Military Technical College
Kobry El-Kobbah,
Cairo, Egypt

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FINITE ELEMENT ANALYSIS OF ISOTROPIC AND ANISOTROPIC BEAMS USING A SIMPLE HIGHER ORDER SHEAR DEFORMATION THEORY

By

F.M. Alraiess**, M. Adnan Elshafei* and S.S. Kossa*

ABSTRACT

In the present work, a finite element modeling and analysis is introduced for isotropic and anisotropic beams subjected to different mechanical loads. The assumed field displacements of the beam are represented by a simple higher order shear deformation theory made by Reddy [1]. The equation of motion is obtained using the principle of virtual work. A hermit cubic shape function is used to represent the transverse displacement w and its derivatives. The axial displacement *u*, and the normal rotation ϕ_x are represented by a linear shape function. A MATLAB code is developed to compute the natural frequency, and the static deformations of the structure due to the applied loads of different boundary condition. The results of the proposed model are compared with the available results of other investigators; good agreement is generally obtained.

KEY WORDS: Finite element method – Reddy's Theory - higher order shear deformation theory – mechanics of composite materials – solid mechanics.

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* Egyptian Armed Forces.
** Libyan Armed Forces.

INTRODUCTION

Several researchers are interested in solving the beam structures using different theories. Reddy [1] developed a simple higher-order shear deformation theory of laminated composite plates which gave parabolic distribution of the transverse shear strains. He obtained the exact closed-form solutions of symmetric cross-ply laminates. His results were compared with three-dimensional elasticity solutions and first-order deformation theory solutions. The results gave more accurate prediction of deflections and stresses, and satisfied the zero tangential traction boundary condition on the surfaces of the plate.

Khdeir and Reddy [2] presented the solution of the governing equations for the bending of cross-ply laminated beams using the state-space concept in conjunction with the Jordan canonical form. They used classical, first-order, second-order and third-order theories in their analysis. They determined the exact solutions for symmetric and asymmetric cross-ply beams with arbitrary boundary conditions subjected to arbitrary loadings. They studied the effect of shear deformation, number of layers and orthotropic ratio on the static response of composite beams. They found that the effect of shear deformation caused large differences between the predicted deflections by the classical beam theory and the higher order theories, especially when the ratio of beam length to its height decreased. They also deduced that the symmetric cross-ply stacking sequence gave a smaller response than those of asymmetric ones. In asymmetric cross-ply arrangements, increasing the number of layers for the same thickness decreased the beam deflection. Finally, they deduced that the increase of the orthotropic ratio decreased the beam deflection.

Chandrashekhara and Bangera [3] developed a finite element model based on a higher-order shear deformation theory to study the free vibration characteristics of laminated composite beams. They incorporated the Poisson's effect, the in-plane inertia and rotary inertia in their formulation. They concluded that: (i) shear deformations decrease natural frequencies of the beam, (ii) the natural frequencies increase by increasing the number of layers, (iii) the clamped-free boundary conditions exhibited the lowest natural frequency, (iv) the increase of fiber orientation angle decreases the natural frequency, and (v) the natural frequency decreases by increasing the material anisotropy.
Lee and Schultz [4] presented a study of free vibration of Timoshenko beams and axi-symmetric Mindlin plates. Their analysis is based on the Chebyshev pseudo spectral method. They deduced that their method has merits over other semi-analytic methods. They concluded that rapid convergences, good accuracy as well as the conceptual simplicity characterize the pseudo-spectral method. The results from this method agreed with those of Bernoulli-Euler beams and Kirchhoff plates when the thickness-to-length ratio was very small. However, they deviated considerably as the thickness-to-length ratio grew larger.

Jafari and Ahmadian [5] investigated free vibration analysis of a cross-ply laminated composite beam (LCB) on Pasternak foundation. Their finite element model was based on Timoshenko beam theory. They designed the model on such a way that it could be used for single-stepped, and cross-section-stepped foundation and multi-span beams. Their results indicate acceptable accuracy, good agreement, and indicated that as lamina deviated from symmetric to non-symmetric order, the natural frequencies decreased.

Raghu and Pavan [6] presented a mathematical model for the stress analysis of symmetric composite end notch flexure specimen using CBT, FOBT, SOBT and TOBT to determine the strain energy release rate. In their formulation, appropriate matching conditions have been applied at the crack tip by enforcing the displacement continuity at the crack tip in conjunction with the variational equation. They found that the third order shear deformation model was better than other beam models in determining the strain energy release rate for unidirectional cross-ply and multidirectional composites interlaminar fracture.

Yunhua Luo [7] presented an efficient three-dimensional Timoshenko beam element with consistent shape functions for two-nodes, constructed from the general solution to the homogeneous Euler-Lagrangian equations. Their numerical results showed that the developed 3D Timoshenko beam element was completely free from shear locking, and furthermore, the performance of the element in convergence was superior to the isoparametric Timoshenko beam element with reduced integration.

Elshafei et al. [8,9] proposed a finite element model, to study the static and the free vibration response of isotropic and anisotropic beams subjected to axial, bending, and torsion loads with warping effect using the classical beam theory. They found that an additional node in the middle of the beam element was required to give a better presenting in the torsion deformation. The obtained results founded reasonable in comparison with FOBT and HOBT.

In the present work, a finite element model has been proposed, based on Reddy beam theory [1], to predict the static and dynamic responses of advanced isotropic and anisotropic beams. A MATLAB code is constructed to compute the structure response due to different applied loads at different boundary conditions.

THEORETICAL FORMULATION

The displacements field equations of the beam are presented as [2]:

$$u(x,z)=u_{\circ}(x)+z\left[c_{\circ}\frac{dw}{dx}+c_1\phi(x)\right]+c_2z^2\psi(x)+c_3\left(z/h\right)^3\left[\phi(x)+\frac{dw}{dx}\right], \qquad \textbf{(1)a}$$

$$v(x,z)=0, \qquad \textbf{(1)b}$$

and

$$w(x,z)=w_{\circ}(x). \qquad \textbf{(1)c}$$

u, v and w are the displacements field equations along the x, y and z coordinates, respectively, u_0 and w_o denote the displacements of a point $(x,y,0)$ at the mid plane, and $\phi(x)$ and $\psi(x)$ are the rotation angles of the cross-section as shown in Fig. 1. Selecting the constant values of Eqn. (1) a as: $c_{\circ}=0,\quad c_1=1\quad c_2=0\quad c_3=-\left(4/3\right)h$. The displacements field equations for Third -order theory (HOBT), made by Reddy, at any point through the thickness can be expressed by [1]:

$$u(x,y,z)=u_{\circ}(x)+z\,\phi_x-\frac{4}{3h^2}z^3\left[\phi_x+\frac{\partial w}{\partial x}\right]$$

$$v(x,y,z)=0$$
$$w(x,y,z)=w_{\circ}(x) \quad \textbf{(2)}$$

where the third term accounts for warping and permits a parabolic shear strain distribution. To get the strain displacement equations, the following assumptions are also considered:

1- Plane stress, where transverse components of normal stresses σ_{yy} and σ_{zz} are negligible compared to the axial stress σ_{xx}.

2- As the beam length is too long compared to the other dimensions thus the following values of the strains components will be applied, $\varepsilon_{yy}=\varepsilon_{zz}=\gamma_{xy}=\gamma_{yz}=0$

3- After deformation, the cross sections of the beam don't remain planar or normal to the centroidal axis, but become parabolic [10] as shown in Fig. (1).

By applying the above assumptions, the remaining strains components are ε_{xx}, and γ_{xz} , can be represented by [11]:

$$\varepsilon_{xx}(x,y,z)\equiv\frac{\partial u(x,z)}{\partial x}=\frac{\partial u}{\partial x}+z\frac{\partial\phi_x}{\partial x}-z^3\frac{4}{3h^2}\left[\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right]=\varepsilon_{\circ(x)}+z\kappa_{\circ(x)}+z^3\kappa_{2(x)} \quad \textbf{(3)a}$$

$$\gamma_{xz}(x,y,z)\equiv\frac{\partial w(x,y,z)}{\partial x}+\frac{\partial u(x,y,z)}{\partial z}=\phi_x-z^2\frac{4}{h^2}\left[\phi_x+\frac{\partial w}{\partial x}\right]+\frac{\partial w}{\partial x}=\gamma_{\circ(xz)}+z^2\kappa_{2(xz)} \quad \textbf{(3)b}$$

where;

$$\varepsilon_{\circ(x)}=\frac{\partial u}{\partial x}\,,\ \gamma_{\circ(xz)}=\phi_x+\frac{\partial w}{\partial x}\,,\ \kappa_{\circ(x)}=\frac{\partial\phi_x}{\partial x} \quad \textbf{(4)a}$$

$$\kappa_{2(x)}=-c_1\left[\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right],\ \kappa_{2(xz)}=-c_2\left[\phi_x+\frac{\partial w}{\partial x}\right] \quad \textbf{(4)b}$$

and $c_1=\dfrac{4}{3h^2}$, $c_2=\dfrac{4}{h^2}$, $\varepsilon_{\circ(x)}$ is the reference surface extensional strain in the x-direction, $\gamma_{\circ(xz)}$ is the in-plane shear strain , $\kappa_{\circ(x)}$ and $\kappa_{2(x)}$ are the reference surface curvatures in the x-direction, $\kappa_{2(xz)}$ is the reference surface curvature in the z-direction. Thus the strains components ε_{xx}, and γ_{xz} can be expressed as:

$$\varepsilon_{xx}(x,y,z)=\varepsilon_{\circ(x)}+z\kappa_{\circ(x)}+z^3\kappa_{2(x)}$$
$$\gamma_{xz}(x,y,z)=\gamma_{\circ(xz)}+z^2\kappa_{2(xz)} \quad \textbf{(5)}$$

VARIATIONAL FORMULATION

The equation of motion of the structure is derived herein using the principle of minimum potential energy. The total potential energy of the structure, Π, is represented by [11]:

$$\Pi = U + W \tag{6}$$

The internal strain energy for a beam element, U, is represented by [11]:

$$U = \frac{1}{2}\iiint_v (\sigma_{xx}\varepsilon_{xx} + \sigma_{xz}\gamma_{xz})dv \tag{7}$$

Case I: Isotropic beam

The stress-strain relation is given as [12]:

$$\sigma_{xx} = \mathrm{E}\ \varepsilon_{xx} \qquad \text{(8)a}$$

$$\sigma_{xz} = G\gamma_{xz} \qquad \text{(8)b}$$

Substituting equation (8) into equation (7)

$$U = \frac{1}{2}\int_v \left(E\varepsilon_{xx}^2 + G\gamma_{xz}^2\right)dV \tag{9}$$

Substituting equation (5) into equation (9) results in:

$$U = \frac{1}{2}\int_v \left[E\left(\varepsilon_{\circ(x)} + z\kappa_{\circ(x)} + z^3\kappa_{2(x)}\right)^2 + G\left(\varepsilon_{\circ(xz)} + z^2\kappa_{2(xz)}\right)^2\right]dV \tag{10}$$

Substituting equation (4) into equation (10) results in:

$$U = \frac{1}{2}\int_v \left\{ \begin{aligned} &E\left[\begin{aligned}&\left(\frac{\partial u_\circ}{\partial x}\right)^2 + 2z\left(\frac{\partial u_\circ}{\partial x}\frac{\partial \phi_x}{\partial x}\right) + z^2\left(\frac{\partial \phi_x}{\partial x}\right)^2 - 2z^3\frac{4}{3h^2}\left[\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial u_\circ}{\partial x}\right)\right] \\ &-2z^4\left(\frac{\partial \phi_x}{\partial x}\right)\frac{4}{3h^2}\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + z^6\frac{16}{9h^4}\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right)^2\end{aligned}\right] \\ &+G\left[\left(\phi_x + \frac{\partial w}{\partial x}\right)^2 - z^2\frac{8}{h^2}\left(\phi_x + \frac{\partial w}{\partial x}\right)^2 + z^4\frac{16}{h^4}\left(\phi_x + \frac{\partial w}{\partial x}\right)^2\right] \end{aligned} \right\} dV \tag{11}$$

