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EFFECT OF THE LINK-LENGTH RATIO ON PAINLEVÉ PARADOX FOR A SLIDING TWO-LINK ROBOT

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ABSTRACT

For a rigid body sliding on a rough surface, a range of uncertainty or non-uniqueness of solution could be found, which is termed: Painlevé paradox. Painlevé paradox is the reason of a wide range of bouncing motion, observed during sliding of robotic manipulators on rough surfaces [1,2]. In this research work, the existence of the paradox zone during the sliding motion of a two-link manipulator is investigated. Parametric study is performed to investigate the effect of friction, link-length ratio, and link-mass ratio on the paradox zone.

KEY WORDS

Two-link robot, Painlevé paradox, friction, sliding robots, multibody system, unilateral constraint, and Dynamical system.

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NOMENCLATURE

F	Generalized contact forces expressed with respect to joint coordinates
F_n	Normal component of the contact force
F_t	Tangential component of the contact force
f	Vector of the generalized contact force
H	Vertical distance between the surface and the fixed pivot joint
J	Manipulator Jacobian, which depends on system orientation
l_i	Length of the i^{th} link of the manipulator
q	Vector of generalized joint coordinates
$\dot{\mathbf{q}}$	Vector of generalized joint velocity
u	Vector of Coriolis and centripetal forces
V	Potential energy
T	Kinetic energy
W	Generalized active forces
$\dot{x}$	Tangential (sliding) component of the velocity of the contact point
$\dot{y}$	Normal component of the velocity of the contact point.
$\boldsymbol{\alpha}, \boldsymbol{\beta}$	Partitioned vectors of the Jacobian
θ_i	Orientation angle of the i^{th} link of the manipulator. In the case study θ_i is chosen as the i^{th} generalized coordinate.
γ	Link-length ratio; $\gamma = \dfrac{l_2}{l_1}$
γ^*	Critical link-length ration; for $\gamma \geq \gamma^*$ the paradox zone disappear.
δ	Link-mass ratio; $\delta = m_2 / m_1$
σ	Sliding direction indicator; σ=1 if sliding to right and σ=-1 if sliding to left
μ	Coefficient of friction
μ^*	Critical coefficient of friction; for $\mu \leq \mu^*$ the paradox zone disappear
$\boldsymbol{\Phi}$	Inertia matrix
χ	Mass per unit length

INTRODUCTION

When a tip of a manipulator link slides against a rough surface, a bouncing motion could occur. If the center of mass trails the tip point, a region of uncertainty or non-uniqueness in the solution could exist. Because of its extraordinary importance, this phenomenon has recently drawn a substantial intension.

Génot and Brogliato [3] performed a detailed study of the Painlevé example, where a single rigid slender rod slides on a rough surface. They focused on the inconsistencies and the indeterminacies of the dynamics in the sliding regimes. Zhao et al. [1,2] proved experimentally that Painlevé paradox is behind a wide range of bouncing motion observed during sliding of robotic manipulators on rough surfaces. Their experiment used a two-link robotic system that comes in contact with a rough moving belt. Analytically, tangential impact is used to solve the paradox. Numerical simulation was carried out and the experimental results were in good agreements with the numerical results. An [4] suggested eliminating the paradox by considering an elastic deformation in the contact zone. A new dynamical system is obtained when replacing the rigid contact constraint by an elastic one. He also discussed the case of tangential impact. Yu et al. studied the motion control of a two link manipulator actuated at the second joint only. He derived a first integral of the dynamical system through which the free and forced motions are analyzed [5]. Elkaranshawy [6] focused on reducing the degrading effects of Painlevé paradox on robotic manipulators during their functional operation. He used the self-motion, inherited in redundant manipulators, to escape the paradox zone or to extend the span of the motion of the contact point before interring to the paradox zone.

In this paper, the Painlevé paradox for a general two link manipulator is studied. The effect of the length and mass ratios of the first link to the second one are emphasized. In the second section the equation of motion of a two link manipulator with different lengths and different masses is derived. In the third section the phenomena known as Painlevé paradox is introduced. In the fourth section the effect of the system parameters on the paradox region is considered. Finally, conclusion is drawn in the last section.

