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 = 2U - C \qquad (1.12)$$

where C is a constant of integration , and

$$V^2 = \dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2$$

is the square of the velocity of the infinitesimal mass in the rotating frame, we have

$$V^2 = 2U - C \qquad (1.13)$$

or, using equation (1.10),

$$\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 = \xi^2 + \eta^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C \qquad (1.14)$$

Equation (1.13) or (1.14) is the Jacobi integral, sometimes called the integral of relative energy. It is important to note that this is not energy integral because in the restricted problem neither energy nor angular momentum is conserved.

1.3 Lagrange Points

the Lagrangian points are the five stationary solutions of the circular restricted three-body problem, i.e. given two massive bodies in circular orbits around their common center of mass, there are five positions in space where a third body, of negligible mass, could be placed which would then maintain its position relative to the two massive bodies.

Example: The Sun-Earth L_4 and L_5 points lie 60° ahead of and 60° behind the Earth in its orbit around the Sun. The three libration points L1, L2 and L3 are collinear libration with the primary masses, see Fig. 1.2

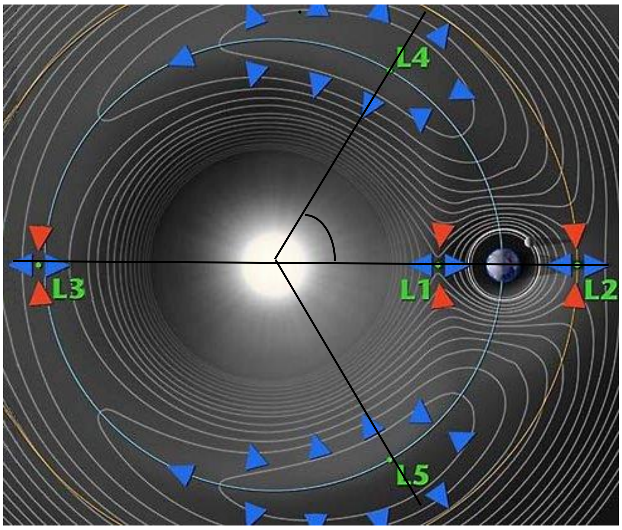

Fig. 1.2: A diagram showing the five Lagrangian points in a two-body system.
(e.g., the Sun and the Earth)

However, not all of the five Lagrange points are stable. For example, with respect to the Earth-Sun Lagrange system, there exist three unstable collinear libration points L_1, L_2, L_3 (aligned on the Earth-Sun vector and located one each on the opposite side of the Sun, the Earth, and in between). An object placed at any of these points will not stay there indefinitely; a small perturbation will cause the object to leave the libration point.

1.4 The Surface of Zero Relative Velocity

The Jacobi integral (1.14) is a relation between the square velocity of the infinitesimal body and its coordinates relative to the rotating (Synodic) frame. If C is known from the initial condition (1.14) determines the velocity of the infinitesimal body. Conversely, equation (1.14) determines the loci of given velocities. In particular if we put $V = 0$, it determines the surface on which the velocity will be zero. Then

$$\xi^2+\eta^2+\frac{2(1-\mu)}{r_1}+\frac{2\mu}{r_2}=C \qquad (1.15)$$

Equation (1.15) defines for a given value of G the boundaries of regions in which the particles must be found, these regions are those for which $2U > C$, since otherwise V_2 would be negative, giving imaginary values for the velocity. Equation (1.15) is called Hills limiting surface, dose not tell us anything about the orbits of the particle within the volumes of space available to it. Before proceeding further, we first consider the following theorem.

1.5 Permissible Motions and Equation of the Surfaces

The above theorem implies that on one side of the surfaces of zero velocity, the velocity will be real and on the other side is imaginary. In other words, it is possible for the body to move on one side and impossible to move on the other. Recalling equation (1.15),

$$\xi^2+\eta^2+\frac{2(1-\mu)}{r_1}+\frac{2\mu}{r_2}=C \qquad (1.16)$$

where,

$$\left.\begin{aligned} r_1^2&=(\xi_1-\xi)^2+\eta^2+\zeta^2=(\mu_1+\xi)^2+\eta^2+\zeta^2,\\ r_2^2&=(\xi_2-\xi)^2+\eta^2+\zeta^2=(1-\mu-\xi)^2+\eta^2+\zeta^2.\end{aligned}\right\}$$

