MILITARY TECHNICAL COLLEGE

CAIRO - EGYPT

EFFECT OF SHEAR DEFORMATIONS ON THE BENDING OF MODERATELY THICK PLATES

BY

HAMED ABDALLA* and KAMAL HASSAN**

ABSTRACT

The Bergan and Wang approach for the shear inclusion in plate deformation has led to an energy expression which is a function of the only lateral deflection. The corresponding Euler equation has been deduced and applied to a simply supported square plate with sinusoidal, uniform and concentrated loads. Numerical calculations have been made for different thickness to span ratios and the results agree well with those of other investigators.

INTRODUCTION

Reissner [1] and Mindlin [2] plate theories are the widely used ones which take the shear effect into consideration. Both theories are characterized by the existence of three independent functions: the lateral deflection and the two rotations due to shear. The corresponding energy expressions do not lead to the classical thin plate theory as the thickness becomes very thin [3 -6] .

Bergan and Wang [7] have recently adopted an approach according to which, the potential energy of the plate is expressed as a function of only the lateral deflection. Moreover, the obtained expression converges to the classical thin plate energy as the thickness decreases. Based on Bergan-Wang energy expression, we have deduced the corresponding Euler equation with its natural boundary conditions. This equation has been solved analytically for the case of simply supported square plate under different types

* Lt. Col. Dr., Chair of Rockets, Military Technical College, Cairo, Egypt.

**Dr. Eng., Investigator, Ministry of Higher Education, Cairo, Egypt.

of loading with different thickness to span ratios. The numerical results are compared with those based on Reissner, Mindlin and three dimensional theories.

BERGAN-WANG ENERGY EXPRESSION

According to Bergan and Wang, the strain energy per unit area F, can be written as the sum of bending contribution F_b and shear contribution F_s:

$$F = F_b + F_s$$

where

$$F_b = \tfrac{1}{2} K_b^T D_b K_b$$

and

$$F_s = \tfrac{1}{2} K_s^T D_s K_s$$

in which $(\ \ )^T$ denotes the transpose of a matrix, and the suffix b(s) stands for bending (shear). For a homogenous and isotropic material the above matrices are given by [7]:

$$K_b = \begin{bmatrix} w_{xx} \\ w_{yy} \\ 2w_{xy} \end{bmatrix} + h_o^2 \begin{bmatrix} w_{xxxx} + w_{xxyy} \\ w_{yyyy} + w_{xxyy} \\ 2(w_{xxxy} + w_{xyyy}) \end{bmatrix}$$

$$K_s = h_o^2 \begin{bmatrix} w_{xxx} + w_{xyy} \\ w_{yyy} + w_{xxy} \end{bmatrix}$$

$$D_b = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & o \\ \nu & 1 & o \\ o & o & \frac{1-\nu}{2} \end{bmatrix}$$

$$D_s = \frac{5E}{12(1+\nu)} \begin{bmatrix} 1 & o \\ o & 1 \end{bmatrix}$$

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with $h_o^2 = \frac{h^2}{5(1-\nu)}$

E is the Young's modulus, ν is the Poisson's ratio and h is the plate thickness. Subscripts x and y denote partial derivatives.

EULER EQUATION AND ITS BOUNDARY CONDITIONS

In order to get the Euler equation, it is convenient to write the potential energy of the plate in the following form:

$$V = \iint F(x,y,w,\ldots,w_{yyyy})\, dx\, dy - \iint pw\, dxdy$$

in which, the first integral is the strain energy (see the previous section), the second integral is the potential energy of the external force p and x and y denote the cartesian coordinates of the mid-plane of the plate. Following the known technique of calculus of variations [8] , we allow the function w(x,y) to receive a variation $\varepsilon\eta(x,y)$, where ε is an arbitrary constant and η is an arbitrary admissible function, then we calculate the first variation of V which leads us finally to the required Euler equation:

$$-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial w_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial w_{xx}}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial w_{xy}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial w_{yy}}\right)$$