EQUATION OF MOTION

The equation of motion of two-link robot during sliding contact is [1,2]:

$$\mathbf{\Phi}\ddot{\mathbf{q}} + \mathbf{W}(\dot{\mathbf{q}},\mathbf{q}) = \mathbf{u} + \mathbf{F} \qquad (1)$$

where

q: vector of generalized joint coordinates $\mathbf{q} \in R^{2\times1}$

Φ(q): inertia matrix , $\mathbf{\Phi}(.) \in R^{2\times2}$

$\mathbf{u}(\dot{\mathbf{q}},\mathbf{q})$: vector of Coriolis and centripetal forces, $\mathbf{u}(.,.) \in R^{2\times1}$

W: generalized active forces, $\mathbf{W} \in R^{2\times1}$

F: generalized contact forces expressed with respect to joint coordinates, $\mathbf{F} \in R^{2\times1}$.

F is related to the components of the contact force at the contact point through the relation:

$$\mathbf{F} = \mathbf{J}^T\mathbf{f} \quad \text{with} \quad \mathbf{f} = \begin{Bmatrix} F_n \\ F_t \end{Bmatrix} \qquad (2)$$

where

f: the contact force with respect to normal and tangential coordinates, $\mathbf{f} \in R^{2\times1}$

F_n: the normal component of the contact force

F_t: the tangential component of the contact force

J(q): the manipulator Jacobian, which depends on system orientation, $\mathbf{J}(.) \in R^{2\times2}$. The Jacobian can be written as:

$$\mathbf{J} = \begin{Bmatrix} \boldsymbol{\alpha}^T \\ \boldsymbol{\beta}^T \end{Bmatrix} \tag{3}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}$: the partitioned vectors of $\mathbf{J}^T$, $\boldsymbol{\alpha} \in R^{2\times1}$ and $\boldsymbol{\beta} \in R^{2\times1}$

In sliding motion, F_t can be expressed in terms of F_n, the friction coefficient μ, and the constant σ, as follows [4]:

$$F_t = -\mu\sigma F_n \tag{3a}$$

with

$$\sigma = \frac{\dot{x}}{\|\dot{x}\|} \tag{3b}$$

where

σ: sliding direction indicator; σ=1 if sliding to right and σ=-1 if sliding to left

$\dot{x}$: tangential (sliding) component of the velocity of the contact point

Consequently, Eq. (1) can be written as:

$$\ddot{\mathbf{q}} = \boldsymbol{\varepsilon} + F_n\boldsymbol{\delta} \tag{4a}$$

with

$$\boldsymbol{\varepsilon} = \boldsymbol{\Phi}^{-1}(\mathbf{W} - \mathbf{u}) \tag{4b}$$

and

$$\boldsymbol{\delta} = \boldsymbol{\Phi}^{-1}(\boldsymbol{\alpha} - \mu\sigma\boldsymbol{\beta}) \tag{4c}$$

PAINLEVÉ PARADOX

The velocity of the contact point is given as:

$$\mathbf{v} = \mathbf{J}\dot{\mathbf{q}} \quad \text{with} \quad \mathbf{v} = \begin{Bmatrix} \dot{y} \\ \dot{x} \end{Bmatrix} \tag{5}$$

where

$\dot{y}$: normal component of the velocity of the contact point. It follows that:

$$\dot{y} = \boldsymbol{\alpha}^T\dot{\mathbf{q}} \tag{6}$$

Time differentiation of Eq.(6) combined with substitution of Eq.(4) leads to:

$$\ddot{y} = -A + BF_n \tag{7a}$$

where

$$A = -\dot{\boldsymbol{\alpha}}^T\dot{\mathbf{q}} - \boldsymbol{\alpha}^T\boldsymbol{\varepsilon} \tag{7b}$$

and

$$B = \boldsymbol{\alpha}^T\boldsymbol{\delta} \tag{7c}$$

The normal component of the acceleration and the normal component of the contact force, in Eq. (7), should satisfy the following condition to continue the sliding [3]:

$$\ddot{y} = 0 \text{ and } F_N \geq 0 \tag{8}$$

If the contact tip leaves the surface, then [3]:

$$\ddot{y} \geq 0 \text{ and } F_N = 0 \tag{9}$$

According to the signs of *A* and *B*, the states of solution give n in Table 1 arise. Hence, if *B* is positive a unique solution can be found and a range of inconsistency or non-uniqueness of solution could be found if *B<0*. This phenomenon is called Painlevé paradox. Therefore, it is ultimately important to carefully look into the sign of B. It is worth to notice from Eqs. 4c and 7c that *B* does not depend on applied loads or motion velocities. It only depends upon system configuration (orientation: $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and mass properties: Φ), sliding direction (σ), and coefficient of friction (μ). Hence, for any specific coefficient of friction, a paradoxical configuration can be specified.