1.6 The particular Solution of Lagrange

As shown previously, the equations of motion of the restricted problem are.

$$\begin{aligned}\ddot{\xi}-2n\dot{\eta}&=\frac{\partial U}{\partial \xi},\\ \ddot{\eta}+2n\dot{\xi}&=\frac{\partial U}{\partial \eta},\\ \ddot{\zeta}&=\frac{\partial U}{\partial \zeta}\end{aligned} \qquad (1.17)$$

where $$U=(1-\mu)\,(r_1^2+\frac{2}{r_1})+\mu\,(r_2^2+\frac{2}{r_2}) \qquad (1.18)$$

The planer problem is satisfied by constant values $\xi=\xi_1,\ \eta=\eta_1$ if the L.H.S of the 1st two equations of (1.17) are zero. In view of equation (1.18) these may be written as:

$$\left.\begin{aligned}\frac{\partial U}{\partial \xi}&=\frac{\partial U}{\partial r_1}\frac{\partial r}{\partial \xi}+\frac{\partial U}{\partial r_2}\frac{\partial r_2}{\partial \xi}=0,\\ \frac{\partial U}{\partial \eta}&=\frac{\partial U}{\partial r_1}\frac{\partial r}{\partial \eta}+\frac{\partial U}{\partial r_2}\frac{\partial r_2}{\partial \eta}=0.\end{aligned}\right\} \qquad (1.19)$$

Before going further, we note that.

(1) Since $\xi=\xi_1,\eta=\eta_1 \quad (\dot{\xi}=\ddot{\xi}=\dot{\eta}=\ddot{\eta}=0)$ then if the infinitesimal body placed at one of the resulting points will remain there forever.

(2) Since $\frac{\partial U}{\partial \xi}=\frac{\partial U}{\partial \eta}=0$ the resulting points correspond to the double points of the function U. They clearly lie on the $\xi\eta$-plane.

1.7.1 The Equilateral Solutions

The last equations are satisfied by the trivial solution,

$$\frac{\partial U}{\partial r_1}=\frac{\partial U}{\partial r_2}=0, \qquad (1.20)$$

which by using (1.18) give the equilateral solutions representing by the points L4 and L5 for which $r_1 = r_2 = 1$.

1.7.2 The Straight Line Solution

Other solutions are obtained from

$$\frac{\partial(r_1, r_2)}{\partial(\xi,\eta)} = \begin{vmatrix} \frac{\partial r_1}{\partial \xi} & \frac{\partial r_1}{\partial \eta} \\ \frac{\partial r_2}{\partial \xi} & \frac{\partial r_2}{\partial \eta} \end{vmatrix} = 0 \qquad (1.21)$$

but

$$r_1^2 = (\xi+\mu)^2 + \eta^2 + \zeta^2 = (\xi+\mu)^2 + \eta^2,$$
$$r_2^2 = (\xi-1+\mu)^2 + \eta^2 + \zeta^2 = (\xi-1+\mu)^2 + \eta^2.$$

whence

$$\frac{\partial(r_1, r_2)}{\partial(\xi,\eta)} = \begin{vmatrix} \xi+\mu & \eta \\ \xi-1+\mu & \eta \end{vmatrix} = 0 \qquad (1.22)$$

for these solutions η = 0, they therefore lie on the ξ - axis and the values of ξ must satisfy the condition,

$$\frac{\partial U}{\partial \xi} = \frac{\partial U}{\partial r_1}\frac{\partial r}{\partial \xi} + \frac{\partial U}{\partial r_2}\frac{\partial r}{\partial \xi} = 0 \quad .$$

To investigate the location of the roots of this equation we have $(\eta = \zeta = 0)$,

$$\left. \begin{aligned} U &= \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\xi^2, \\ r_1^2 &= (\xi - \xi_1)^2, \\ r_2^2 &= (\xi - \xi_2)^2. \end{aligned} \right\} \qquad (1.23)$$

So that

$$\frac{\partial U}{\partial \xi} = \xi - \frac{1-\mu}{(\xi-\xi_1)^2} - \frac{\mu}{(\xi-\xi_2)^2}, \qquad (1.24)$$