$$-\frac{\partial^3}{\partial x^3}\left(\frac{\partial F}{\partial w_{xxx}}\right) - \frac{\partial^3}{\partial x^2 \partial y}\left(\frac{\partial F}{\partial w_{xxy}}\right) - \frac{\partial^3}{\partial x \partial y^2}\left(\frac{\partial F}{\partial w_{xyy}}\right) - \frac{\partial^3}{\partial y^3}\left(\frac{\partial F}{\partial w_{yyy}}\right)$$

$$+\frac{\partial^4}{\partial x^4}\left(\frac{\partial F}{\partial w_{xxxx}}\right) + \frac{\partial^4}{\partial x^3 \partial y}\left(\frac{\partial F}{\partial w_{xxxy}}\right) + \frac{\partial^4}{\partial x^2 \partial y^2}\left(\frac{\partial F}{\partial w_{xxyy}}\right) + \frac{\partial^4}{\partial x \partial y^3}\left(\frac{\partial F}{\partial w_{xyyy}}\right) +$$

$$+\frac{\partial^4}{\partial y^4}\left(\frac{\partial F}{\partial w_{yyyy}}\right) = p$$

For the simple case where the boundaries of the plate coincide with coordinate lines, we can express the natural boundary conditions as follows:

for x = constant

. . .

$$\eta\left[\frac{\partial F}{\partial w_x} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w_{xx}}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial w_{xy}}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial w_{xxx}}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial w_{xxy}}\right)\right.$$

$$+ \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial w_{xyy}}\right) - \frac{\partial^3}{\partial x^3}\left(\frac{\partial F}{\partial w_{xxxx}}\right) - \frac{\partial^3}{\partial x^2 \partial y}\left(\frac{\partial F}{\partial w_{xxxy}}\right) - \frac{\partial^3}{\partial x \partial y^2}\left(\frac{\partial F}{\partial w_{xxyy}}\right)$$

$$\left. - \frac{\partial^3}{\partial y^3}\left(\frac{\partial F}{\partial w_{xyyy}}\right)\right] = 0$$

$$\eta_x\left[\frac{\partial F}{\partial w_{xx}} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w_{xxx}}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial w_{xxy}}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial w_{xxxx}}\right) + \right.$$

$$\left. \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial w_{xxxy}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial w_{xxyy}}\right)\right] = 0$$

$$\eta_{xx}\left[\frac{\partial F}{\partial w_{xxx}} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w_{xxxx}}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial w_{xxxy}}\right)\right] = 0$$

$$\eta_{xxx}\left(\frac{\partial F}{\partial w_{xxxx}}\right) = 0$$

Similar expressions are easily obtained for y=constant.

Substituting the expression of F in Euler equation and carrying out the derivatives, lead to the following rather simple form :

$$D\left(\Delta^2 w + h_o^2\, \Delta^3 w + h_o^4\, \Delta^4 w\right) = p$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ is the flexural rigidity

and $\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is the laplacian operator.

The boundary conditions can be also expressed in the simple following forms:

For a clamped case :

$$w = w_x = w_{xx} = w_{xxx} = 0 \text{ on edge } x = \text{constant}$$

For a simply supported case :

$$w = w_{xx} = w_{xxxx} = w_{xxxxxx} = 0 \text{ on edge } x = \text{constant}$$

Similar expressions can be written for y=constant.

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Remarks :

1- It is clear that the deduced Euler equation converges to the classical thin plate equation:

$D \Delta^2 w = p$ as the thickness decreases.

2- It can be easily seen that the first two boundary conditions for both clamped and simply supported cases are equivalent to those of classical thin theory.

EXAMPLES

In this section we determine the maximum deflection of simply supported square plate of various thickness to span ratios for sinusoidal, uniformly distributed and central concentrated loads. The solution is obtained by application of Navier's approach [9,10] to the following system of fourth order partial differential equations:

$$D\Delta^2 u = p$$

$$h_o^4 \Delta^2 w + h_o^2 \Delta w + w = u$$

with the following boundary conditions :

$u = u_{xx} = 0$ on an edge x = constant for the first equation. and

$w = w_{xx} = 0$ on an edge x = constant for the second equation.