Rearranging equation (11) gives,

$$U = \frac{1}{2}\int_v \left\{ \begin{aligned} &E\left[\begin{aligned}&\left(\frac{\partial u_\circ}{\partial x}\right)^2 + 2z\left(\frac{\partial u_\circ}{\partial x}\frac{\partial \phi_x}{\partial x}\right) + z^2\left(\frac{\partial \phi_x}{\partial x}\right)^2 - z^3\frac{8}{3h^2}\left[\left(\frac{\partial u_\circ}{\partial x}\frac{\partial \phi_x}{\partial x} + \frac{\partial u_\circ}{\partial x}\frac{\partial^2 w}{\partial x^2}\right)\right] \\ &-z^4\frac{8}{3h^2}\left(\left(\frac{\partial \phi_x}{\partial x}\right)^2 + \frac{\partial \phi_x}{\partial x}\frac{\partial^2 w}{\partial x^2}\right) + z^6\frac{16}{9h^4}\left(\left(\frac{\partial \phi_x}{\partial x}\right)^2 + 2\frac{\partial \phi_x}{\partial x}\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial^2 w}{\partial x^2}\right)^2\right)\end{aligned}\right] \\ &+G\left[\left(1 - z^2\frac{8}{h^2} + z^4\frac{16}{h^4}\right)\left(\phi_x^{\ 2} + 2\phi_x\frac{\partial w}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2\right)\right] \end{aligned} \right\} dV \tag{12}$$

By taking the first variation of Eqn. (12), the incremental change of the strain energy of the beam is represented by:

$$\delta U=\frac{1}{2}\int_{v}\left\{\begin{aligned}&E\left[\begin{aligned}&\left(2\frac{\partial u}{\partial x}\frac{\partial\delta u}{\partial x}\right)+2z\left(\frac{\partial\delta u}{\partial x}\frac{\partial\phi_x}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial\delta\phi_x}{\partial x}\right)+2z^2\left(\frac{\partial\phi_x}{\partial x}\frac{\partial\delta\phi_x}{\partial x}\right)\\&-z^3\frac{8}{3h^2}\left(\frac{\partial\delta u}{\partial x}\frac{\partial\phi_x}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial\delta\phi_x}{\partial x}+\frac{\partial\delta u}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial u}{\partial x}\frac{\partial^2\delta w}{\partial x^2}\right)\\&-z^4\frac{8}{3h^2}\left(2\frac{\partial\phi_x}{\partial x}\frac{\partial\delta\phi_x}{\partial x}+\frac{\partial\phi_x}{\partial x}\frac{\partial^2\delta w}{\partial x^2}+\frac{\partial\delta\phi_x}{\partial x}\frac{\partial^2 w}{\partial x^2}\right)\\&+z^6\frac{32}{9h^4}\left(\frac{\partial\phi_x}{\partial x}\frac{\partial\delta\phi_x}{\partial x}+\frac{\partial\phi_x}{\partial x}\frac{\partial^2\delta w}{\partial x^2}+\frac{\partial\delta\phi_x}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2\delta w}{\partial x^2}\right)\end{aligned}\right]\\&+G\left[\left(1-z^2\frac{8}{h^2}+z^4\frac{16}{h^4}\right)\left(2\phi_x\delta\phi_x+2\phi_x\frac{\partial\delta w}{\partial x}+2\frac{\partial w}{\partial x}\delta\phi_x+2\frac{\partial w}{\partial x}\frac{\partial\delta w}{\partial x}\right)\right]\end{aligned}\right\}dV \qquad \textbf{(13)}$$

Equation (13) defines the stiffness matrix constituents of the beam element as:

$$k_{11}=\int_v E\left(\frac{\partial u}{\partial x}\frac{\partial\delta u}{\partial x}\right)dV$$

$$k_{12}=k_{21}^T=-\int_v E\left[z^3\frac{4}{3h^2}\left(\frac{\partial\delta u}{\partial x}\frac{\partial^2 w}{\partial x^2}\right)\right]dV$$

$$k_{13}=k_{31}^T=\int_v E\left[z\left(\frac{\partial\delta u}{\partial x}\frac{\partial\phi_x}{\partial x}\right)-z^3\frac{4}{3h^2}\left(\frac{\partial\delta u}{\partial x}\frac{\partial\phi_x}{\partial x}\right)\right]dV \qquad \textbf{(14)}$$

$$k_{22}=\int_v\left\{E\left[z^6\frac{16}{9h^4}\left(\frac{\partial^2\delta w}{\partial x^2}\frac{\partial^2 w}{\partial x^2}\right)\right]+G\left[\left(\frac{\partial\delta w}{\partial x}\frac{\partial w}{\partial x}\right)-z^2\frac{8}{h^2}\left(\frac{\partial\delta w}{\partial x}\frac{\partial w}{\partial x}\right)+z^4\frac{16}{h^4}\left(\frac{\partial\delta w}{\partial x}\frac{\partial w}{\partial x}\right)\right]\right\}dV$$

$$k_{23}=k_{32}^T=\int_v\left\{\begin{aligned}&E\left[-z^4\frac{4}{3h^2}\left(\frac{\partial^2\delta w}{\partial x^2}\frac{\partial\phi_x}{\partial x}\right)+z^6\frac{16}{9h^4}\left(\frac{\partial^2\delta w}{\partial x^2}\frac{\partial\phi_x}{\partial x}\right)\right]\\&+G\left[\left(\frac{\partial\delta w}{\partial x}\phi_x\right)-z^2\frac{8}{h^2}\left(\frac{\partial\delta w}{\partial x}\phi_x\right)+z^4\frac{16}{h^4}\left(\frac{\partial\delta w}{\partial x}\phi_x\right)\right]\end{aligned}\right\}dV$$

$$k_{33}=\int_v\left\{\begin{aligned}&E\left[z^2\left(\frac{\partial\delta\phi_x}{\partial x}\frac{\partial\phi_x}{\partial x}\right)-z^4\frac{8}{3h^2}\left(\frac{\partial\delta\phi_x}{\partial x}\frac{\partial\phi_x}{\partial x}\right)+z^6\frac{16}{9h^4}\left(\frac{\partial\delta\phi_x}{\partial x}\frac{\partial\phi_x}{\partial x}\right)\right]\\&+G\left[\left(\delta\phi_x\phi_x\right)-z^2\frac{8}{h^2}\left(\delta\phi_x\phi_x\right)+z^4\frac{16}{h^4}\left(\delta\phi_x\phi_x\right)\right]\end{aligned}\right\}dV$$

Case II: Anisotropic Beam

The stress-strain relation of a lamina in matrix notation is given by [12-13]:

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{xz}\end{Bmatrix}=\begin{bmatrix}\tilde{Q}_{11} & \\ & \tilde{Q}_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{xz}\end{Bmatrix} \qquad \textbf{(15)}$$

The complete derivation of Eqn. (15) can be seen in Appendix A.

The strain energy for the laminate is obtained by substituting Eqn. (15) into Eqn. (7) as:

$$U=\frac{1}{2}\int_v\left(\tilde{Q}_{11}\varepsilon_{xx}^2+\tilde{Q}_{55}\gamma_{xz}^2\right)dV \qquad \textbf{(16)}$$

Substituting by Equations (3),(4), and (5) into Equation (16), one can obtain:

$$U=\frac{1}{2}\left\{\begin{array}{l}\tilde{Q}_{11}\left[\begin{array}{l}\left(\frac{\partial u}{\partial x}\right)^2+2z\left(\frac{\partial u}{\partial x}\frac{\partial \phi_x}{\partial x}\right)+z^2\left(\frac{\partial \phi_x}{\partial x}\right)^2-2z^3\frac{4}{3h^2}\left[\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial u}{\partial x}\right)\right]\\ -2z^4\left(\frac{\partial \phi_x}{\partial x}\right)\frac{4}{3h^2}\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+z^6\frac{16}{9h^4}\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)^2\end{array}\right]\\ +\tilde{Q}_{55}\left[\left(\phi_x+\frac{\partial w}{\partial x}\right)^2-z^2\frac{8}{h^2}\left(\phi_x+\frac{\partial w}{\partial x}\right)^2+z^4\frac{16}{h^4}\left(\phi_x+\frac{\partial w}{\partial x}\right)^2\right]\end{array}\right\}dV \qquad \textbf{(17)}$$

Integrating Eqn. (17) through the thickness of the laminate, the strain energy for anisotropic beam element is represented by:

$$U=\frac{1}{2}\iint_A\left\{\begin{array}{l}\left[\begin{array}{l}A_{11}\left(\frac{\partial u}{\partial x}\right)^2+2B_{11}\left(\frac{\partial u}{\partial x}\frac{\partial \phi_x}{\partial x}\right)+D_{11}\left(\frac{\partial \phi_x}{\partial x}\right)^2-E_{11}\frac{8}{3h^2}\left[\left(\frac{\partial u}{\partial x}\frac{\partial \phi_x}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}\right)\right]\\ -F_{11}\frac{8}{3h^2}\left(\left(\frac{\partial \phi_x}{\partial x}\right)^2+\frac{\partial \phi_x}{\partial x}\frac{\partial^2 w}{\partial x^2}\right)+H_{11}\frac{16}{9h^4}\left(\left(\frac{\partial \phi_x}{\partial x}\right)^2+2\frac{\partial \phi_x}{\partial x}\frac{\partial^2 w}{\partial x^2}+\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right)\end{array}\right]\\ +\left[\left(A_{55}-D_{55}\frac{8}{h^2}+F_{55}\frac{16}{h^4}\right)\left(\phi_x^2+2\phi_x\frac{\partial w}{\partial x}+\left(\frac{\partial w}{\partial x}\right)^2\right)\right]\end{array}\right\}dA \qquad \textbf{(18)}$$

By taking the first variation of equation (18), incremental strain energy for the laminated beam element is represented by:

$$\delta U=\frac{1}{2}\int_A\left\{\begin{array}{l}\left[\begin{array}{l}2A_{11}\left(\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}\right)+2B_{11}\left(\frac{\partial u}{\partial x}\frac{\partial \delta\phi_x}{\partial x}+\frac{\partial \phi_x}{\partial x}\frac{\partial \delta u}{\partial x}\right)+2D_{11}\left(\frac{\partial \phi_x}{\partial x}\frac{\partial \delta\phi_x}{\partial x}\right)\\ -E_{11}\frac{8}{3h^2}\left[\left(\frac{\partial u}{\partial x}\frac{\partial \delta\phi_x}{\partial x}+\frac{\partial \phi_x}{\partial x}\frac{\partial \delta u}{\partial x}\right)+\left(\frac{\partial u}{\partial x}\frac{\partial^2 \delta w}{\partial x^2}+\frac{\partial \delta u}{\partial x}\frac{\partial^2 w}{\partial x^2}\right)\right]\\ -F_{11}\frac{8}{3h^2}\left[2\left(\frac{\partial \phi_x}{\partial x}\frac{\partial \delta\phi_x}{\partial x}\right)+\left(\frac{\partial \phi_x}{\partial x}\frac{\partial^2 \delta w}{\partial x^2}+\frac{\partial \delta\phi_x}{\partial x}\frac{\partial^2 w}{\partial x^2}\right)\right]\\ +H_{11}\frac{16}{9h^4}\left[\begin{array}{l}2\left(\frac{\partial \phi_x}{\partial x}\frac{\partial \delta\phi_x}{\partial x}\right)+2\left(\frac{\partial \phi_x}{\partial x}\frac{\partial^2 \delta w}{\partial x^2}+\frac{\partial \delta\phi_x}{\partial x}\frac{\partial^2 w}{\partial x^2}\right)\\ +2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 \delta w}{\partial x^2}\right)\end{array}\right]\end{array}\right]\\ +\left[\left(A_{55}-D_{55}\frac{8}{h^2}+F_{55}\frac{16}{h^4}\right)\left[\begin{array}{l}2\left[\phi_x\,\delta\phi_x\right]+2\left(\phi_x\frac{\partial \delta w}{\partial x}+\delta\phi_x\frac{\partial w}{\partial x}\right)\\ +2\left(\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\right)\end{array}\right]\right]\end{array}\right\}dA$$

(19) a

Equation (19) defines the elements stiffness matrix constituents as:

$$k_{11} = \int_A A_{11}\left(\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}\right)dA$$

$$k_{12} = k_{21}^T = -\int_A E_{11}\frac{4}{3h^2}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial \delta u}{\partial x}\right)dA$$

$$k_{13} = k_{31}^T = \int_A \left(B_{11} - E_{11}\frac{4}{3h^2}\right)\left(\frac{\partial u}{\partial x}\frac{\partial \delta \phi_x}{\partial x}\right)dA \qquad \text{(19) b}$$

$$k_{22} = \int_A \begin{pmatrix} H_{11}\dfrac{16}{9h^4} + A_{55} \\ -D_{55}\dfrac{8}{h^2} + F_{55}\dfrac{16}{h^4} \end{pmatrix}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \delta w}{\partial x^2}\right)dA$$

$$k_{23} = k_{32}^T = \int_A \left(-F_{11}\frac{4}{3h^2} + H_{11}\frac{16}{9h^4}\right)\left(\frac{\partial \phi_x}{\partial x}\frac{\partial^2 \delta w}{\partial x^2}\right)dA + \int_A \left(A_{55} - D_{55}\frac{8}{h^2} + F_{55}\frac{16}{h^4}\right)\left(\phi_x \frac{\partial \delta w}{\partial x}\right)dA$$

$$k_{33} = \int_A \left(D_{11} - F_{11}\frac{8}{3h^2} + H_{11}\frac{16}{9h^4}\right)\left(\frac{\partial \phi_x}{\partial x}\frac{\partial \delta \phi_x}{\partial x}\right)dA + \int_A \left(A_{55} - D_{55}\frac{8}{h^2} + F_{55}\frac{16}{h^4}\right)(\phi_x \delta \phi_x)dA$$