CASE STUDY

The two degrees of freedom manipulator shown in Fig. 1, consists of two uniform links connected by a revolute joint. The first link has a length l_1 and mass m_1 and the second link has a length l_2 and mass m_2. The free end of the manipulator slides on a rough surface with coefficient of friction μ. The vertical distance between the surface and the fixed pivot joint is *H*. Torques τ_1 and τ_2 are applied at the two joints and the orientation angles $\theta_1$and θ_2 are the chosen two generalized coordinates.

The vector of generalized joint coordinates for this manipulator is

$$\mathbf{q} = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

the kinetic energy is:

$$T = \frac{1}{2}(\frac{1}{3}m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{6}m_2 l_2^2\dot{\theta}_2^2 + \frac{1}{2}m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)$$

and the potential energy is:

$$V = \tau_1(\frac{\pi}{2} - \theta_1) + \tau_2(\theta_1 - \theta_2) - \frac{1}{2}m_1 g l_1 \cos\theta_1 - m_2 g(l_1\cos\theta_1 + \frac{1}{2}l_2\cos\theta_2)$$

Consequently, using Euler-Lagrange model to derive the equations of motion, the matrices in Eq.(1) are obtained as follow:

$$\Phi = \begin{bmatrix} (\frac{1}{3}m_1 + m_2)l_1^2 & \frac{1}{2}m_2 l_1 l_2\cos(\theta_1 - \theta_2) \\ \frac{1}{2}m_2 l_1 l_2\cos(\theta_1 - \theta_2) & \frac{1}{3}m_2 l_2^2 \end{bmatrix},\ \mathbf{u} = \begin{bmatrix} \frac{1}{2}m_2 l_1 l_2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) \\ -\frac{1}{2}m_2 l_1 l_2\dot{\theta}_1^2\sin(\theta_1 - \theta_2) \end{bmatrix} \text{ and}$$

$$\mathbf{W} = \begin{bmatrix} \tau_1 - \tau_2 - \frac{1}{2} g l_1 \sin\theta_1 (m_1 + 2m_2) \\ \tau_2 - \frac{1}{2} m_2 g l_2 \sin\theta_1 \end{bmatrix}$$

Kinematics of the manipulator gives:

$$\boldsymbol{\alpha} = \begin{Bmatrix} l_1 \sin\theta_1 \\ l_2 \sin\theta_2 \end{Bmatrix} \text{ and } \boldsymbol{\beta} = \begin{Bmatrix} l_1 \cos\theta_1 \\ l_2 \cos\theta_2 \end{Bmatrix}$$

Painlevé paradox occurs only when the center of mass trails the contact point. Hence, it could occur for the forward motion to the right i.e. σ=1. Since **α** and **β** are functions of θ_1 and θ_2 the substitution in Eqs.(4 and 7) leads to expression for $B(\theta_1,\theta_2)$ as:

$$B(\theta_1,\theta_2) = \frac{l_1}{6f}(\sin\theta_1 - \mu\cos\theta_1)(2h_1 - 3h_2) - \frac{l_2}{2f}(\sin\theta_2 - \mu\cos\theta_2)(h_3 - 2h_4)$$

where

$$f = l_1^2 l_2^2 (\frac{1}{9} m_1 m_2 + \frac{1}{3} m_2^2 - \frac{1}{4} m_2^2 \cos(\theta_1 - \theta_2)),$$

$$h_1 = m_2 l_1 l_2^2 \sin\theta_1, \qquad h_2 = m_2 l_1 l_2^2 \sin\theta_2 \cos(\theta_1 - \theta_2),$$

$$h_3 = m_2 l_1^2 l_2 \sin\theta_1 \cos(\theta_1 - \theta_2), \quad h_4 = l_1^2 l_2 \sin\theta_2 (\frac{m_1}{3} + m_2)$$

