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MILITARY TECHNICAL COLLEGE

CAIRO - EGYPT

OPTIMUM CONTROL OF A GYROSCOPIC STABILIZER WITH STOCHASTIC DISTURBANCE

F.A. TOLBAH*

ABSTRACT

The problem of optimum control of a moment compensating vertical gyro frame is considered. The transfer function matrix of the stabilizer has been derived. The optimum compensator of the stabilizer had been obtained such that the dispersion of the error signal is minimized for a certain value of the settling time. The proposed technique had been compared by the conventional methods. The analytical methods as well as the simulation results proved the superiority of the proposed method over the conventional approaches.

INTRODUCTION

The problem of improving the accuracy of the gyroscopic stabilizers had been developed by several authers. However, the design problem of optimum compensators minimizing the dispersion of the error signal-in case of stochastic disturbance-and providing certain predetermined settling time of the gyroscopic stabilizer, has not yet developed. The present research deals with the problem of optimum control of a moment compensating stabilized vertical gyro frame. The gyro system consists of two identical gyro elements suspended on the stabilized frame. The two gyro elements are spining in opposite directions and are coupled such that the spining axis of each gyro can precess by the same angle but in opposite directions. Such elements can be used for stabilization purposes on a swinging navigation vehicles.

DERIVATION OF THE TRANSFER FUNCTION MATRIX OF THE STABILIZER

Fig. 1 shows a schematic representation of the stabilizer with its two gyroscopic elements. Determining the magnitudes and directions of the inertia, gyroscopic and the

* Associate Professor, Dept. of Machine Design & Production Eng., The Faculty of Eng., Ain Shams University, Cairo, Egypt.