FINITE ELEMENT FORMULATION

A two node beam element with four mechanical degrees of freedom at each node is used and shown in Fig. (2).The axial displacement u is expressed through the nodal displacement in the present model as follows [14-15]:

$$u(x) = u_1\varsigma_1^e + u_2\varsigma_2^e = \sum_{j=1}^{2} u_j^e \varsigma_j^e \qquad \textbf{(20)}$$

where the linear interpolation functions ς_1 and ς_2 are given by:

$$\varsigma_1 = 1 - \frac{x}{L}\ ,\ \ \varsigma_2 = \frac{x}{L} \qquad \textbf{(21)}$$

The transverse displacement w and the rotation θ are expressed in the finite element model by a Hermite cubic interpolation shape functions such as [16]:

$$w(x) = w_1\xi_1^e + w_2\xi_2^e + w_3\xi_3^e + w_4\xi_4^e = \sum_{j=1}^{4} w_j^e \xi_j^e \qquad \textbf{(22)}$$

where;

$$\begin{aligned}
\xi_1 &= 1-3\left(\frac{x}{L}\right)^2+2\left(\frac{x}{L}\right)^3 \\
\xi_2 &= -x\left(1-\frac{x}{L}\right)^2 \\
\xi_3 &= 3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3 \\
\xi_4 &= -x\left[\left(\frac{x}{L}\right)^2-\frac{x}{L}\right]
\end{aligned} \tag{23}$$

The rotation of the normal to the mid-plane about the *y* axis ϕ_x is expressed in the proposed model as follows [17]:

$$\phi_x(x)=\phi_1\zeta^e_1+\phi_2\zeta^e_2=\sum_{j=1}^{2}\phi^e_j\zeta^e_j \tag{24}$$

where the linear interpolation functions ζ_1 and ζ_2 are given by:

$$\zeta_1=1-\frac{x}{L}\ ,\ \zeta_2=\frac{x}{L} \tag{25}$$

Case I: Isotropic beam

Substituting by the shape functions, Eqn.(20) to Eqn.(25), into Eqn (14) and perform the integrating by considering the beam element has a length L, a width b and a height h, the isotropic beam element stiffness matrix is obtained and given in Appendix B.

Case II: Anisotropic beam

Substituting by the shape functions, Eqn.(20) to Eqn.(25), into Eqn (20)a and perform the integrating by considering the beam element has a length L, a width b and a height h, the

anisotropic beam element stiffness matrix is obtained and given in Appendix B.

Mass matrix:

The kinetic energy is introduced to obtain the consistent mass matrix for a beam element and is given by [18]:

$$T=\frac{1}{2}\int_v \rho_\circ\left(\dot{u}^2+\dot{w}^2\right)dV \tag{28}$$

By substituting Eqn. (2) into Eqn. (28) yields:

$$T=\int_A\int_{-h/2}^{h/2}\rho_\circ\left[u_\circ+z^2\phi_x\delta\phi_x-c_1z^4\left(\phi_x+\frac{\partial w}{\partial x}\right)+c_1^2z^6\left(\phi_x+\frac{\partial w}{\partial x}\right)+w_\circ\right]dV \tag{29}$$

where; ρ_0 is the mass density of the material. By taking the first variation of Eqn. (29) yields [10]:

$$\delta T=\int_A\int_{-h/2}^{h/2}\rho_\circ\begin{bmatrix} u\delta u+z^2\phi_x\delta\phi_x-c_1z^4\left(\phi_x+\frac{\partial w}{\partial x}\right)\delta\phi_x-c_1z^4\left(\delta\phi_x+\frac{\partial\delta w}{\partial x}\right)\phi_x \\ +c_1^2z^6\left(\phi_x+\frac{\partial w}{\partial x}\right)\left(\delta\phi_x+\frac{\partial\delta w}{\partial x}\right)+w_\circ\delta w_\circ \end{bmatrix}dV \tag{30}$$

Perform the integration Eqn. (30) through the thickness of the beam yields:

$$\delta T=\int_A\left[\begin{array}{l} I_\circ u\delta u+I_\circ w\delta w-\left(c_1I_4\phi_x-c_1^2I_6\phi_x-c_1^2I_6\dfrac{\partial w}{\partial x}\right)\dfrac{\partial \delta w}{\partial x}\\ +\left(I_2\phi_x-2c_1I_4\phi_x-c_1I_4\dfrac{\partial w}{\partial x}+c_1^2I_6\phi_x+c_1^2I_6\dfrac{\partial w}{\partial x}\right)\delta\phi_x\end{array}\right]dA \quad \textbf{(31)}$$

Equation (31) defines the mass matrix constituents of the beam element such as:

$$\begin{gathered} m_{11}=\int_A I_\circ\left(\partial u\partial\delta u\right)dA\\ m_{22}=\int_A I_\circ\left(\partial w\partial\delta w\right)dA+\int_A I_6c_1^2\left(\frac{\partial w}{\partial x}\frac{\partial\delta w}{\partial x}\right)dA\\ m_{23}=\int_A I_6c_1^2\left(\phi_x\frac{\partial\delta w}{\partial x}\right)dA-\int_A I_4c_1\left(\phi_x\frac{\partial\delta w}{\partial x}\right)dA\\ m_{33}=\int_A I_6c_1^2\left(\phi_x\delta\phi_x\right)dA-2\int_A I_4c_1\left(\phi_x\delta\phi_x\right)dA+\int_A I_2\left(\phi_x\delta\phi_x\right)dA\\ m_{12}=m_{13}=m_{21}=m_{31}=0 \end{gathered} \quad \textbf{(32)}$$

where; $\left(I_\circ,I_2,I_4,I_6\right)=\int_{-h/2}^{h/2}\rho_\circ\left(1,z^2,z^4,z^6\right)dz$

By substituting the shape functions equations (22), (24), and (26) into Eqn. (32) yield the mass matrix of the beam element given in Appendix B.

Load vector:

The virtual work done by external loads is given by [11]:

$$\delta W=\int_0^L P_{axial}\left(x\right).\left(u_\circ\delta u_\circ\right).dx+\int_0^L P_{shear}\left(x\right).\left(w_\circ\delta w_\circ\right).dx\,, \quad \textbf{(33)}$$

where; P_{shear} and P_{axial} are transverse and axial forces

By substituting the shape functions equations (22), (24) and (26) into Eqn. (33) and perform the integration over the length of the beam yields the element load vector given in Appendix B:

Equation of Motion:

The system equation of motion is obtained by summing the individual matrices of whole elements; mass, stiffness, and load such as [18]:

$$[\mathrm{M}]\{\ddot{\mathrm{q}}\}+[\mathrm{K}]\{\mathrm{q}\}=\{\mathrm{F}\} \quad \textbf{(34)}$$

where; $[M]$ is the global mass matrix, $[K]$ is the global stiffness matrix, $\{F\}$ is the global nodal forces vector, $\{\ddot{q}\}$ is the global nodal accelerations, and $\{q\}$is the global nodal displacements. The solution of the equation of motion Eqn. (35) gives the static as well as the dynamic responses of the structure system.

Numerical Examples

A MATLAB code is constructed to perform the analysis of isotropic and anisotropic beams using the present finite element model. The static and free vibration analyses are preformed for beams subjected to different kinds of mechanical loads. The model inputs are the beam dimensions, material properties and number of beam layers. The present model is capable of predicting the nodal (axial and transversal) deflections, the fundamental natural frequency and the mode shape of the beam, respectively.

RESULTS AND DISCUSSIONS

Case I: Isotropic beam results

Model convergence

The effect of number of elements was studied to converge the model results of cantilever beam shown in Figure (3) and its properties are given in Table (1). The obtained results are shown in Figure (4).

Table (1): Properties of an isotropic beam

Property	Value	Unit	Property	Value	Unit
E	100	GPa	Length (L)	1	m
G	40	GPa	Width (b)	0.2	m
ρ	1000	Kg/m^3	Height (h)	0.02	m

a) Static Analysis

Table (2) compares the maximum transverse displacement at mid span (x = L / 2) for the given cantilever beam subjected to uniform distributed load with various boundary conditions using eight number of elements. The obtained results are compared with the exact solution of Ref. [10].

Table (2): Mid span deflections of isotropic beam under uniform load

Parameter	Mid span deflections, [mm]			
Boundary Conditions	clamped – clamped		hinged – hinged	
load (N/m)	Ref.[10]	Present Model	Ref.[10]	Present Model
175.126	0.00262	0.00264	0.0132	0.0132
875.634	0.0115	0.0132	0.0661	0.0661
1751.268	0.0188	0.0264	0.1322	0.1323

b) Dynamic Analysis

A free vibration analysis is performed for a given cantilever beam. The first natural frequency is obtained by the proposed model using eight elements and compared to the exact solution proposed by Ref. [19] and given in Table (3).

Table (3): Natural frequency (rad/s) for isotropic cantilever beam

mode	Ref.[19]	Present Model	Exact solution
1	200 (rad/s)	205 (rad/s)	203 (rad/s)

Figures (5)a - (5)d show the first four mode shapes for aluminum beam with a properties given in Table(4) using twenty number of elements.

Table (4): Material and geometric properties for isotropic beam.

Property	Aluminum	Unit	Property	Aluminum	Unit
E	206.82	*Gpa*	Width (b)	0.0254	*m*
ν	0.25	-	Height (h)	0.0254	*m*
Length (L)	2.54	*m*			*m*

Case II: Anisotropic Beam Results

Model convergence

The effect of number of elements was studied to convergence the model results of a composite beam under various boundary conditions as shown in Figure (6). The dimensionless material properties are given as, $E_1/E_2 = 25$, $G_{12} = 0.5E_2$, $\gamma_{12} = 0.25$, $L/h = 100$, and it is subjected to a uniformly distributed load. The model convergence was obtained at (20) elements for clamped – free beam, and (50) elements of clamped – clamped beam.

Figures (7)a and (7)b present the effect of number of elements on the non-dimensional transverse deflection of the beam with length to height ratio of 50 as clamped – free and clamped – clamped boundary conditions, respectively. The beam is solved for anti-symmetric cross-ply with fiber orientation angles $(0^\circ, 90^\circ)$. The laminate is assumed to be of the same thickness and made of the same material. The transverse deflection is non-dimensioned and given as; $\overline{w} = \dfrac{wAE_2h^2 10^2}{f_\circ L^4}$; where *w* is the actual transverse deflection. It can be seen that the transverse deflection reaches an asymptotic value at small number of elements, which proves convergence of the proposed model results.

a) Static Analysis

The predicted displacements by the proposed model for anisotropic beams with various boundary conditions for the case of symmetric cross-ply $(0^\circ, 90^\circ, 0^\circ)$ and anti-symmetric cross-ply $(0^\circ, 90^\circ)$ laminations with L/h=10, respectively are listed in Tables (5) and (6) in comparison with the respective results of Ref. [2].

The predicted displacements of the proposed model for a beam subjected to uniformly distributed load and has different number of layers with L/h=10 are compared with the corresponding results of Ref. [2] and represented in Figures (8)a, and (8)b which show the effect of ply-orientation and number of layers on non-dimensional transverse deflection of composite beams with clamped – free and hinged - hinged boundary conditions with the same laminate thickness

Table (5): Non- dimensional mid-span deflection of anti-symmetric cross-ply (0/90/0) composite beams for various boundary conditions.

Parameter	Non- dimensional mid-span deflection							
Boundary conditions	H - H		C - H		C - C		C - F	
L/h	Ref. [2]	Present Model	Ref.[2]	Model	Ref.[2]	Present Model	Ref.[2]	Present Model
5	2.412	2.417	1.952	1.955	1.537	1.539	6.824	6.840
10	1.096	1.098	0.740	0.740	0.532	0.532	3.455	3.461
50	0.665	0.668	0.280	0.281	0.147	0.147	2.251	2.263

Table (6): Non- dimensional mid-span deflection of anti-symmetric cross-ply (0/90) composite beams for various boundary conditions.