Due to the unilateral constraint, the following relation between θ_1 and θ_2 is applied

$$l_1 \cos(\theta_1) + l_2 \cos(\theta_2) = H \tag{10}$$

Consequently, θ_2 can be eliminated in the expression of B which becomes a function of θ_1 only. Putting $B=0$ leads to $B(\theta_1^1) = B(\theta_1^2) = 0$ with $\forall\theta_1 \in [0,\theta_1^1) \cup (\theta_1^2,\theta_1^F), B(\theta_1) > 0$ the case is out from Painlevé paradox i.e. the solution is consistent, and $\forall\theta_1 \in (\theta_1^1,\theta_1^2), B(\theta_1) < 0$ Painlevé paradox is found. θ_1^F is the maximum possible value for θ_1 to keep the touching between the robot end and the rough surface. It has to be noticed that these results are applicable when *σ=1* i.e. the motion is to right. Typically similar results can be obtained when the motion is to left i.e. *σ=-1* but with negative signs for the angles. Therefore, the vertical line passing through the pivot joint works as a mirror in that regard.

SIMULATION AND DISCUSSION

To investigate the effect of friction, link-length ratio, and link-mass ratio on the dynamical behaviour of the system, a numerical experiment is conducted.

In the first bunch of tests, the following numerical values are assigned to the robot parameters: $\mu = 0.6$, $m_1 = m_2 = 0.12\ kg$, $l_1 = 0.21\ m$, and $H = 0.3775\ m$. The following range of link-length ratio $\gamma = \frac{l_2}{l_1}$ is considered: $0.7976 \leq \gamma \leq 1.7976$. Figure 2 shows the variation of *B* with respect to the angular displacement θ_1 for the specified range of link-length ratio. For each curve the motion starts with $\theta_1 = 0$ and ends when $\theta_1 = \theta_1^F$. Angles θ_1^1 and θ_1^2, which specify the paradox zone, are indicated by the intersection of each curve with the horizontal straight line; *B=0*. It is clear that when γ increases the paradox region, *B<0*, decreases. For $\gamma \geq \gamma^*; \gamma^* = 1.2435$ the paradox region totally disappears. Increasing length ratio also increases the corresponding maximum possible angle θ_1^F.

In the second bunch of tests, same numerical values of H, μ, m_1, and $l_1 = l_2 = 0.21\, m$ are used and the effect of the link-mass ratio $\delta = m_2 / m_1$ is investigated, see Fig. 3. The paradox zone decreases slightly by increasing the mass ratio. However, if there is a paradox zone corresponding to the specified length ratio, the variation of mass ratio does not eliminate that zone. It can also be noticed that, the curves become closer to the horizontal line $B = 0$ with the increase of mass ratio. Thus, at any angular displacement θ_1 the higher mass ratio makes the absolute value of B smaller and the trade-off of using the mass ratio to reduce the paradox region will be moving the whole working conditions of the system closer to the boundary of the paradox.

In the third bunch of tests, the mass per unit length is considered constant i.e. $m_1 = \chi l_1$, $m_2 = \chi l_2$ and $\chi = cont.$. In this case the mass ratio equals to length ratio. For $\chi = 0.5714$, $\mu = 0.6$, $m_1 = 0.12\, kg$, $l_1 = 0.21\, m$, and $H = 0.3775\, m$, Fig.4 shows, as expected, that the overall behavior of the relation between B and θ_1 is the same as first case, see Fig.2, except that in the current case $\gamma^* = 1.2365$.

For two identical links with $m = 0.12\, kg$, $l = 0.21\, m$, and $H = 0.3775\, m$, Fig. 5 shows a family of curves represent the relation between the angular position of the first link θ_1 and the coefficient of friction μ at different values of B. For any given value of θ_1, there is a minimum value for the coefficient of friction to enter the paradox region. For any specified system configuration there is a value μ^* represents the global minimum of the coefficient of friction below which the paradox zone totally disappears. For the current system configuration this value is $\mu^* = 0.3650$.

When $\mu = 0.6$, $m = 0.12\, kg$, $l = 0.21\, m$, Fig. 6 shows the relation between θ_1 and *B* at different values of (H/l). Obviously, the paradox region increases at higher values of the ratio H/l, and totally disappears when $(\mathrm{H}/l) < (\mathrm{H}/l)^*$, where $(\mathrm{H}/l)^* = 1.515$.