(a) The Solution L1

$$\left. \begin{aligned} & r_1 + r_2 = 1, \\ & \xi + \mu = r_1, \\ & 1 - \mu - \xi = r_2, \\ & \frac{\partial r_1}{\partial \xi} = -\frac{\partial r_2}{\partial \xi} = 1 \end{aligned} \right\} \qquad (1.25)$$

from (1.18) it follows that

$$\left. \begin{aligned} \frac{\partial U}{\partial r_1} &= (1-\mu)\,(-r_1^{-2} + r_1), \\ \frac{\partial U}{\partial r_2} &= \mu\ \ (-r_2^{-2} + r_2)\,. \end{aligned} \right\} \qquad (1.26)$$

The equation for $\frac{\partial U}{\partial \xi}$ becomes, expressed in r2.

$$\frac{\mu}{1-\mu} = \frac{(3r_2 - 3r_2^2 + r_2^3)\ r_2^2}{(1-r_2)^2\ (1-r_2^3)}, \tag{1.27}$$

If $\frac{\mu}{1-\mu}$ is small, this equation has a root in the vicinity of $r_2 = \alpha$ where

$$\alpha = \left[\frac{\mu}{3\,(1-\mu)}\right]^{1/3}. \tag{1.28}$$

Then α can be written as.

$$\alpha = r_2\ \ (1 + \frac{1}{3} r_2 + \frac{1}{3} r_2^2 + \cdots\cdots). \tag{1.29}$$

Let successive approximations of r_2 be r_{2i} ,

$$r_{20} = \alpha,$$

$$\alpha = r_{21}\,(1 + \frac{1}{3}\alpha),$$

$$r_{21} = \alpha\,(1 - \frac{1}{3}\alpha),$$

$$\alpha = r_{22}\,[1 + \frac{1}{3}\alpha + \frac{2}{9}\alpha^2 + \cdots\cdots]$$

Then successive approximation yields

$$r_{22} \cong r_2' \cong \alpha\ \ (1 - \frac{1}{3}\alpha - \frac{1}{9}\alpha^2 + \ldots\ldots). \tag{1.30}$$

(b) The solution L2

It lies beyond the smaller mass. Here

$$\left.\begin{aligned} & r_1 - r_2 = 1, \\ & \xi + \mu = r_1, \\ & \xi + \mu - 1 = r_2, \\ & \frac{\partial r_1}{\partial x} = \frac{\partial r_2}{\partial x} = 1 \end{aligned}\right\} \tag{1.31}$$

The equation for $\frac{\partial U}{\partial \xi}$ becomes.

$$\frac{\mu}{1-\mu} = \frac{(3r_2 + 3r_2^2 + r_2^3)\ r_2^2}{(1+r_2)^2\ (1-r_2^3)}. \tag{1.32}$$

Again $\frac{\mu}{1-\mu}$ is small. This equation has a root in the vicinity of $r_2 = \alpha$, expanded in power of r_2

$$\left.\begin{aligned} & \alpha = r_2\ (1 - \frac{1}{3} r_2 + \frac{1}{3} r_2^2 \cdots\cdots\cdots), \\ & r_2 = \alpha\ (1 + \frac{1}{3}\alpha - \frac{1}{9}\alpha^2 + \cdots\cdots) \end{aligned}\right\}. \tag{1.33}$$

(c) The Solution L3

It lies beyond the larger mass. Here

$$\left.\begin{aligned} & r_2 - r_1 = 1, \\ & -\xi - \mu = r_1, \\ & 1 - \mu - \xi = r_2, \\ & \frac{\partial r_1}{\partial \xi} = \frac{\partial r_2}{\partial \xi} = -1 \end{aligned}\right\} \qquad (1.34)$$

The equation1.35 for $\frac{\partial U}{\partial \xi}$ becomes

$$\frac{\mu}{1-\mu} = \frac{r_1^{-2} - r_1}{r_2^{-2} - r_2}.$$

Put

$$r_1 = 1 + \beta,$$
$$r_2 = 2 + \beta$$

Expressing in β we get

$$-\mu \approx \frac{12}{7}\beta\,(1 - \frac{23}{84}\beta^2 + \cdots).$$

Successive approximations yield

$$\beta^{(0)} = -\frac{7}{12}\mu,$$

$$-\mu = \frac{12}{7}\beta^{(1)}\,[1 - \frac{23}{84}(\frac{7}{12}\mu)^2],$$

$$\beta^{(1)} = -\frac{7}{12}\mu\;[1 + \frac{23}{84}(\frac{7}{12}\mu)^2].$$

Finally 1.35 will becomes

$$\boxed{r_2 = 2 - \frac{7}{12}\mu\left[1 + \frac{23}{84}(\frac{7}{12}\mu)^2\right]}.$$