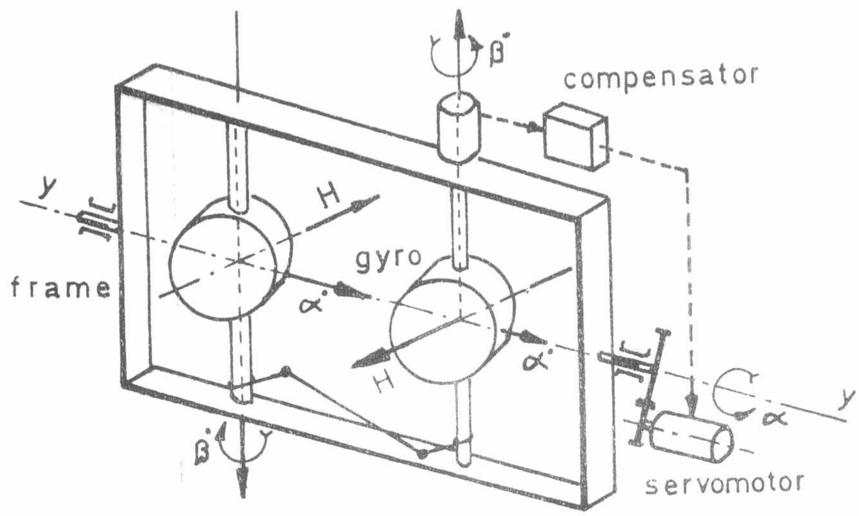

Fig. 1
The gyroscopic system

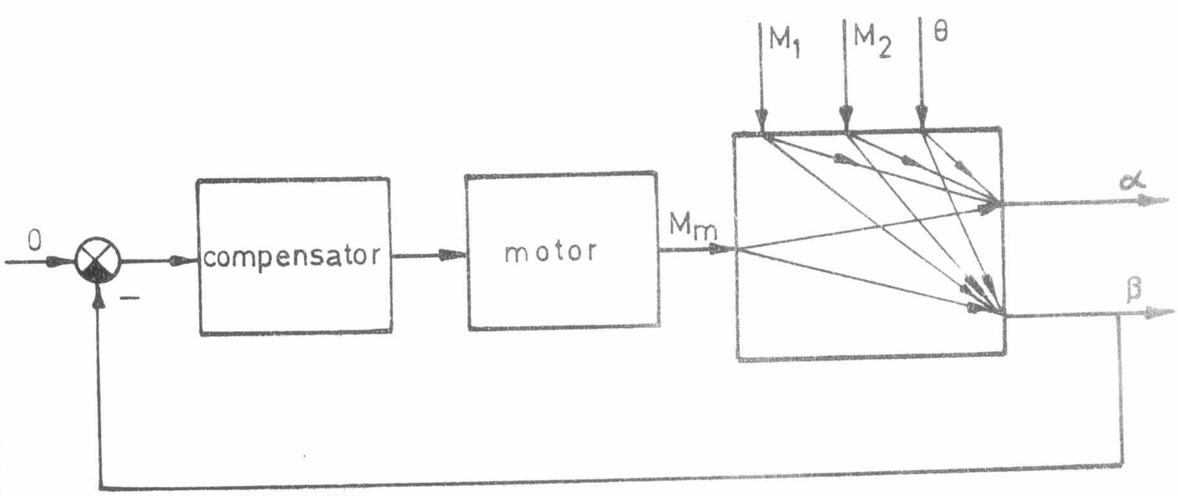

Fig. 2
The eqivalent block diagram

viscous friction moments [1], the equations of motion can be obtained in the following forms:

$$I_y \ddot{\alpha} + 2 H \dot{\beta} + C_1 \dot{\alpha} - M_n = I_o \ddot{\theta} + C_1 \dot{\theta} + M_1$$
$$2 I_g \ddot{\beta} - 2 H \dot{\alpha} + C_2 \dot{\beta} = -M_2 \qquad (1)$$

where
- H - the kinetic moment of the gyroscop
- C_1, C_2 - the coefficients of viscous friction
- M_m - the servomotor moment
- M_1, M_2 - external disturbing moments
- I_g, I_o - the moments of inertia of the gyro element and the motor
- $\dot{\theta}, \ddot{\theta}$ - the angular velocity and acceleration of rolling.

The polar moment of inertia I_y represents the moment of inertia of the frame, gyro elements and the equivalent moment of inertia of the servomotor refered to the frame axis. Fig. 2 shows the equivalent block diagram for the gyroscopic stabilizer with its compensator and servomotor. The transfer function matrix of the gyro frame can be obtained in the following form:

$$\begin{bmatrix} \alpha(s) \\ \beta(s) \end{bmatrix} = \begin{bmatrix} W_{\alpha\theta}(s) & W_{\alpha M1}(s) & W_{\alpha M2}(s) & W_{\alpha Mm}(s) \\ W_{\beta\theta}(s) & W_{\beta M1}(s) & W_{\beta M2}(s) & W_{\beta Mm}(s) \end{bmatrix} \begin{bmatrix} \theta(s) \\ M1(s) \\ M2(s) \\ M_m(s) \end{bmatrix} \qquad (2)$$

Expressions of the different components of the transfer function matrix can be deduced as follows:

$$W_{\alpha\theta}(s) = \frac{s^2(I_o s + C_1)(2 I_g s + C_2)}{D(s)} \; ; \; W_{\alpha M1}(s) = M_{\alpha Mm}(s) = \frac{s(2 I_g s + C_2)}{D(s)}$$

$$W_{\alpha M2}(s) = W_{\beta M1}(s) = W_{\beta Mm}(s) = \frac{2 Hs}{D(s)} \; ; \; W_{\beta M2}(s) = \frac{s(I_y s + C_1)}{D(s)}$$

$$W_{\beta\theta}(s) = \frac{2 Hs^2(I_o s + C_1)}{D(s)} ; D(s) = (I_y s^2 + C_1 s)(2 I_g s^2 + C_2 s) + 4H^2 s^2$$