Parameter	Non- dimensional mid-span deflection							
Boundary Conditions	H - H		C - H		C - C		C - F	
L/h	Ref.[2]	Present Model	Ref.[2]	Present Model	Ref.[2]	Present Model	Ref.[2]	Present Model
5	4.777	4.784	2.863	2.864	1.922	1.921	15.279	15.303
10	3.688	3.696	1.740	1.742	1.005	1.006	12.343	12.369
50	3.336	3.344	1.346	1.348	0.679	0.680	11.337	11.364

b) Dynamic Analysis

The predicted values for the fundamental natural frequency of composite beams are compared with the results given by Ref. [3]. The geometric properties of the beam are shown in Figure (9) and the material properties of AS4/3501-6 graphite/epoxy composite are, $E_1 = 144.80$ GPa, $E_2 = 9.65$ GPa, $G_{12} = G_{13} = 4.14$ GPa, $G_{23} = 3.45$ GPa, $\gamma_{12} = 0.3$ and $\rho = 1389.23 Kg/m^3$. The beam has length to height ratio of $L/h =$15. The fundamental natural frequency is non-dimensionized as; $\overline{\omega} = \omega L^2 \sqrt{(\rho / E_1 h^2)}$, where, ω is the fundamental natural frequency.

Tables (7) and (8) show a good agreement between the predicted results of the fundamental natural frequencies of the proposed model with the results given by Ref. [3] for the cases symmetrically laminated [0/90/90/0], [45/-45/-45/45] and anti-symmetric laminated [0/90/0/90], [45/-45/45/-45] and [30/50/30/50] composite beams. Figures (10)a, (10)b and (10)c represent the effect of ply-orientation angles on the first three mode shapes of a composite beam with clamped – free boundary condition. It is clear from the obtained results that the effect of the shear deformation are greater for the higher modes.

Table (7): Comparison of non-dimensional fundamental natural frequencies of symmetric laminated beams under various boundary conditions.

Parameter	**Non-dimensional fundamental natural frequencies**			
Ply Orientation Angles	**[0/90/90/0]**		**[45/-45/-45/45]**	
Boundary Conditions	**Ref.[3]**	**Present Model**	**Ref. [3]**	**Present Model**
H - H	2.5023	2.4957	0.8295	0.9084
C – C	4.5940	4.6432	1.8472	1.9958
C – H	3.5254	3.5369	1.2855	1.3992
C - F	0.9241	0.9223	0.2965	0.3253

Table (8): Ply orientation angles effect on the non-dimensional fundamental natural frequencies of composite beams with clamped - clamped edges.

Parameter	**Non-dimensional fundamental natural frequencies**					
Ply Orientation Angles	**[0/90/0/90]**		**[45/-45/45/-45]**		**[30/50/30/50]**	
Mode No.	**Ref.[3]**	**Present Model**	**Ref.[3]**	**Present Model**	**Ref.[3]**	**Present Model**
1	3.7244	3.7147	1.9807	1.9958	2.2526	2.5937
2	8.9275	8.9024	5.2165	5.2462	5.8624	6.6352
3	15.3408	15.2940	9.6912	9.7264	10.7609	11.9859
4	22.3940	22.3279	10.5345	15.1215	11.9506	18.1997
5	24.3155	29.7372	15.0981	15.1915	16.5747	20.9952

CONCLUSIONS

The following conclusions have been drawn:

1. The good agreement between the deflections and the natural frequencies predicted by the present model and the corresponding predictions of other investigators using other theories proves the predictive capabilities of such a model.
2. The transversal displacements predicted by the present finite element model are found to converge towards an asymptote at reasonable number of elements.
3. As the number of layers increases, the transversal deflection decreases and the accuracy of the present model for the natural frequencies increases.
4. The predicted results found reasonable compared with the other theories without using the shear correction factor.

5. Further work is needed to improve the predictive capabilities of the present model by covering the following:
   (a) Using a higher order displacement theory made by E. Wu et al. [21-22], which consider the effect of transverse shear deformation to obtain an accurate response.
   (b) Taking into account the geometric nonlinearities.

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Appendix A

The stress-strain relation for a thin orthotropic lamina of an anisotropic beam having coincidence of principal axis on geometric axis is given by [1]:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}$$

(A-1)

$$\begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix}$$

where, Q_{ij} is the reduced stiffness coefficient.
The components of the lamina stiffness matrix in terms of the engineering constants are given as [20]:

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{66} = G_{12}$$

(A-2)

$$Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{44} = G_{23}, \quad Q_{55} = G_{13},$$

where; E_1 and E_2 are the Young's modulus in the longitudinal and the transversal directions of the fiber, respectively, and ν_{12} , and ν_{21} are Poisson's ratios in the two directions. The stress-strain relation of a lamina in the geometric directions x, y and z is given by [12]:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}$$

(A-3)

$$\begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix}$$

where; $\bar{Q}_{ij}$ is the transformed reduced stiffness coefficient. The stress-strain relation of a lamina is rewritten as:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} \tilde{Q}_{11} & \\ & \tilde{Q}_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{xz} \end{Bmatrix} \qquad \textbf{(A-4)}$$

where, $\tilde{Q}_{ij}$ is the transformed reduced stiffness coefficient and given by:

$$\begin{gathered} \tilde{Q}_{ij} = \bar{Q}_{ij} - \bar{Q}_{i3}\bar{Q}_{j3} / \bar{Q}_{33} \quad \text{For i, j = 1} \\ \tilde{Q}_{ij} = \bar{Q}_{ij} \quad \text{For i, j = 5} \end{gathered} \qquad \textbf{(A-5)}$$

The resultants of normal force, moment, and shear force acting on the laminate thickness are calculated as:

$$\begin{Bmatrix} N_{xx} \\ M_{xx} \\ P_{xx} \end{Bmatrix} = \sum_{n=1}^{N} \left\{ \int_{z_{k-1}}^{z_k} \begin{Bmatrix} 1 \\ z \\ z^3 \end{Bmatrix} (\sigma_x)_k \, dz \right\} \quad , \begin{Bmatrix} Q_x \\ R_x \end{Bmatrix} = \sum_{n=1}^{N} \left\{ \int_{z_{k-1}}^{z_k} \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} (\sigma_{xz})_k \, dz \right\} \qquad \textbf{(A-6)}$$

where N_{xx}, M_{xx}, Q_x are force, moment and shear acting on a laminate, P_{xx} is an additional higher order resultant related to warping, R_x is an additional higher order shear resultant.

The laminate stiffness coefficients are defined as [1]:

$$\begin{gathered} (A_{11}, B_{11}, D_{11}, E_{11}, F_{11}, H_{11}) = \sum_{n=1}^{N} \int_{-h/2}^{h/2} (\tilde{Q}_{11})_k \, (1, z, z^2, z^3, z^4, z^6) dz \\ (A_{55}, D_{55}, F_{55}) = \sum_{n=1}^{N} \int_{-h/2}^{h/2} (\tilde{Q}_{55})_k \, (1, z^2, z^4) dz \end{gathered} \qquad \textbf{(A-7)}$$

where h is the laminate thickness, z is the distance from the middle surface to the surface of the *n*th lamina.

Appendix B

B.1: Mass matrix for the Beam element:

$$M_e = \frac{\rho A h^3}{420}\begin{bmatrix} \frac{140}{h^2} & 0 & 0 & 0 & \frac{70}{h^2} & 0 & 0 & 0 \\ 0 & \frac{156}{h^2}+\frac{2}{L^2} & \frac{22L}{h^2}+\frac{1}{6L} & \frac{8}{3L} & 0 & \frac{54}{h^2}-\frac{2}{L^2} & \frac{-13L}{h^2}+\frac{1}{6L} & \frac{8}{3L} \\ 0 & \frac{22L}{h^2}+\frac{1}{6L} & \frac{4L^2}{h^2}+\frac{2}{9} & \frac{-4}{9} & 0 & \frac{13L}{h^2}-\frac{1}{6L} & \frac{-3L^2}{h^2}-\frac{1}{18} & \frac{4}{9} \\ 0 & \frac{8}{3L} & \frac{-4}{9} & \frac{68}{9} & 0 & \frac{-8}{3L} & \frac{4}{9} & \frac{34}{9} \\ \frac{70}{h^2} & 0 & 0 & 0 & \frac{140}{h^2} & 0 & 0 & 0 \\ 0 & \frac{54}{h^2}-\frac{2}{L^2} & \frac{13L}{h^2}-\frac{1}{6L} & \frac{-8}{3L} & 0 & \frac{156}{h^2}+\frac{2}{L^2} & \frac{-22L}{h^2}-\frac{1}{6L} & \frac{-8}{3L} \\ 0 & \frac{-13L}{h^2}+\frac{1}{6L} & \frac{-3L^2}{h^2}-\frac{1}{18} & \frac{4}{9} & 0 & \frac{-22L}{h^2}-\frac{1}{6L} & \frac{4L^2}{h^2}+\frac{2}{9} & \frac{-4}{9} \\ 0 & \frac{8}{3L} & \frac{4}{9} & \frac{34}{9} & 0 & \frac{-8}{3L} & \frac{-4}{9} & \frac{68}{9} \end{bmatrix} \quad \textbf{(B-1)}$$

B.2: load vector for the beam element

$$F_e = \left\{ -P_{axial}/L \quad P_{shear}L/2 \quad P_{shear}L^2/12 \quad 0 \quad P_{axial}/L \quad P_{shear}L/2 \quad -P_{shear}L^2/12 \quad 0 \right\}^T \quad \textbf{(B-2)}$$

B.3: Stiffness matrix for isotropic beam element:

$$K_e = \begin{bmatrix} \frac{Ebh}{L} & 0 & 0 & 0 & \frac{-Ebh}{L} & 0 & 0 & 0 \\ 0 & \frac{Ebh^3}{21L^3}+\frac{16Gbh}{25L} & \frac{Ebh^3}{42L^2}+\frac{4Gbh}{75} & \frac{-4Gbh}{15L} & 0 & \frac{-Ebh^3}{21L^3}-\frac{16Gbh}{25L} & \frac{Ebh^3}{42L^2}+\frac{4Gbh}{75} & \frac{-4Gbh}{15L} \\ 0 & \frac{Ebh^3}{42L^2}+\frac{4Gbh}{75} & \frac{Ebh^3}{63L}+\frac{16GbhL}{225} & \frac{-4Ebh^3}{315L}+\frac{2GbhL}{45} & 0 & \frac{-Ebh^3}{42L^2}-\frac{4Gbh}{75} & \frac{Ebh^3}{126L}-\frac{4GbhL}{225} & \frac{4Ebh^3}{315L}-\frac{2GbhL}{45} \\ 0 & \frac{-4Gbh}{15L} & \frac{-4Ebh^3}{315L}+\frac{2GbhL}{45} & \frac{8Gbh}{15L}+\frac{17Ebh^3}{315L} & 0 & \frac{4Gbh}{15L} & \frac{4Ebh^3}{315L}-\frac{2GbhL}{45} & \frac{-8Gbh}{15L}-\frac{17Ebh^3}{315L} \\ \frac{-Ebh}{L} & 0 & 0 & 0 & \frac{Ebh}{L} & 0 & 0 & 0 \\ 0 & \frac{-Ebh^3}{21L^3}-\frac{16Gbh}{25L} & \frac{-Ebh^3}{42L^2}-\frac{4Gbh}{75} & \frac{4Gbh}{15L} & 0 & \frac{Ebh^3}{21L^3}+\frac{16Gbh}{25L} & \frac{-Ebh^3}{42L^2}-\frac{4Gbh}{75} & \frac{4Gbh}{15L} \\ 0 & \frac{Ebh^3}{42L^2}+\frac{4Gbh}{75} & \frac{Ebh^3}{126L}-\frac{4GbhL}{225} & \frac{4Ebh^3}{315L}-\frac{2GbhL}{45} & 0 & \frac{-Ebh^3}{42L^2}-\frac{4Gbh}{75} & \frac{Ebh^3}{63L}+\frac{16GbhL}{225} & \frac{-4Ebh^3}{315L}+\frac{2GbhL}{45} \\ 0 & \frac{-4Gbh}{15L} & \frac{4Ebh^3}{315L}-\frac{2GbhL}{45} & \frac{-8Gbh}{15L}-\frac{17Ebh^3}{315L} & 0 & \frac{4Gbh}{15L} & \frac{-4Ebh^3}{315L}+\frac{2GbhL}{45} & \frac{8Gbh}{15L}+\frac{17Ebh^3}{315L} \end{bmatrix} \quad \textbf{(B-3)}$$