2.2 Libration Points Location

As we go through the previous sections in which it becomes clear that the equilibirum solution only exists when the relative rotating frame exists and also the partial derivatives of the pseudopotential function (U_ξ, U_η, U_ζ) are all zero , i.e., U = const. These points correspond to the positions in the rotating frame at which the gravitational forces and the centrifugal force associated with the rotation of the synodic frame all cancel , with the result that a particle located at one of these points appears stationary in synodic frame.

Where U is the potential which can be written as composed of two components, namely the potential of the classical (RTBP) photo-gravitational potential U_{ph} and the relativistic correction U_{rph}

$$U = Uph + Urph \qquad (2.1)$$

Where U_{ph} and U_{rph} are given by

$$U_{ph} = \frac{r^2}{2} + \frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2} \qquad (2.2)$$

$$U_{rph}=\frac{r^2}{2c^2}(\mu(1-\mu)-3)+\frac{1}{8c^2}((\xi+\dot{\eta})^2+(\eta-\dot{\xi})^2)^2+\frac{3}{2c^2}\left[\frac{q_1(1-\mu)}{r_1}+\frac{q_2\mu}{r_2}\right]((\xi+\dot{\eta})^2+(\eta-\dot{\xi})^2)$$

$$-\frac{1}{2c^2}\left[\frac{q_1(1-\mu)}{r_1}+\frac{q_2\mu}{r_2}\right]^2-\frac{q_1q_2\mu(1-\mu)}{2c^2}\left[\frac{1}{r_1}+\left(\frac{1}{r_1}-\frac{1}{r_2}\right)(1-3\mu-7\xi-8\dot{\eta})+\eta^2\left(\frac{q_2\mu}{r_1^3}+\frac{q_1(1-\mu)}{r_2^3}\right)\right]$$

$$\left.\begin{aligned}n&=1+\frac{1}{2c^2}(\mu(1-\mu)-3)\\ r&=\sqrt{(\xi^2+\eta^2)}\\ r_1&=\sqrt{(\xi+\mu)^2+\eta^2}\\ r_2&=\sqrt{(\xi+\mu-1)^2+\eta^2}\end{aligned}\right\} \qquad (2.4)$$

The liberation point are obtained from equations of motion after setting

$\ddot{\xi}=\ddot{\eta}=\dot{\xi}=\dot{\eta}\doteq 0$ These points represent particular solutions of equations of motion after setting

With $\dot{\xi}=\dot{\eta}=0$

The explicit formulas are

(Remembering that $U=U_{ph}+U_{rph}$

Equation (2.6) will be

$$\left.\begin{aligned}\frac{\partial U}{\partial \xi}&=\frac{\partial U_C}{\partial \xi}+\frac{\partial U_r}{\partial \xi}=0\\ \frac{\partial U}{\partial \eta}&=\frac{\partial U_c}{\partial \eta}+\frac{\partial U_r}{\partial \eta}=0\end{aligned}\right\} \qquad (2.5)$$

$$\frac{\partial U}{\partial \xi}=\xi-\frac{q_1(1-\mu)(\xi+\mu)}{r_1^3}-\frac{q_2\mu(\xi+\mu-1)}{r_2^3}+\frac{1}{c^2}\left\{(\mu-\mu^2-3)\xi+\left(\frac{q_1(1-\mu)}{r_1}+\frac{q_2\mu}{r_2}\right)\right.$$

$$\times\left(\frac{q_1(1-\mu)(\xi+\mu)}{r_1^3}+\frac{q_2\mu(\xi+\mu-1)}{r_2^3}\right)+\frac{1}{2}(\eta^2+\xi^2)\xi$$

$$-\frac{3}{2}\left(\frac{q_1(1-\mu)(\xi+\mu)}{r_1^3}+\frac{q_2\mu(\xi+\mu-1)}{r_2^3}\right)(\eta^2+\xi^2)+3\left(\frac{q_1(1-\mu)}{r_1}+\frac{q_2\mu}{r_2}\right)$$