DETERMINATION OF THE TRANSFER FUNCTION OF THE OPTIMUM SERVOMECHANISM

Consider the system shown in Fig. 3, in which the impulse response of the considered filter is of limited memory T; i.e. $w(t) = 0$ for $t \leq 0^-$ and $t > T$. An expression of the error signal can be obtained in the following form:

$$\varepsilon(t) = \varepsilon_g(t) + \varepsilon_m(t) \; ; \tag{3}$$

$$\varepsilon_g(t) = g(t) - \int_0^T g(t-\tau)\, w(\tau)\, d\tau \; ; \tag{4}$$

$$\varepsilon_m(t) = m(t) - \int_0^T m(t-\tau)\, w(\tau)\, d\tau \tag{5}$$

Expanding the function $g(t-\tau)$ in the form:

$$g(t-\tau) = g(t) - \tau \dot{g}(t) + \frac{\tau^2}{2!}\ddot{g}(t) - \dots + (-1)^r \frac{\tau^r}{r!} g^{(r)}(t); \tag{6}$$

the necessary condition for $\varepsilon_g(t) = 0$ can be formulated as follows:

$$g(t) = \mu_0 g(t) - \mu_1 \dot{g}(t) + \frac{\mu_2}{2!}\ddot{g}(t) - \dots + (-1)^r \frac{\mu_r}{r!} g^{(r)}(t) \tag{7}$$

where: $$\mu_i = \int_0^T \tau^i\, w(\tau)\, d\tau \; ; \quad i = 1, 2, \dots, r \tag{8}$$

Equation (7) leads to the following constraints:

$$\mu_o = 1, \; \mu_1 = \mu_2 = \dots = \mu_r = 0 \tag{9}$$

It is possible to get the impulse response of the optimum filter $w(t)$ which minimizes the dispersion of the error signal $\varepsilon_m(t)$ when taking into consideration the constraints given by equation set (9). In the present research, the signal $g(t)$ is considered to be in the following form:

$$g(t) = \sum_{i=0}^{N} a_i\, t^i \tag{10}$$

We assume further that the signal $g(t)$ has a spectral density function in the region $(-w_o, w_o)$ and the signal $m(t)$ has a spectral density function in the region $|w| \geq w_o$. Accordingly, the signal $x(t)$ can be splited as shown in Fig. 4. The considered assumption simplifies the solution given in[2] and the filter $w(t)$ can be obtained in the form:

$$w(t) = A_o + A_1 t + A_2 t^2 + \dots + A_r t^N \tag{11}$$

The impulse response of the other filter $w^{\infty}(t)$ can be obtained by the same approach which was proposed by Wiener. Using the constraints given by (9) and equations (8), (11), it is possible to derive the following set of equations which are linear in A_i:

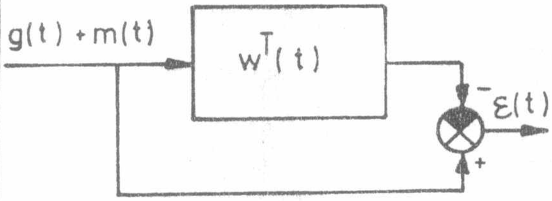

Fig. 3
The optimum filter

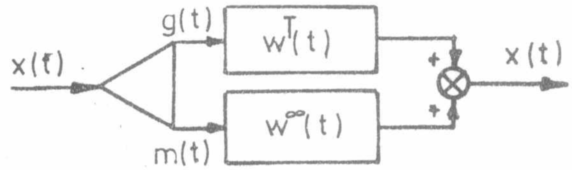

Fig. 4
Spliting the signal

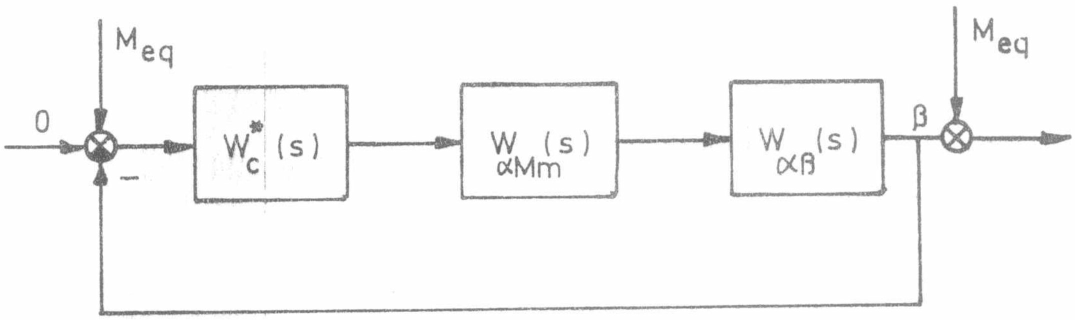

Fig. 5
Gyroscopic system transformation

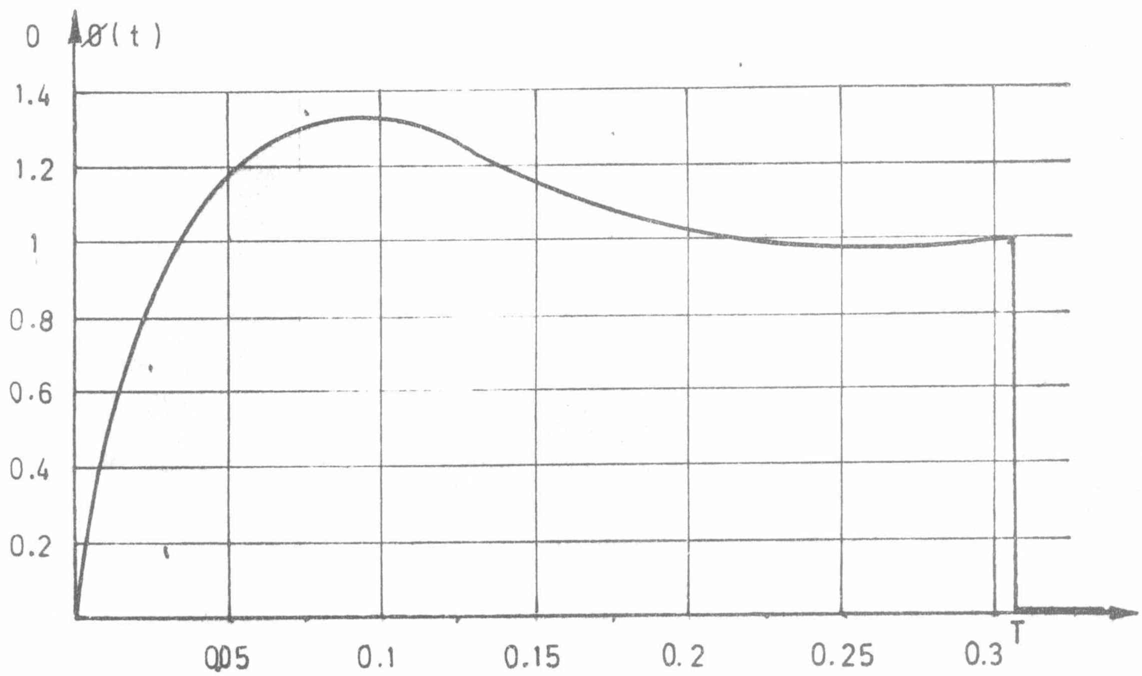

Fig. 6
Optimum system step response

$$\left.\begin{aligned} A_oT + \frac{A_1T^2}{2} + \dots + \frac{A_rT^{(r+1)}}{r+1} &= 1 \\ A_o\frac{T^2}{2} + A_1\frac{T^3}{3} + \dots + \frac{A_rT^{(r+2)}}{r+2} &= 0 \\ \vdots \qquad\qquad & \vdots \end{aligned}\right\} \qquad (12)$$

For any value of N, the previous set of equations can be solved w.r.t. A_i, e.g. for N = 2 we get:

$$A_o = \frac{9}{T}, \quad A_1 = -\frac{36}{T^2}, \quad A_2 = \frac{30}{T^3} \qquad (13)$$

For N = 3;

$$A_o = \frac{1}{T}, \quad A_1 = -\frac{120}{T^2}, \quad A_2 = \frac{240}{T^3}, \quad A_4 = -\frac{140}{T^4} \qquad (14)$$