It might be interesting to redraw the same relation using a higher value for the coefficient of friction. A value of $\mu = 4/3$ is used in Fig. 7, hence, all the curves have their lower parts below the horizontal line $B = 0$ and the paradox zone exists despite the value of (H/l).

In the described numerical experiment, the effect of system parameters on the paradox zone has been investigated. Specifically, link-length ratio, link-mass ratio, the coefficient of friction, and the height to length ratio (*H/l*) have been considered. The paradox zone decreases with the increase of the link-length ratio and link-mass ratio but no critical ratio can be specified to eliminate the paradox. Though the effect of mass ratio is inconsequential, the effect of length ratio is significant; if its value is greater than a critical ratio the paradox zone disappears. When the mass per unit length is considered constant, the length and mass ratios are the same and the change in the ratio is more significant. Friction also has a crucial rule in Painlevé paradox. A minimum value for the friction coefficient can be specified to produce the paradox. The height to link-length ratio is also critical; decreasing this ratio decreases the range for the paradox. Depending on the value of the friction coefficient, a critical value could be specified to eliminate the paradox zone.

In general, these parameters can be used to reduce or eliminate the Painlevé paradox effect in the two link robotic system. Though the coefficient of friction can be used to avoid the paradox, it might be practically difficult to change this coefficient. Hence the most practical and effective choice is to increase the link-length ratio to decrease the paradox zone or even to eliminate it.

CONCLUSION

In this research work, the equation of motion of a two link robot has been constructed and the condition leading to Painlevé paradox has been derived. A numerical experiment has been conducted to investigate the effect of link-length ratio, link-mass ratio, friction, and height to length ratio on the dynamical behaviour of the system. Different scenarios are possible through the dynamical system parameters. The paradox zone decreases with the increase of the length ratio and mass ratio. In the other hand it decreases with the decrease of the coefficient of friction and the height to length ratio. Generally, critical values, to eliminate the paradox, can be specified to length ratio, friction coefficient, and height to length ratio. Practically the link-length ratio can be regulated by making the length of the second link adjustable.

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Table 1. States of solution.

	A	*B*	solution	state of the solution
1	+	+	$F_N = \frac{A}{B}$	sliding
2	-	+	$F_N = 0$	flying
3	+	-	ϕ	Inconsistency (no solution)
4	-	-	$F_N = 0$ and $F_N = \frac{A}{B}$	Indeterminacy (non-uniqueness)

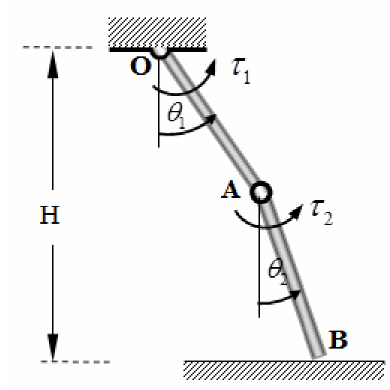

Fig. 1. Two Degrees of Freedom Manipulator.

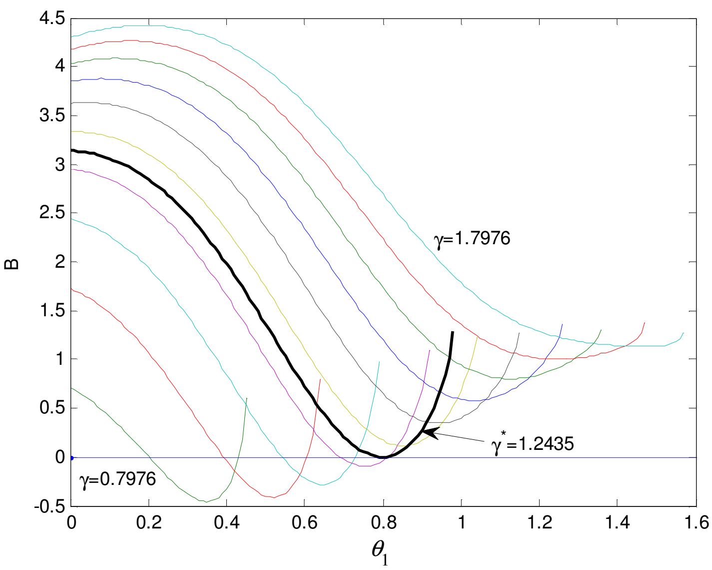

Fig. 2. The Angular Position θ_1 vs *B* for a Range of Values of $\gamma = l_2 / l_1$.