$$-\frac{1}{2}q_1q_2\mu(1-\mu)\left[-\frac{1}{r_1^3}(\xi+\mu)+\left(\frac{(\xi+\mu)}{r_1^3}+\frac{(\xi+\mu-1)}{r_2^3}\right)(-1+3\mu+7\xi)\right.$$

$$\left.\left.-7\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-3\eta^2\left(\frac{q_2\mu(\xi+\mu)}{r_1^5}+\frac{q_1(1-\mu)(\xi+\mu-1)}{r_2^5}\right)\right]\right\}=0$$

and

$$\frac{\partial U}{\partial \eta}=$$

$$\eta-\frac{q_1(1-\mu)}{r_1^3}\eta-\frac{q_2\mu}{r_2^3}\eta+\frac{1}{c^2}\Bigg\{(\mu-\mu^2-3)\eta+\left(\frac{q_1(1-\mu)}{r_1}+\frac{q_2\mu}{r_2}\right)\left(\frac{q_1(1-\mu)}{r_1^3}+\frac{q_2\mu}{r_2^3}\right)\eta$$

$$+\frac{1}{2}(\xi^2+\eta^2)\eta-\frac{3}{2}(\xi^2+\eta^2)\left(\frac{q_1(1-\mu)}{r_1^3}+\frac{q_2\mu}{r_2^3}\right)\eta+3(\frac{q_1(1-\mu)}{r_1}+\frac{q_2\mu}{r_2})\eta$$

$$+\frac{q_1q_2\mu(1-\mu)}{2}\eta+\frac{1}{2}q_1q_2\mu(1-\mu)\eta\left[\left(-\frac{1}{r_1^3}+\frac{1}{r_3^3}\right)(-1+3\mu+7\xi)\right.$$

$$\left.\left.+3\left(\frac{q_1(1-\mu)}{r_2^5}+\frac{q_2\mu}{r_1^5}\right)\eta^2+\frac{1}{r_1^3}-2\left(\frac{q_2\mu}{r_1^3}+\frac{q_1(1-\mu)}{r_2^3}\right)\right]\right\}=0$$

2.3 Location of collinear Libration Points

(a) Location of L1

The collinear points must, by definition , have $\xi=\eta=0$ and the solution of the classical (RTBP) since $\frac{1}{c^2}\angle\angle 1$

(see Fig. 2.2)

$$\left.\begin{aligned} r_1 + r_2 &= 1 \\ r_1 &= \xi + \mu \\ r_2 &= 1 - \mu - \xi \\ \frac{\partial r_1}{\partial \xi} &= -\frac{\partial r_2}{\partial \xi} = 1 \end{aligned}\right\}$$

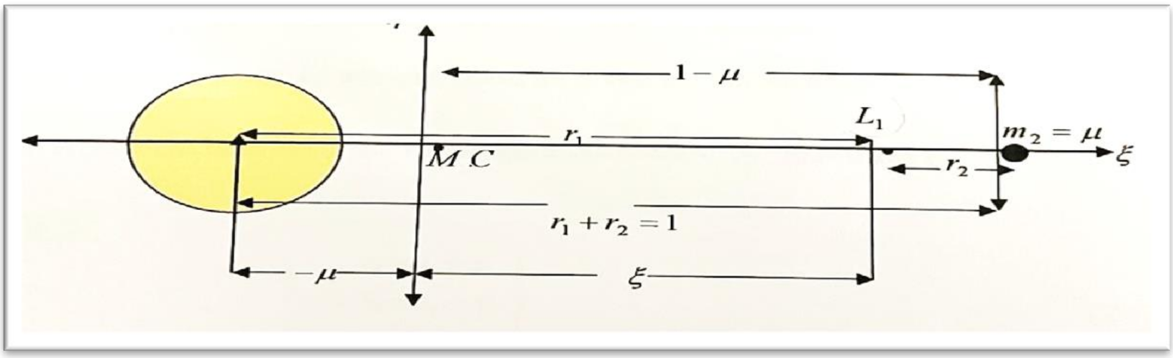

Fig . 2.1 : show the location of L1

Substituting from (2,7) into (2,6) we get (2,8)

$$\frac{\partial U}{\partial \xi} = \xi - \left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2}\right) +$$
$$\frac{1}{c^2}\left\{(\mu-\mu^2-3)\xi + \left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2}\right)\left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2}\right)\right.$$
$$-\frac{3}{2}\left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2}\right)\xi^2 + 3\left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2}\right)\xi + \frac{1}{2}\xi^3$$
$$\left.-\frac{1}{2}q_1q_2\mu(1-\mu)\left[-\frac{1}{r_1^2} + \left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)(-1+3\mu+7\xi) - 7\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right]\right\} = 0$$