Similar conditions are established for y = constant. It is obvious that this system of equations is equivalent to the eighth order partial differential equation which has been already deduced.

The maximum deflection of the symmetrically loaded plate occurs at the center and will be expressed in the form :

$$w_{max} = (1 + \delta)\, w_c$$

where w_c is the maximum deflection calculated by classical thin plate theory and δ is a correction term that shows the shear effect. In all examples we take poisson's ratio to be 0.3.

Example 1. Sinusoidal loading

Consider $p(x,y) = P_o \sin(\pi x/a) \sin(\pi y/a)$ where

P_o is the amplitude of the load and a is the plate side length. The maximum deflection in this case is given by :

$$w_{max} = \frac{a^4 P_o}{4 \pi^4 D} \quad \frac{1}{1 - \frac{2 \pi^2}{5(1-\nu)} \left(\frac{h}{a}\right)^2 + \frac{4 \pi^4}{25(1-\nu)^2} \left(\frac{h}{a}\right)^4}$$

The results are presented in Table 1 and compared with the values of Schäfer [11] who used Reissner theory, and Levinson and Cooke [12] who used Mindlin theory.

Example 2. Uniformly distributed load

In this case $p(x,y)=P_o$ and the corresponding solution is expressed as

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a}$$

where

$$a_{mn} = \frac{P_{mn}}{1-\left(\frac{\pi h_o}{a}\right)^2 (m^2+n^2) + \left(\frac{\pi h_o}{a}\right)^4 (m^2+n^2)^2}$$

and

$$P_{mn} = \frac{16 a^4 P_o}{\pi^6 D} \quad \frac{1}{mn(m^2+n^2)^2}$$

The maximum deflection is then given by

$$w_{max} = \sum_{m=1,3,\ldots} \sum_{n=1,3,\ldots} (-1)^{(m+n-2)/2} a_{mn}$$

Table 2. shows a comparison of the results with Salerno and Goldberg [13] solution of Reissner equations, Levinson and Cooke solution of Mindlin equations and the three dimensional solution of Srinivas and Rao [14].

Example 3. Central concentrated load

The solution of this example is given by the same expressions of the previous one, except for P_{mn} which is given by :

$$P_{mn} = \frac{4 a^2 P}{\pi^4 D} \quad \frac{(-1)^{(m+n-2)/2}}{(m^2+n^2)^2}$$

In table 3, the results are compared with two types of finite element solutions of Reissner theory:conforming finite elements of Rao et al [3] and hybrid finite elements of Wu [15].

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Table 1. The correction term δ for a sinusoidal load ($w_c=0.00257\ a^4P_o/D$)

h/a	0.05	0.1	0.15	0.2
Authors	0.014	0.056	0.125	0.212
Schäfer [11]	0.012	0.048	0.108	0.192
Levinson and Cooke [12]	-	0.056	-	0.226

Table 2. The correction term δ for a uniformly distributed load ($w_c= 0.00406\ a^4P_o/D$)

h/a	0.05	0.1	0.15	0.2
Authors	0.013	0.052	0.119	0.219
Salerno and Goldberg [13]	0.011	0.044	0.099	0.176
Levinson and Cooke [12]	-	0.052	-	0.208
Srinivas and Rao [14]	0.054	0.129	-	0.333

Table 3. The correction term δ for a central concentrated load ($w_c= 0.0116\ a^2P/D$)

h/a	0.05	0.1	0.15	0.2
Authors	0.022	0.062	0.105	0.140
Rao et al [3]	0.019	0.082	0.266	0.329
Wu [15]	0.008	0.075	0.181	0.322

CONCLUSION

Based on Bergan-Wang approach , an eight order partial differential equation for the deflection of thick plate has been derived. This equation which accounts for both bending and shear effects has been solved for a simply supported square plate subject to different types of loading with various thickness to span ratios. The results agree well with other solutions based on different theories.

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