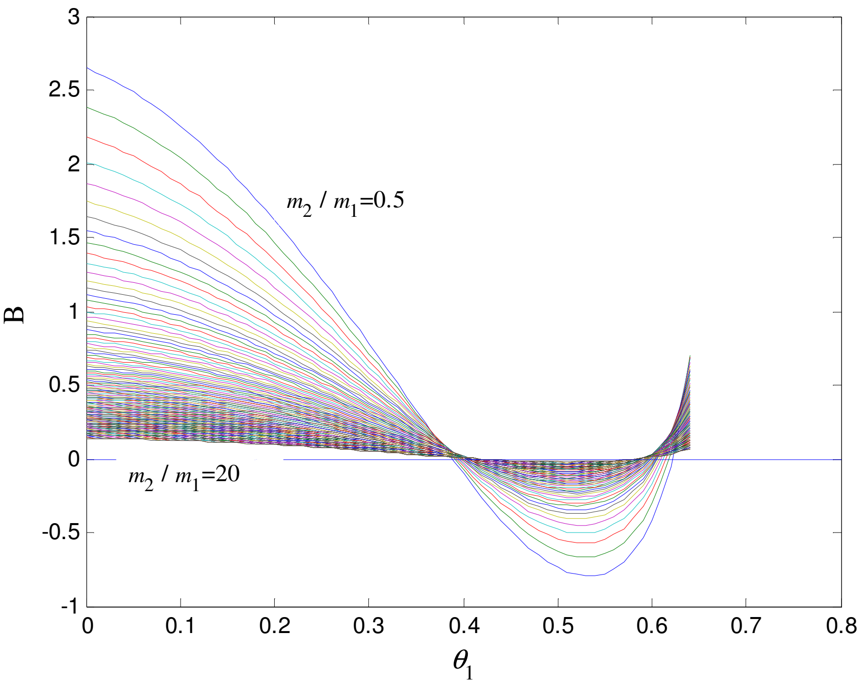

Fig. 3. The Angular Position θ_1 vs *B* for a Range of Values of $\delta = m_2 / m_1$.

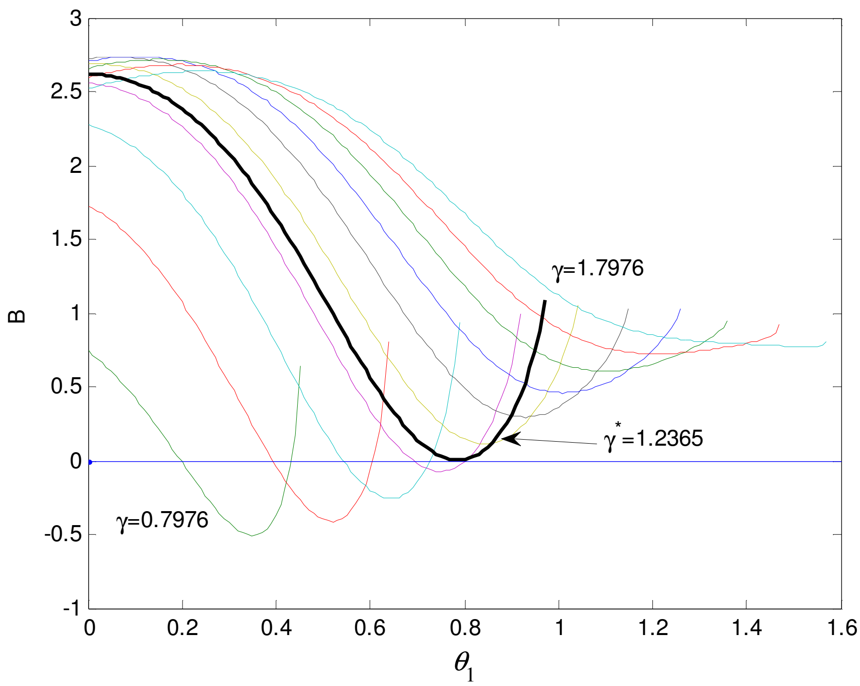

Fig. 4. The Angular Position θ_1 vs *B* for a Range of Values of $\gamma = l_2 / l_1$ considering the mass density $\chi = 0.5714$.