$$\frac{\partial U}{\partial \xi} = 1-\mu-r_2 - \left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2}\right) + \frac{1}{c^2}\left\{\left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2}\right)\left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2}\right)\right.$$
$$+(\mu-\mu^2-3)(1-\mu-r_2) + \frac{1}{2}(1-\mu-r_2)^3 - \frac{3}{2}\left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2}\right)(1-\mu-r_2)^2$$
$$+3\left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2}\right)(1-\mu-r_2) - \frac{1}{2}q_1q_2\mu(1-\mu)\left[-\frac{1}{r_1^2} + \left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)(-1+3\mu\right.$$
$$\left.\left.+7(1-\mu-r_2)) - 7\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right]\right\} = 0$$

Which can be-written as a function of r_1, r_2 as equation 2

Then it may be reasonable in our case to assume that positions of the equilibrium points L1 are the same as given by classical (RTBP) but perturbed due to the inclusion of the relativistic correction by quantities $\varepsilon = (0(\frac{1}{c^2}))$

$$\left.\begin{aligned} r_1 &= a_1 + \varepsilon_1 \\ r_2 &= b_1 - \varepsilon_1 \\ a_1 &= 1 - b_1 \end{aligned}\right\} \qquad (2,10)$$

Where a_1 and b_1 are unperturbed positions of r_1 and r_2 respectively, and b_1 is given after some successive approximation by the relation.

$$\left.\begin{aligned} b_1 &= \alpha\left(1 - \frac{\alpha}{3} - \frac{\alpha^2}{9} + \frac{2}{27}\alpha^3 + \frac{2}{81}\alpha^4\right) \\ \alpha &= \left(\frac{q_2\mu}{3q_1(1-\mu)}\right)^{\frac{1}{3}} \end{aligned}\right\} \qquad (2,11)$$

Substituting from equation (2,10) into equation (2,11)

$$\begin{aligned} \frac{\partial U}{\partial \xi} &= 1 - \mu - b_1 + \varepsilon_1 - \frac{q_1(1-\mu)}{(1-b_1)^2}\left(1 - \frac{2\varepsilon_1}{1-b_1}\right) + \frac{q_1\mu}{b_1^2}\left(1 + \frac{2\varepsilon_1}{b_1}\right) \\ &+ \frac{1}{c^2}\Big\{(\mu - \mu^2 - 3)(1-\mu-b_1) + \left(\frac{q_1(1-\mu)}{(1-b_1)} + \frac{q_2\mu}{b_1}\right) \\ &\times\left(\frac{q_1(1-\mu)}{(1-b_1)^2} - \frac{q_2\mu}{b_1^2}\right) + \frac{1}{2}(1-\mu-b_1)^3 - \frac{3}{2}\left(\frac{q_1(1-\mu)}{(1-b_1)^2} - \frac{q_2\mu}{b_1^2}\right)(1-\mu-b_1)^2 \\ &+ 3\left(\frac{q_1(1-\mu)}{(1-b_1)} + \frac{q_2\mu}{b_1}\right)(1-\mu-b_1) - \frac{1}{2}q_1q_2\mu(1-\mu)\Bigg[-\frac{1}{(1-b_1)^2} - \\ &- 7\left(\frac{1}{(1-b_1)} - \frac{1}{b_1}\right) + \left(\frac{1}{(1-b_1)^2} + \frac{1}{b_1^2}\right)(6 - 7b_1 - 4\mu)\Bigg]\Big\} = 0 \end{aligned}$$