Substituting the previous values in expression (11), the corresponding optimum transfer functions can be derived in the following forms:

$$W_2(s) = \frac{9}{Ts} - \frac{36}{T^2s^2} + \frac{60}{T^3s^3} - e^{-Ts}\left(\frac{3}{Ts} + \frac{24}{T^2s^2} + \frac{60}{T^3s^3}\right) \qquad (15)$$

$$W_3(s) = \frac{1}{Ts} - \frac{120}{T^2s^2} + \frac{480}{T^3s^3} - \frac{840}{T^4s^4} - e^{-sT}\left(\frac{19}{Ts} - \frac{60}{T^2s^2} - \frac{360}{T^3s^3} - \frac{840}{T^4s^4}\right) \qquad (16)$$

The previous transfer functions satisfies the condition that the corresponding impulse responses vanish for $t \leq 0^-$ and $t > T$.

CHOISE OF THE FILTER MEMORY (T)

The settling time of the optimum system should be obtained from the condition that the signal g(t) is to be obtained with certain predeformined accuracy. If the signal g(t) has a spectral density function within the interval $(-w_o, w_o)$ then [3]:

$$|g^{(k)}_{max}| \leq w_o^k \; g_{max} \qquad (17)$$

Considering expression (10) and the previous inequality, we have:

$$a_k \leq \frac{|g_{max}|}{k!} w_o^k ;$$

$$\sum_{k=o}^{N} a_k t^k \leq \sum_{k=o}^{N} |g_{max}| \cdot \frac{w_o^k}{k!} \cdot t^k \qquad (18)$$

Considering the limit as $N \rightarrow \infty$:

$$g(t) \leq \sum_{k=o}^{\infty} \frac{|g_{max}|}{k!} \cdot \left(\frac{2\pi T}{T_o}\right)^k = |g_{max}|\, e^{((2\pi T)/T_o)} \qquad (19)$$

where $T_o = 2\pi/w_o$

Accordingly, the maximum possible relative error can be obtained from the relation:

$$\frac{\varepsilon}{|g_{max}|} = e^{\left(\frac{2\pi T}{T_o}\right)} - \sum_{k=o}^{N} \frac{1}{k!} \left(\frac{2\pi T}{T_o}\right)^k \qquad (20)$$

Assigning an acceptable value of the relative error $\varepsilon/|g_{max}|$, equation (20) can be used for determining the corresponding value of T as a function of T_o and N.

GYROSCOPIC SYSTEM TRANSFORMATION AND THE OPTIMUM COMPENSATOR

Considering the schematic diagram shown in Fig. 2, the gyroscopic system under consideration can be transformed to the equivalent system shown in Fig. 5. The transfer function relating the angles α, β can be obtained by the servomotor channel via $W_{\alpha Mm}(s)$, $W_{\beta Mm}(s)$, i.e.:

$$W_{\alpha\beta}(s) = \frac{\alpha(s)}{\beta(s)} = \frac{2\,H}{2\,I_g s + C_2} \qquad (21)$$

The equivalent signal $m_{eq}(t)$ can be formulated in the form:

$$M_{eq}(s) = \theta(s)\, W_{\alpha\theta}(s) + M_1(s)\, W_{\alpha M1}(s) + M_2 W_{\alpha M2}(s) \qquad (22)$$

The equivalent system in Fig. 5 is used to determine the transfer function of the optimum compensator. The transfer function of the optimum system is related to the equivalent system in Fig. 5 through the relation:

$$\Phi^{opt.}(s) = \frac{W_{\alpha\beta}(s)\, W_c^*(s)\, W_{\alpha Mm}(s)}{1 + W_{\alpha\beta}(s)\, W_c^*(s)\, W_{\alpha Mm}(s)} \qquad (23)$$

Accordingly, the optimum filter $W_c^*(s)$ can be deduced in the form:

$$W_c^*(s) = \frac{\Phi^{opt}(s)}{W_{\alpha\beta}(s)\, W_{\alpha Mm}(s)\, (1 - \Phi^{opt}(s))} \qquad (24)$$

The transfer function $\Phi^{opt}(s)$ can be obtained from expression (11), and the system memory T is to be choosen from expression (20) for certain predetermined accuracy level.