B.4: Stiffness matrix for anisotropic beam element

$$
K_e \left[\begin{matrix}
\frac{A_{11}b}{L} & 0 & \frac{-4E_{11}b}{3h^2L} & \frac{B_{11}b}{L}-\frac{4E_{11}b}{3h^2L} \\
0 & \frac{64H_{11}b}{3h^4L^3}+\frac{6A_{55}b}{5L}-\frac{48D_{55}b}{5h^2L}+\frac{96F_{55}b}{5h^4L} & \frac{32H_{11}b}{3h^4L^2}+\frac{A_{55}b}{10}-\frac{4D_{55}b}{5h^2}+\frac{8F_{55}b}{5h^4} & \frac{-A_{55}b}{2}+\frac{4D_{55}b}{h^2}-\frac{8F_{55}b}{h^4} \\
\frac{-4E_{11}b}{3h^2L} & \frac{32H_{11}b}{3h^4L^2}+\frac{A_{55}b}{10}-\frac{4D_{55}b}{5h^2}+\frac{8F_{55}b}{5h^4} & \frac{64H_{11}b}{9h^4L}+\frac{2A_{55}bL}{15}-\frac{16D_{55}bL}{15h^2}+\frac{32F_{55}bL}{15h^4} & \frac{-4F_{11}b}{3h^2L}+\frac{16H_{11}b}{9h^4L}+\frac{A_{55}bL}{12}-\frac{2D_{55}bL}{3h^2}+\frac{4F_{55}bL}{3h^4} \\
\frac{B_{11}b}{L}-\frac{4E_{11}b}{3h^2L} & \frac{-A_{55}b}{2}+\frac{4D_{55}b}{h^2}-\frac{8F_{55}b}{h^4} & \frac{-4F_{11}b}{3h^2L}+\frac{16H_{11}b}{9h^4L}+\frac{A_{55}bL}{12}-\frac{2D_{55}bL}{3h^2}+\frac{4F_{55}bL}{3h^4} & \frac{D_{11}b}{L}-\frac{8F_{11}b}{3h^2L}+\frac{16H_{11}b}{9h^4L}+\frac{A_{55}bL}{3}-\frac{8D_{55}bL}{3h^2}+\frac{16F_{55}bL}{3h^4} \\
\frac{-A_{11}b}{L} & 0 & \frac{4E_{11}b}{3h^2L} & \frac{-B_{11}b}{L}+\frac{4E_{11}b}{3h^2L} \\
0 & \frac{-64H_{11}b}{3h^4L^3}-\frac{6A_{55}b}{5L}+\frac{48D_{55}b}{5h^2L}-\frac{96F_{55}b}{5h^4L} & \frac{-32H_{11}b}{3h^4L^2}-\frac{A_{55}b}{10}+\frac{4D_{55}b}{5h^2}-\frac{8F_{55}b}{5h^4} & \frac{A_{55}b}{2}-\frac{4D_{55}b}{h^2}+\frac{8F_{55}b}{h^4} \\
\frac{4E_{11}b}{3h^2L} & \frac{32H_{11}b}{3h^4L^2}+\frac{A_{55}b}{10}-\frac{4D_{55}b}{5h^2}+\frac{8F_{55}b}{5h^4} & \frac{32H_{11}b}{9h^4L}-\frac{A_{55}bL}{30}+\frac{4D_{55}bL}{15h^2}-\frac{8F_{55}bL}{15h^4} & \frac{4F_{11}b}{3h^2L}-\frac{16H_{11}b}{9h^4L}-\frac{A_{55}bL}{12}+\frac{2D_{55}bL}{3h^2}-\frac{4F_{55}bL}{3h^4} \\
\frac{-B_{11}b}{L}+\frac{4E_{11}b}{3h^2L} & \frac{-A_{55}b}{2}+\frac{4D_{55}b}{h^2}-\frac{8F_{55}b}{h^4} & \frac{4F_{11}b}{3h^2L}-\frac{16H_{11}b}{9h^4L}-\frac{A_{55}bL}{12}+\frac{2D_{55}bL}{3h^2}-\frac{4F_{55}bL}{3h^4} & \frac{-D_{11}b}{L}+\frac{8F_{11}b}{3h^2L}-\frac{16H_{11}b}{9h^4L}+\frac{A_{55}bL}{6}-\frac{4D_{55}bL}{3h^2}+\frac{8F_{55}bL}{3h^4}
\end{matrix} \right. \quad \textbf{(B-4)}
$$

$$
\left. \begin{matrix}
\frac{-A_{11}b}{L} & 0 & \frac{4E_{11}b}{3h^2L} & \frac{-B_{11}b}{L}+\frac{4E_{11}b}{3h^2L} \\
0 & \frac{-64H_{11}b}{3h^4L^3}-\frac{6A_{55}b}{5L}+\frac{48D_{55}b}{5h^2L}-\frac{96F_{55}b}{5h^4L} & \frac{32H_{11}b}{3h^4L^2}+\frac{A_{55}b}{10}-\frac{4D_{55}b}{5h^2}+\frac{8F_{55}b}{5h^4} & \frac{-A_{55}b}{2}+\frac{4D_{55}b}{h^2}-\frac{8F_{55}b}{h^4} \\
\frac{4E_{11}b}{3h^2L} & \frac{-32H_{11}b}{3h^4L^2}-\frac{A_{55}b}{10}+\frac{4D_{55}b}{5h^2}-\frac{8F_{55}b}{5h^4} & \frac{32H_{11}b}{9h^4L}-\frac{A_{55}bL}{30}+\frac{4D_{55}bL}{15h^2}-\frac{8F_{55}bL}{15h^4} & \frac{4F_{11}b}{3h^2L}-\frac{16H_{11}b}{9h^4L}-\frac{A_{55}bL}{12}+\frac{2D_{55}bL}{3h^2}-\frac{4F_{55}bL}{3h^4} \\
\frac{-B_{11}b}{L}+\frac{4E_{11}b}{3h^2L} & \frac{A_{55}b}{2}-\frac{4D_{55}b}{h^2}+\frac{8F_{55}b}{h^4} & \frac{4F_{11}b}{3h^2L}-\frac{16H_{11}b}{9h^4L}-\frac{A_{55}bL}{12}+\frac{2D_{55}bL}{3h^2}-\frac{4F_{55}bL}{3h^4} & \frac{-D_{11}b}{L}+\frac{8F_{11}b}{3h^2L}-\frac{16H_{11}b}{9h^4L}+\frac{A_{55}bL}{6}-\frac{4D_{55}bL}{3h^2}+\frac{8F_{55}bL}{3h^4} \\
\frac{A_{11}b}{L} & 0 & \frac{-4E_{11}b}{3h^2L} & \frac{B_{11}b}{L}-\frac{4E_{11}b}{3h^2L} \\
0 & \frac{64H_{11}b}{3h^4L^3}+\frac{6A_{55}b}{5L}-\frac{48D_{55}b}{5h^2L}+\frac{96F_{55}b}{5h^4L} & \frac{-32H_{11}b}{3h^4L^2}-\frac{A_{55}b}{10}+\frac{4D_{55}b}{5h^2}-\frac{8F_{55}b}{5h^4} & \frac{A_{55}b}{2}-\frac{4D_{55}b}{h^2}+\frac{8F_{55}b}{h^4} \\
\frac{-4E_{11}b}{3h^2L} & \frac{-32H_{11}b}{3h^4L^2}-\frac{A_{55}b}{10}+\frac{4D_{55}b}{5h^2}-\frac{8F_{55}b}{5h^4} & \frac{64H_{11}b}{9h^4L}+\frac{2A_{55}bL}{15}-\frac{16D_{55}bL}{15h^2}+\frac{32F_{55}bL}{15h^4} & \frac{-4F_{11}b}{3h^2L}+\frac{16H_{11}b}{9h^4L}+\frac{A_{55}bL}{12}-\frac{2D_{55}bL}{3h^2}+\frac{4F_{55}bL}{3h^4} \\
\frac{B_{11}b}{L}-\frac{4E_{11}b}{3h^2L} & \frac{A_{55}b}{2}-\frac{4D_{55}b}{h^2}+\frac{8F_{55}b}{h^4} & \frac{-4F_{11}b}{3h^2L}+\frac{16H_{11}b}{9h^4L}+\frac{A_{55}bL}{12}-\frac{2D_{55}bL}{3h^2}+\frac{4F_{55}bL}{3h^4} & \frac{D_{11}b}{L}-\frac{8F_{11}b}{3h^2L}+\frac{16H_{11}b}{9h^4L}+\frac{A_{55}bL}{3}-\frac{8D_{55}bL}{3h^2}+\frac{16F_{55}bL}{3h^4}
\end{matrix} \right]
$$

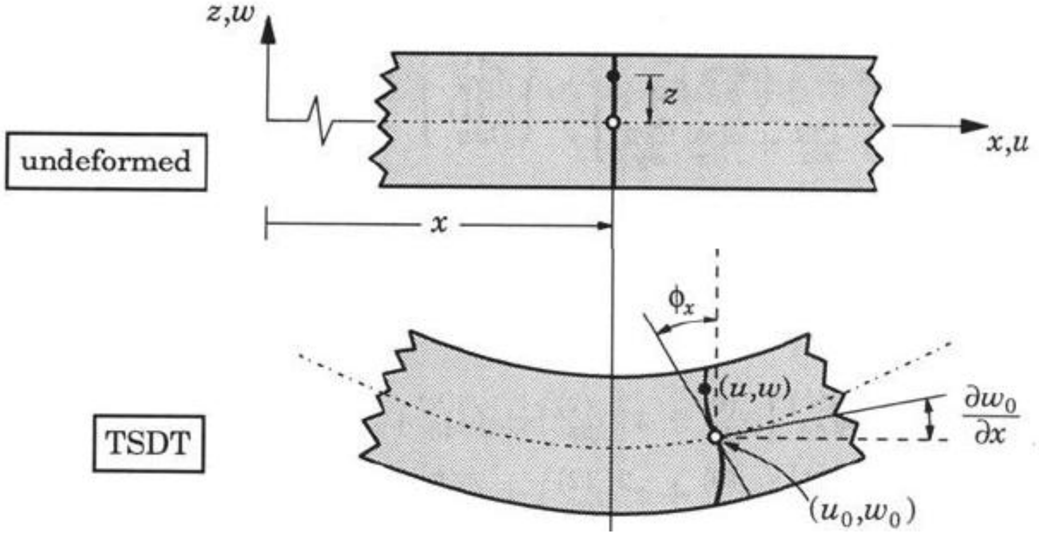

Figure (1): Un-deformed and Deformed cross section [10]