Retaining the terms up to the first order in the small quantities , we get 2.13

$$\begin{aligned} \frac{\partial U}{\partial \xi} &= 1 - \mu - b_1 + \varepsilon_1 - \frac{q_1(1-\mu)}{(1-b_1)^2}\left(1 - \frac{2\varepsilon_1}{1-b_1}\right) + \frac{q_1\mu}{b_1^2}\left(1 + \frac{2\varepsilon_1}{b_1}\right) \\ &+ \frac{1}{c^2}\Big\{(\mu - \mu^2 - 3)(1-\mu-b_1) + \left(\frac{q_1(1-\mu)}{(1-b_1)} + \frac{q_2\mu}{b_1}\right) \\ &\times\left(\frac{q_1(1-\mu)}{(1-b_1)^2} - \frac{q_2\mu}{b_1^2}\right) + \frac{1}{2}(1-\mu-b_1)^3 - \frac{3}{2}\left(\frac{q_1(1-\mu)}{(1-b_1)^2} - \frac{q_2\mu}{b_1^2}\right)(1-\mu-b_1)^2 \\ &+ 3\left(\frac{q_1(1-\mu)}{(1-b_1)} + \frac{q_2\mu}{b_1}\right)(1-\mu-b_1) - \frac{1}{2}q_1q_2\mu(1-\mu)\Bigg[-\frac{1}{(1-b_1)^2} - \\ &- 7\left(\frac{1}{(1-b_1)} - \frac{1}{b_1}\right) + \left(\frac{1}{(1-b_1)^2} + \frac{1}{b_1^2}\right)(6 - 7b_1 - 4\mu)\Bigg]\Big\} = 0 \end{aligned}$$

Equation (2,13) can be solved for ε_1 to yield equation 2.14

$$\varepsilon_1 = \left(1 + \frac{2q_1(1-\mu)}{(1-b_1)^3} + \frac{2q_2\mu}{b_1^3}\right)^{-1} \left(-1 + b_1 + \mu + \frac{q_1(1-\mu)}{(1-b_1)^2} - \frac{q_2\mu}{b_1^2} - \frac{1}{c^2}\left\{\left(\frac{q_1(1-\mu)}{(1-b_1)} + \frac{q_2\mu}{b_1}\right)\left(\frac{q_1(1-\mu)}{(1-b_1)^2} - \frac{q_2\mu}{b_1^2}\right)\right.\right.$$

$$+ (\mu - \mu^2 - 3)(1-\mu-b_1) + \frac{1}{2}(1-\mu-b_1)^3 - \frac{3}{2}\left(\frac{q_1(1-\mu)}{(1-b_1)^2} - \frac{q_2\mu}{b_1^2}\right)(1-\mu-b_1)^2 + 3\left(\frac{q_1(1-\mu)}{(1-b_1)} + \frac{q_2\mu}{b_1}\right)$$

$$\left.\left. \times (1-\mu-b_1) - \frac{1}{2}q_1q_2\mu(1-\mu)\left[\frac{1}{(1-b_1)^2} - 7\left(\frac{1}{(1-b_1)} - \frac{1}{b_1}\right) + \left(\frac{1}{(1-b_1)^2} + \frac{1}{b_1^2}\right)(6 - 7b_1 - 4\mu)\right]\right\}\right)$$

Which can be assumed as composed of two parts, as

$$r_2 = C_1 + R_1 \qquad (2,15)$$

where C_1 represents the position of r_2 as obtained from the potential of the photo-gravitational RTBP Uph , and R_1 is the relativistic contribution to

the problem Urph , thus , C_1 and R_1 are function of μ

The location of L_1 is given by ; $\xi_{0,L_1} = 1 - \mu - r_2$

$$\xi_{0,L_1} = 1-\mu-\left\{\frac{q_2(-1+q_1)}{3q_2+6q_1}+\frac{q_1}{3(2q_1+q_2)^2}(\frac{q_2}{q_1})^{\frac{1}{3}}(4q_1(3+q_2)+q_2(8+3q_2))(\frac{\mu}{3})^{\frac{1}{3}}\right.$$

$$-\frac{1}{9(2q_1+q_2)^3}(\frac{q_2}{q_1})^{\frac{2}{3}}(-6q_2^3+20q_1^2q_2(1+q_2)+8q_1^3(3+2q_2)$$

$$+q_1q_2^2(-2+9q_2)\Big)(\frac{\mu}{3})^{\frac{2}{3}}$$

$$+\frac{q_1}{81(2q_1+q_2)^4}(\frac{q_2}{q_1})(q_1^3(504-580q_2)+8q_1^2q_2(77-122q_2)$$

$$+q_1q_2^2(274-577q_2)10q_2^3(10-9q_2)\Big)(\frac{\mu}{3})$$

$$+\frac{2}{81q_2(2q_1+q_2)^5}(\frac{q_2}{q_1})^{\frac{4}{3}}\Big((-30q_2^6+432q_1^6(3+q_2)+3q_1q_2^5(-67+37q_2)$$