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DESIGN OF AN OPTIMUM COMPENSATOR WITH INFINATE MEMORY

Considering the block diagram in Fig. 2 and the transfer function matrices given by expression (2), it is possible to get the following relation:

$$\alpha(s) = \frac{W_{\alpha\theta}(s)\, W_{\beta\theta}(s)\,(1+W_c(s)\, W_{\beta Mm}(s)) - W_c(s)\, W_{\alpha Mm}(s)}{W_{\beta\theta}(s)\,(1 + W_c(s)\, W_{\beta Mm}(s))} \cdot \theta(s) \quad (25)$$

To remove the effect of θ on α, the following relation should be satisfied:

$$W_{\alpha\theta}(s)\, W_{\beta\theta}(s)\,(1 + W_c(s)\, W_{\beta Mm}(s)) - W_c(s)\, W_{\alpha Mm}(s) = 0 \quad (26)$$

i.e.

$$W_c(s) = \frac{W_{\alpha\theta}(s)\, W_{\beta\theta}(s)}{W_{\alpha Mm}(s) - W_{\beta Mm}(s)\, W_{\alpha\theta}(s)\, W_{\beta\theta}(s)} \quad (27)$$

Using the different expressions given in (2), it is possible to get the following expression for $W_c(s)$:

$$W_c(s) = s \cdot \frac{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_o}{b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_o} \quad (28)$$

where:

$a_4 = 4\, HI_o I_y I_g$; $a_3 = 2\, HI_o(2\, I_g C_1 + I_y C_1) + 8\, HC_1 I_1 I_y I_g$;

$a_2 = 4\, HC_1 I_o(2\, I_g C_1 + I_y C_2) + 4\, HC_1^2 I_y I_g + 2\, HI_o(4\, H^2 + C_1 C_2)$;

$a_1 = 4\, HC_1 I_o(4\, H^2 + C_1 C_2) + 2\, HC_1^2(2\, I_g C_1 + I_y C_2)$;

$a_o = 2\, HC_1^2(4\, H^2 + C_1 C_2)$; $b_4 = 4\, I_g I_y$;

$b_3 = 2\, I_g I_y\,(2\, I_y C_1 + I_y C_2) + (2\, I_g C_1 + I_y C_2)$;

$b_2 = 2\, I_g I_y(2\, I_y I_g + C_1 C_2) + (2\, I_g C_1 + I_y C_2)^2 + {}$
$\quad + 2\, I_g I_y(C_1 C_2 + 4\, H^2) - 4\, H^2 I_o^2$

$b_1 = (2\, I_g C_1 + I_y C_2)(2\, I_y I_g + C_1 C_2) + {}$
$\quad + (C_1 C_2 + 4\, H^2)(2\, I_g C_1 + I_y C_2) - 8\, HC_1 I_o$;

$b_o = (C_1 C_2 + 4\, H^2)(2\, I_g I_y + C_1 C_2) - 4\, H^2 C_1^2$.

EXAMPLE

Consider the following practical data [4] for the gyroscopic system given in Fig. 1:

$I_y = 275.85$ N.m.sec^2 ; $I_o = 24.72$ N.m.sec^2 ;

$I_g = 5.88 \times 10^3$ N.m.sec^2 ; $H = 21.582$ N.m.sec.

$C_1 = 9.71$ N.m.sec. ; $C_2 = 4.9 \times 10^{-3}$ N.m.sec.

It is required to design the gyroscopic system to follow the signal $\alpha(t) = g(t) = a_o + a_1 t + a_2 t^2$ with an error not exceeding 0.05. The spectral density function of the previous signal is intercepted in the region (-2, 2) rad/sec. The disturbances Θ, M_1, M_2 have spectral density functions in the region $w > 2$ rad/sec.