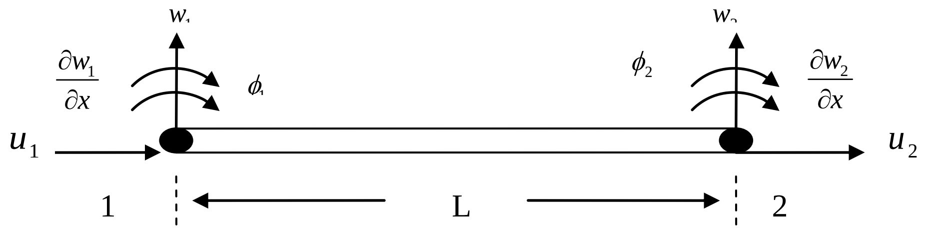

Figure (2): Degrees of freedom at each node of the beam element

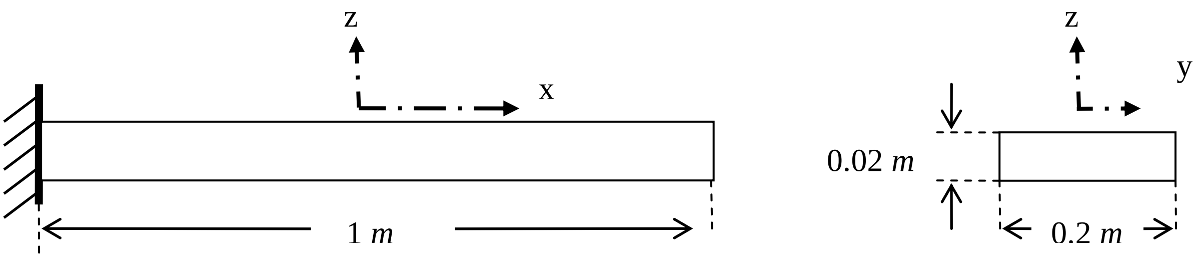

Figure (3): An isotropic cantilever beam

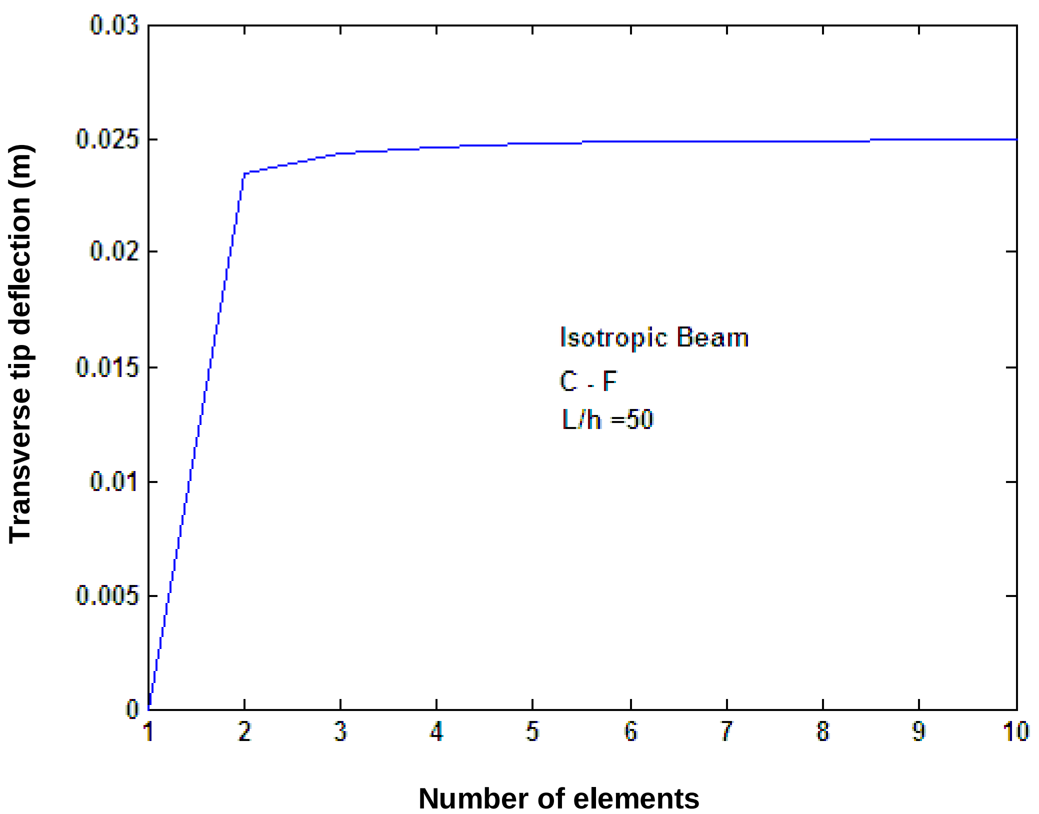

Figure (4): Effect of the number of elements on the transverse deflection

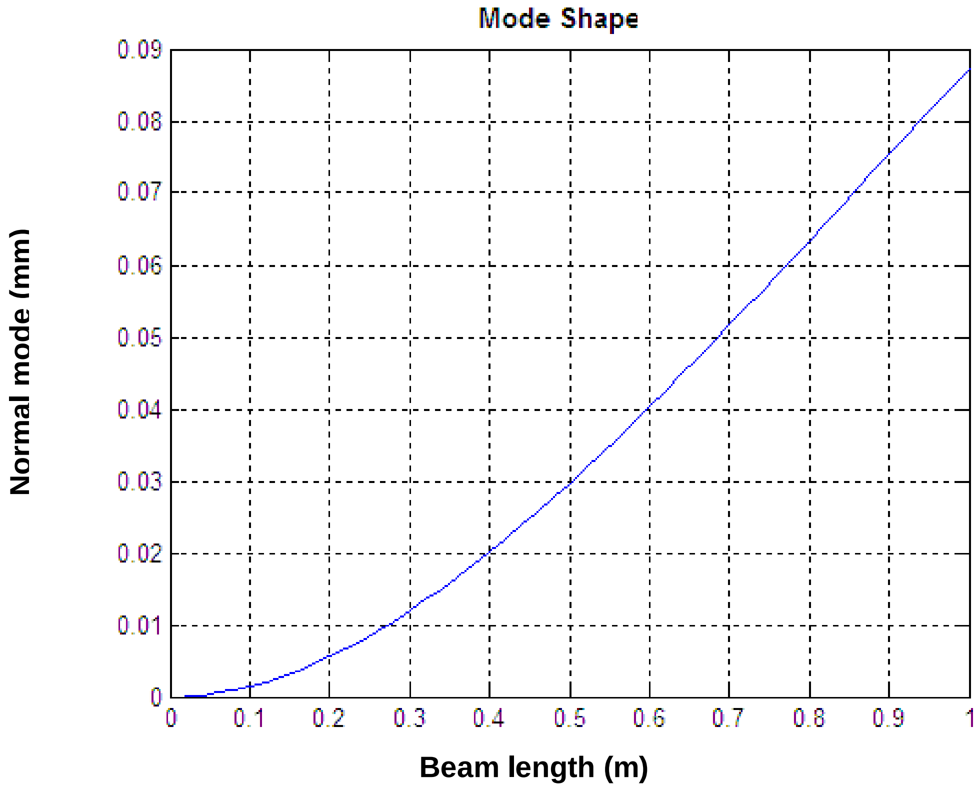

Figure (5) a: The first mode shape**.**

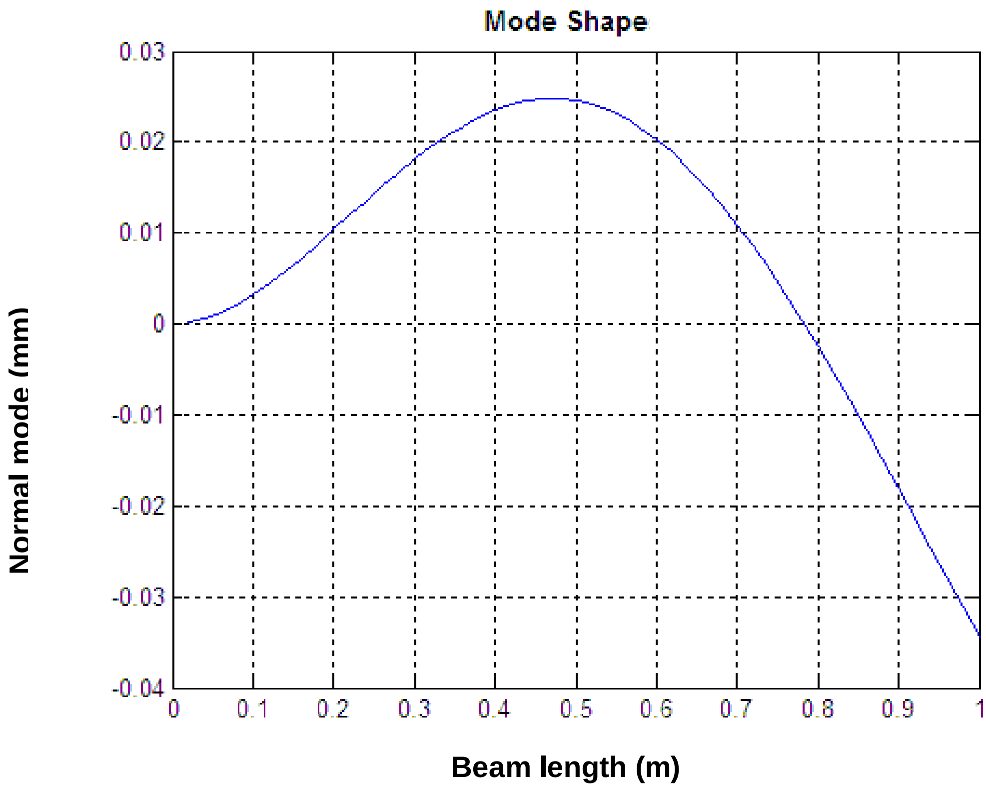

Figure (5) b: The second mode shape**.**

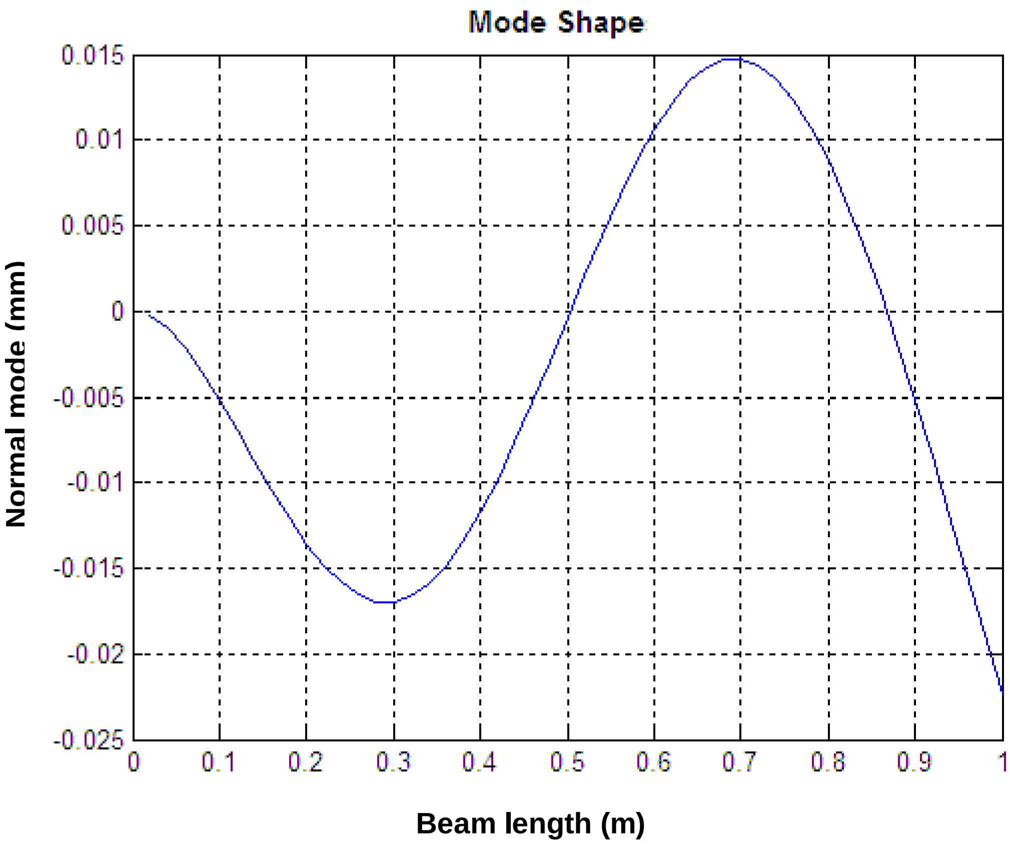

Figure (5) c: The third mode shape**.**

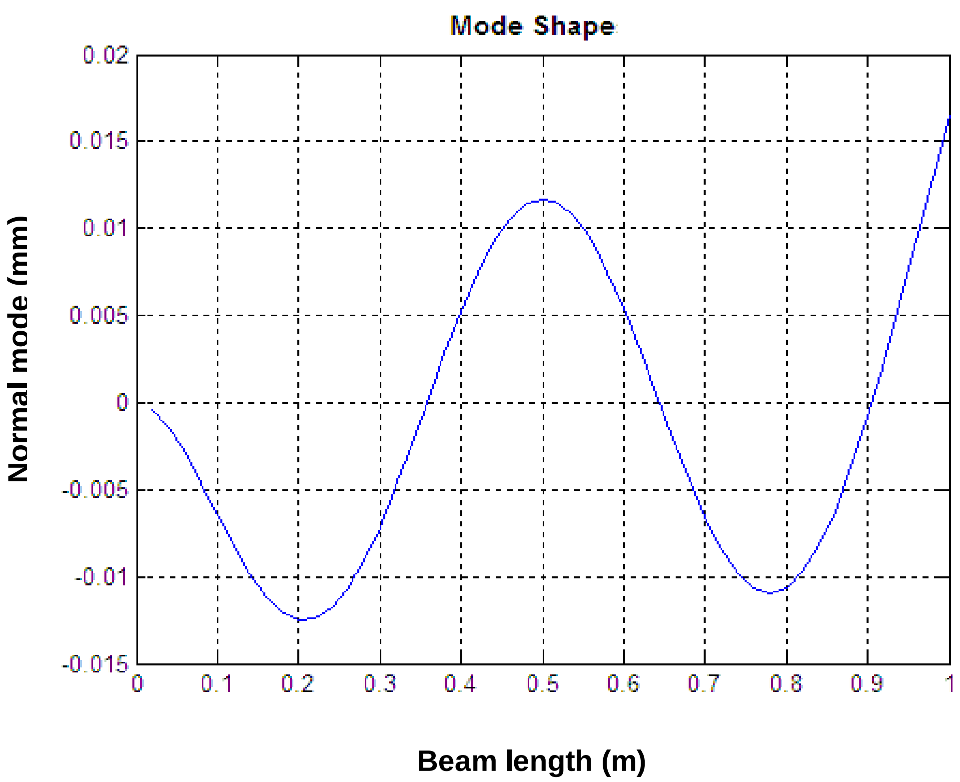

Figure (5) d: The fourth mode shape**.**