$$+8q_1^5q_2(201+176q_2)+4q_1^4q_2^2(-61+520q_2^3)+q_1^2q_2^4(-610+693q_2)$$

$$+q_1^3q_2^3(-563+815q_2)\Big)(\frac{\mu}{3})^{\frac{4}{3}}$$

$$-\frac{1}{243(2q_1+q_2)^6}(\frac{q_2}{q_1})^{\frac{2}{3}}\Big(32q_1^4q_2^2(161-248q_2)-2456(3+2q_2)-2q_2^6(38+51q_2)$$

$$+8q_1^2q_2^4(1629+371q_2)+q_1q_2^5(2222+397q_2)+4q_1^3q_2^3(5002+399q_2)$$

$$-32q_1^5q_2(432+443q_2)\Big)(\frac{\mu}{3})^{\frac{5}{3}}+\ldots-\left(1/c^2\right)\left(-\frac{(-1+q_1)(5+2q_1)q_2}{6(2q_1+q_2)}\right.$$

$$+\frac{q_2}{6(q_2+2q_1)^2}(\frac{q_2}{q_1})^{\frac{1}{3}}(4q_1(1+2q_1)(-4+5q_1)+(-13+q_1(-3+22q_1))q_2\Big)(\frac{\mu}{3})^{\frac{1}{3}}$$

$$-\frac{1}{9(2q_1+q_2)^3}(\frac{q_2}{q_1})^{\frac{2}{3}}\Big((36q_1^4+6q_1^2(-1+3q_1+6q_1^2)q_2$$

$$+q_1(-5+8q_1)(2+11q_1)q_2^2+(-11+q_1(-15+38q_1))q_2^3\Big)(\frac{\mu}{3})^{\frac{2}{3}}$$

$$+\frac{1}{162(2q_1+q_2)^4}(\frac{q_2}{q_1})\Big(432q_1^5(5+3q_1)+4q_1^3(263+q_1(1527+64q_1))q_2$$

$$-4q_1^2(-608+q_1(-1563+929q_1))q_2^2-q_1(-1931+q_1(-1995+2468q_1))q_2^3$$

$$-6(-59-24q_1+50q_1^2)q_2^4\Big)(\frac{\mu}{3})$$

$$-\frac{1}{162(2q_1+q_2)^5}(\frac{q_2}{q_1})^{\frac{4}{3}}\Big((144q_1^5(87+4q_1(-32+57q_1))$$
$$+16q_1^4(1625+q_1(-2109+4585q_1))q_2+8q_1^3(2599$$
$$+q_1(-2247+7793q_1))q_2^2+4q_1^2(2176+q_1(624+4985q_1))q_2^3$$
$$+q_1(2843+3894q_1+250q_1^2)q_2^4-2(-238+q_1(-345+286q_1))q_2^5\Big)\Big)(\frac{\mu}{3})^{\frac{4}{3}}$$
$$-\frac{1}{486q_1(2q_1+q_2)^6}(\frac{q_2}{q_1})^{\frac{2}{3}}(15552q_1^8+1728q_1^6(45+q_1(7+27q_1))q_2$$
$$+432q_1^5(492+q_1(-211+500q_1))q_2^2+112q_1^4(2008+q_1(-1779+3293q_1))q_2^3$$
$$+12q_1^3(8681+12q_1(-1166+2287q_1))q_2^4$$
$$+12q_1^2(949+2q_1(-2334+6133q_1))q_2^5+q_1(-3109+4q_1(-1056+6841q_1))q_2^6$$
$$+6(-69+q_1(114+197q_1))q_2^7\Big)(\frac{\mu}{3})^{\frac{5}{3}}+....\Big)\Big\}$$

Location of L_2 and L_3 We can proceed similarly to evaluate the locations of L_2 and L_3

Conclusion

In the present work , the effect of the gravitational attraction of two bigger primaries and the post Newtonian perturbation on the location of collinear points in the restricted three body problem is carried out. The two primaries are considered also to be radiant sources . These effects appear as additional terms in the classical potential.

New formulas for the locations of the collinear points are obtained , and can be used for further investectigation . Taking these effects into consideration causes a shift in the location of the collinear points.In a forthcoming work these results can be used to investigate the stability of the Points.

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