Using the given data, it is possible to get the following transfer functions:

$$W_{\alpha\Theta}(s) = \frac{1}{\Delta(s)} (0.29\ s^2 + 0.24\ s + 0.05)\ ;$$

$$W_{\alpha Mm}(s) = \frac{1}{s\ \Delta(s)} (0.012\ s + 0.005)\ ; \quad W_{\alpha M2}(s) = \frac{43.1}{s\ \Delta(s)}\ ;$$

$$\Delta(s) = 3.24\ s^2 + 1.45\ s + 465.8\ ;$$

$$W_{\beta\Theta}(s) = \frac{1}{\Delta(s)} (1067\ s + 423.4)\ ;$$

$$W_{\beta M2}(s) = \frac{1}{s\ \Delta(s)} (275.85\ s + 9.81)\ ; \quad W_{\beta M2}(s) = \frac{43.1}{s\ \Delta(s)}\ ;$$

$$W_{\alpha\beta}(s) = \frac{43.1}{0.0117\ s + 4.9 \times 10^{-3}} \cong \frac{3683.76}{s}$$

Using expression (15) it is possible to get the optimum transfer function $\Phi^{opt}(s)$, and the transfer function of the optimum compensator can be obtained from expression (24) in the form:

$$W_c(s) = \frac{s^2\ \Delta(s)}{7.95 \times 10^4} \cdot \frac{L(s)}{T^3\ s^3 - L(s)}$$

where $L(s) = 9\ T^2 s^2 - 36\ Ts + 60 - e^{-sT}(3\ T^2 s^2 + 24\ Ts + 60)$

Considering the following approximation:

$$e^{-sT} = \frac{1 - 0.5\ Ts}{1 + 0.5\ Ts}\ ;$$

an expression for $W_c(s)$ can be obtained in the form:

$$W_c(s) = \frac{s^3}{7.95 \times 10^4} \cdot \frac{9.72\ T^3 s^4 + \bar{a}_3 s^3 + \bar{a}_2 s^2 + \bar{a}_1 s + \bar{a}_o}{T^4 s^4 - 7\ T^3 s^3 + 35\ T^2\ s^2 + 24\ Ts - 240}$$

where:

$\bar{a}_3 = 4.35\ T^3 - 77.75\ T^2$; $\bar{a}_2 = -(81\ T + 34.8\ T^2 - 1397.4\ T^3)$;

$\bar{a}_1 = -(36.25\ T + 11179.2\ T^2)$; $\bar{a}_o = -11645\ T.$

To choose T for an accuracy level of 0.05 and N = 2, expression (20) can be used. Equation (20) is solved graphically for T, and it is found that T = 0.314 sec. Substituting for T in the previous expression we get:

$$W_o(s) = \frac{s^3}{7.95 \times 10^4} \cdot \frac{0.03\ s^4 - 7.52s^3 + 14.4s^2 - 1113.6s - 3656.5}{9.72\times10^{-3}s^4 - 0.21s^3 + 3.84s^2 + 7.53s - 240}$$

The unit-step response of the closed loop system with the previous optimum filter can be obtained using expressions (11) and (13); i.e.

$$\emptyset(t) = A_0 t + 0.5\ A_1 t^2 + 0.333\ A_2 t^3$$

$$= 28.66\ t - 182.56\ t^2 + 323\ t^3 \quad ; \quad 0 \leqslant t \leqslant 0.314\ \text{sec.}$$

The previous unit step response is plotted in Fig. 6.

Now the problem will be solved again using the given data and expression (28), the following transfer function can be obtained:

$$W_c(s) = s \cdot \frac{(500\ s^4 + 357\ s^3 + 198\ s^2 + 151.7\ s + 1) \times 10^{-3}}{0.0012\ s^4 + 0.002\ s^3 + 0.0018\ s^2 + 0.0008\ s + 1}$$

CONCLUSIONS

The proposed technique for optimum compensator design represents a good tool for gyroscopic system accuracy improvements. The simulation results verified that the obtained optimum compensator increases the accuracy of the gyroscopic system if it is compared by the conventional design approaches. The simplified model of the optimum filter with limited memory is due to the adopted assumption that, the spectral density functions of the signal and the noise are not intersecting. The problem of practical implementation of the proposed optimum filter is not considered in the present research, and it represents another problem for further invistigation.

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