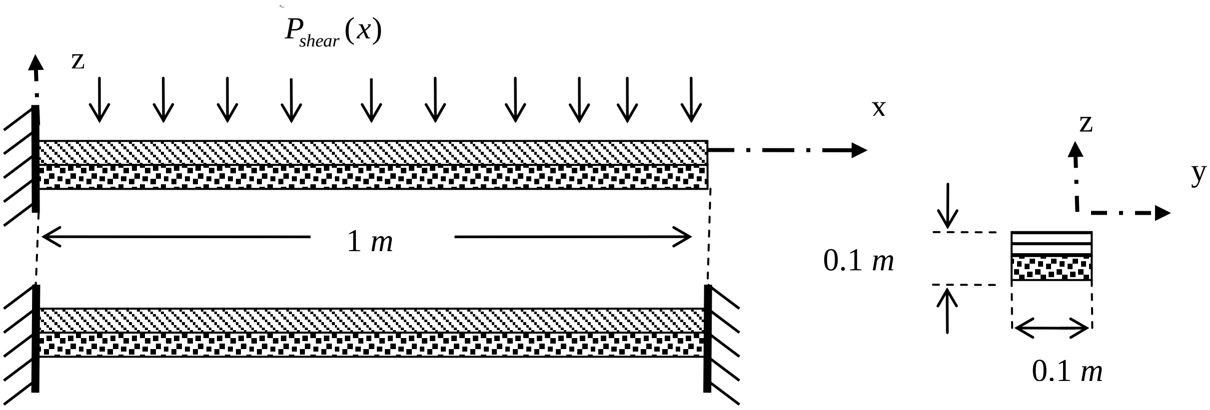

Figure (6): Geometric characteristics of C-F and C-C composite beams

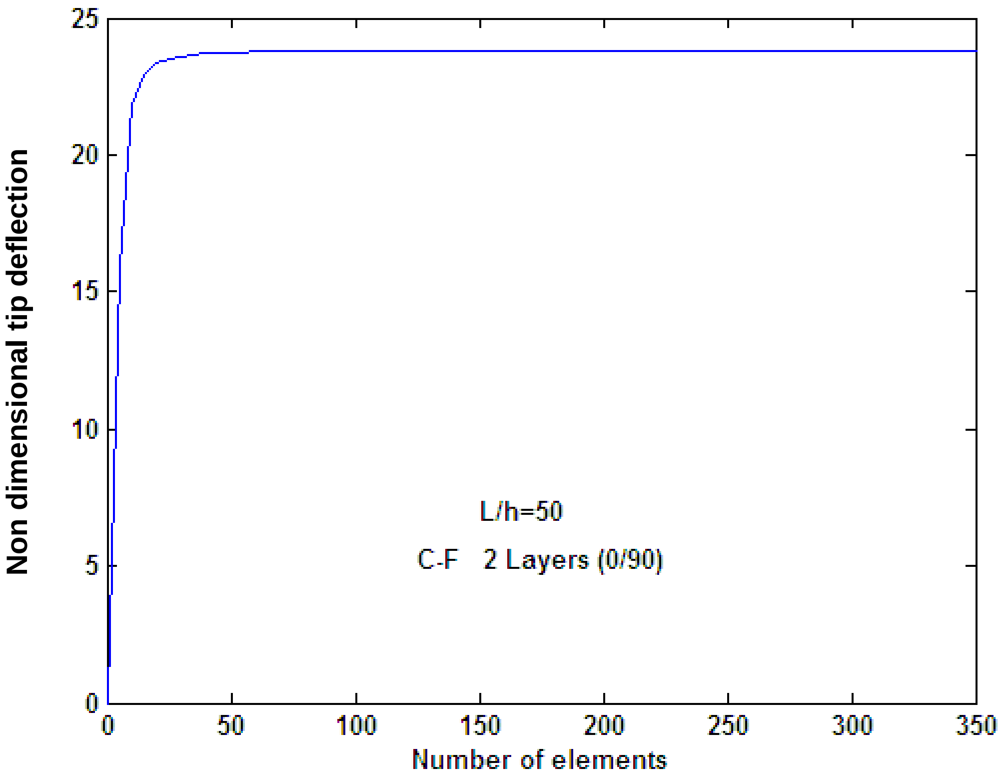

Figure (7) a: Effect of the number of element on tip deflection of clamped-free beam.

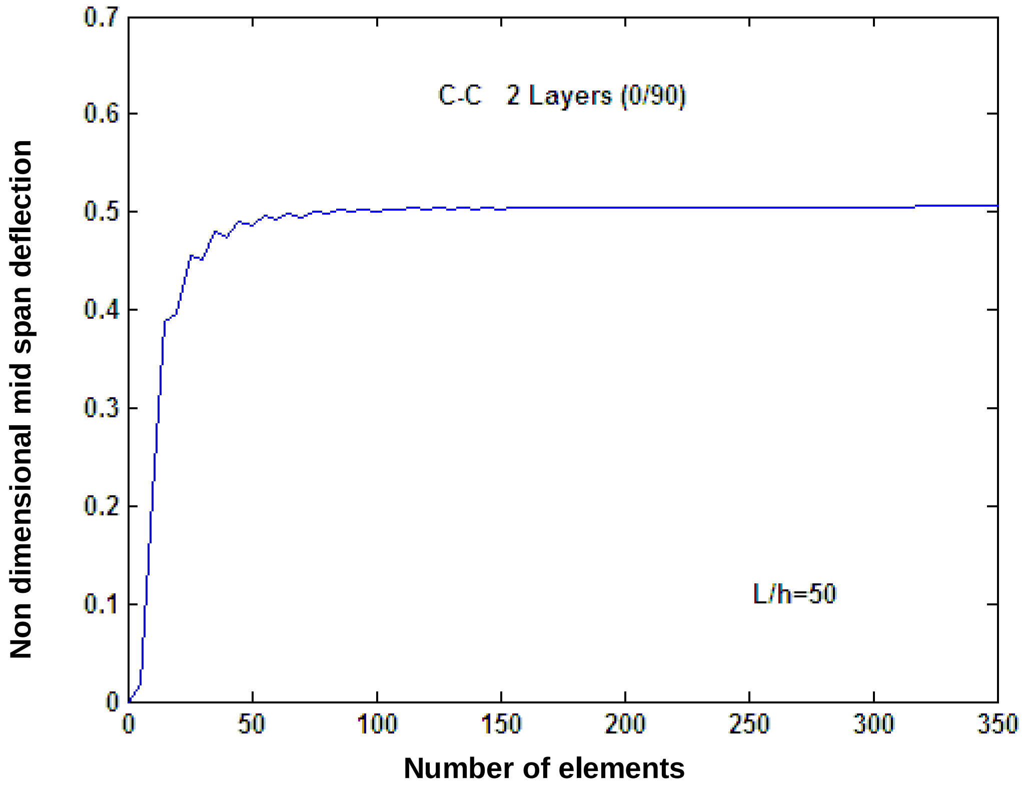

Figure (7) b: Effect of the number of element on mid span deflection of clamped-clamped beam.

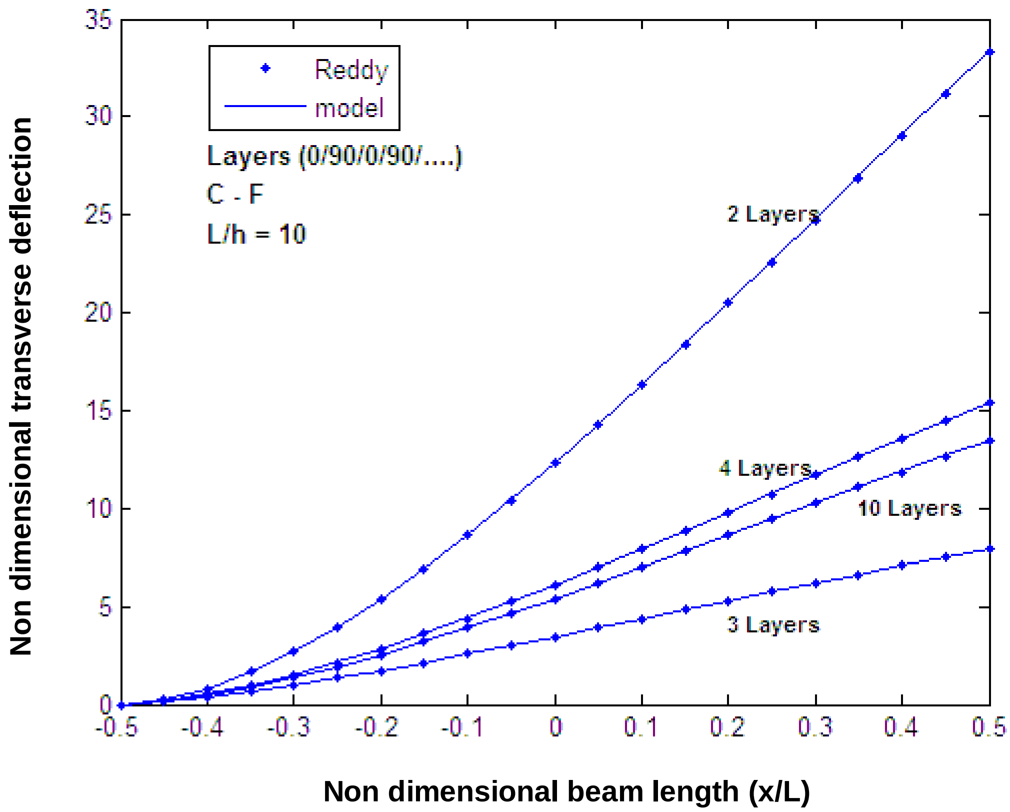

Figure (8) a: **Non dimensional** transverse deflection of cantilever composite beam.

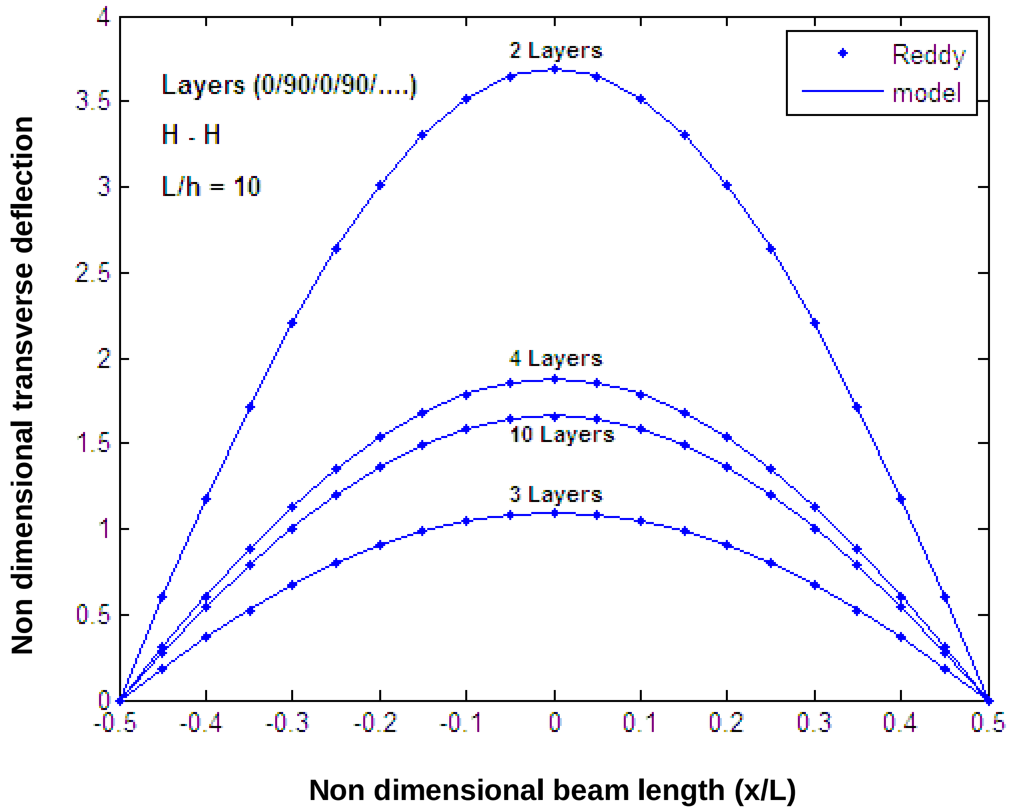

Figure (8) b: **Non dimensional** transverse deflection of hinged-hinged composite beam.

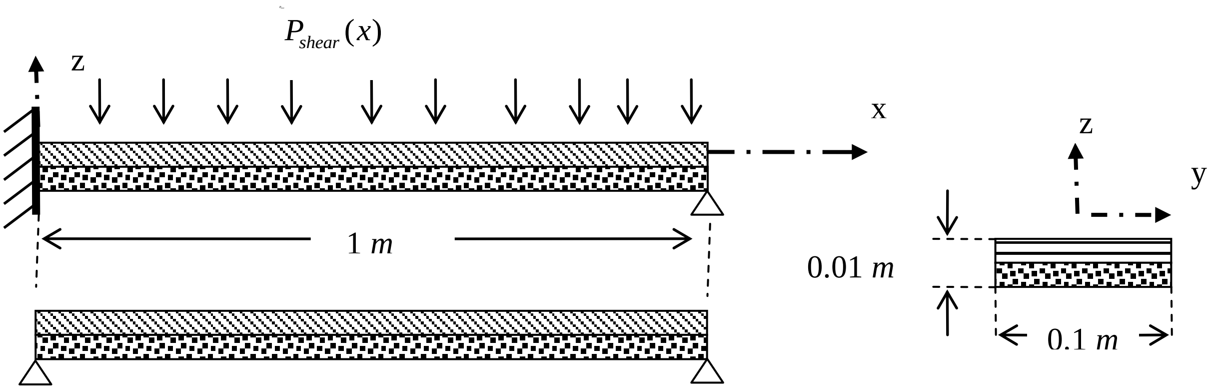

Figure (9): Geometry of C-H and H-H composite beams

Mode Shape

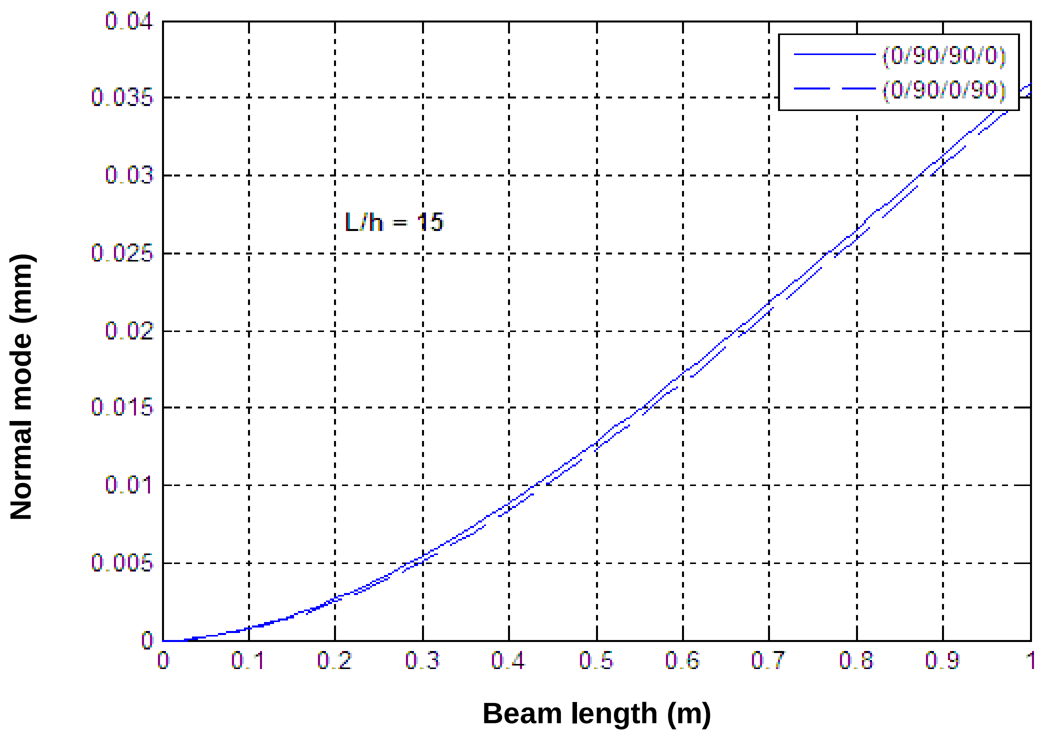

Figure (10) a: The first mode shape**.**

Mode Shape

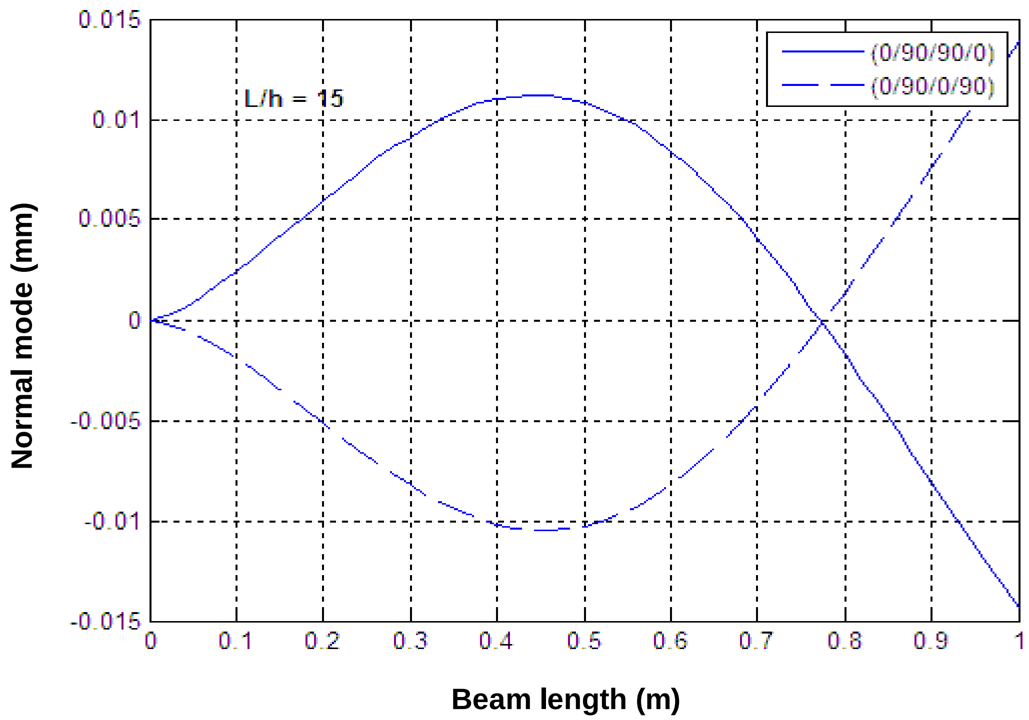

Figure (10) b: The second mode shape**.**

Mode Shape

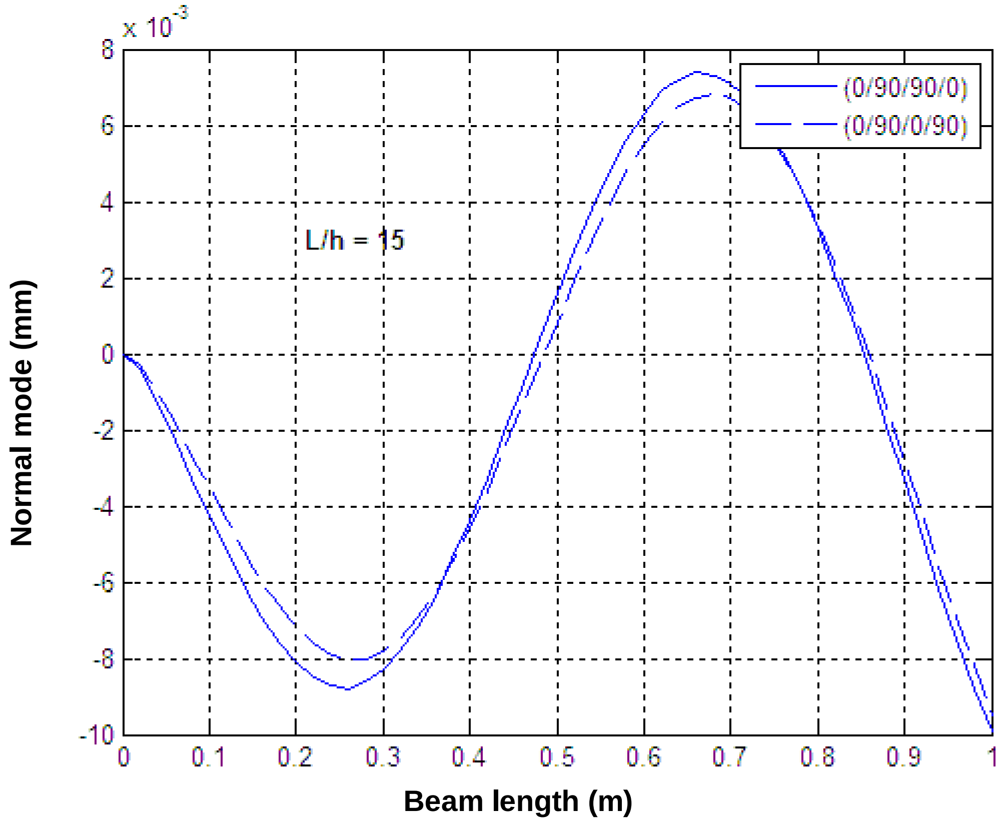

Figure (10) c: The third mode shape**.**

NOMENCLATURE

Symbols	**Definition**
$[m_e]$	Mass matrix of the beam element.
$[k_e]$	Element stiffness matrix.
$[F_e]$	Element nodal forces.
Π	Total potential energy.
q_e	Nodal displacement.
$\ddot{q}_e$	The second derivative of the nodal displacement.
U	Internal strain energy.
W	External work.
u, v and w	Displacements of a point in the x, y, &z directions.
$u_\circ, w_\circ$	Displacements at reference surface along x and y axes.
$\varepsilon_{xx}, \varepsilon_{yy}, and \varepsilon_{zz}$	Linear strains in the x, y, and z directions.
$\varepsilon_{\circ(x)}$	Reference surface extensional strain in the x-direction.
$\gamma_{\circ(xz)}$	The in-plane shear strains.
$\kappa_{\circ(x)}, \kappa_{2(x)}$	Reference surface curvatures in x-direction.
$\kappa_{2(xz)}$	Reference surface twisting curvature.
γ_{xy} , γ_{yz}	Transverse shear strains.
n	Number of layers in a laminate.

N	Total number of beam layers.
A	The beam surface area.
σ_{xx}	Stress in x-direction.
σ_{xz}	Shear stress in x-z plane.
E,G	Isotropic young's and shear modulus.
$c_1, \& c_2, c_3 \quad c_4$	Constant values.
$u_1 \ , u_2$	Displacements at the boundaries of the beam element.
$w_1 \ , w_3$	Transverse displacements.
$w_2 \ , w_4$	Slopes at the boundaries.
L	Length of a beam element.
b	Width of a beam element.
h	Height of a beam.
T	Kinetic energy.
ρ	Mass density of material.
ς_i	Axial displacement shape function.
ζ_i	Transverse displacement shape function.
ξ_i	Rotation displacement shape function.
P_{shear}	Transverse forces.
P_{axil}	Axial forces.
Q_{ij}	Components of the lamina stiffness matrix.
s_{ij}	Compliances of the lamina.
E_1, E_2	Young's modulus in the fiber and normal directions.
N_x	Force per unit length.
M_x	Moment per unit length.
A_{ij}	Extensional stiffness matrix element.
B_{ij}	Bending-extension coupling stiffness matrix element.
D_{ij}	Bending stiffness.
E_{ij}	Warping-extension coupling stiffness.
F_{ij}	Warping-bending coupling stiffness.
H_{ij}	Warping - higher order bending coupling stiffness.
c_{ijkl}	Elastic constants.
ω	Circular frequency of the system.
HOBT	Third-order Beam Theory.
SOBT	Second-order Beam Theory.
FOBT	First-order Beam Theory.
CBT	Classical Beam Theory.