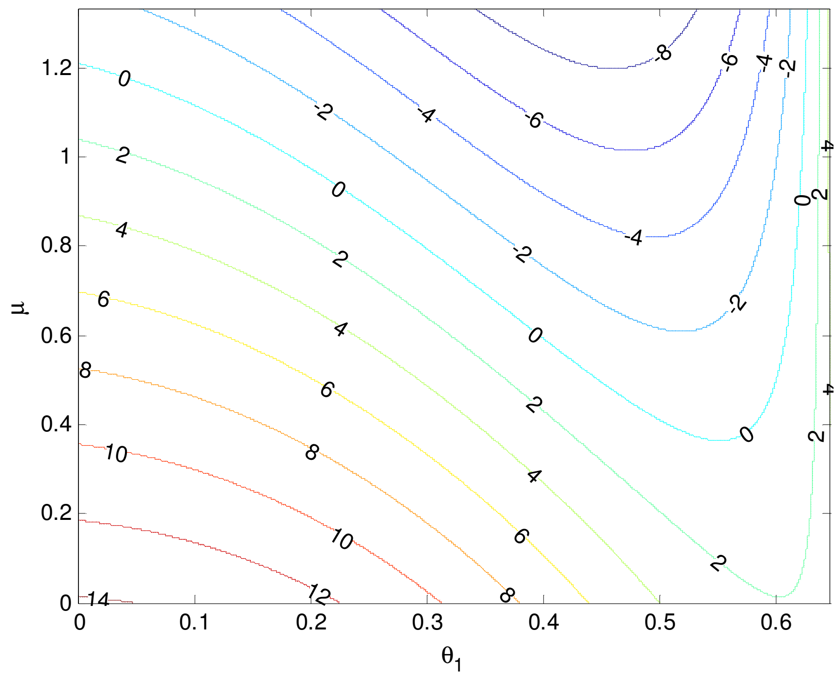

Fig. 5. The Coefficient of Friction vs Angular Displacement.

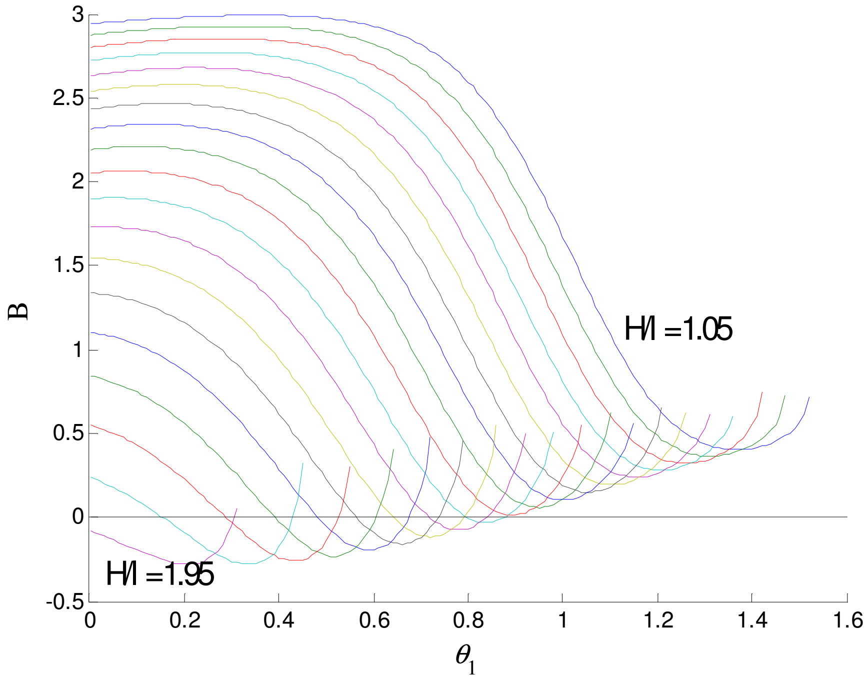

Fig. 6. The Angular Position θ_1 vs *B* for a Range of Values of H/l $(\mu = 0.6)$.

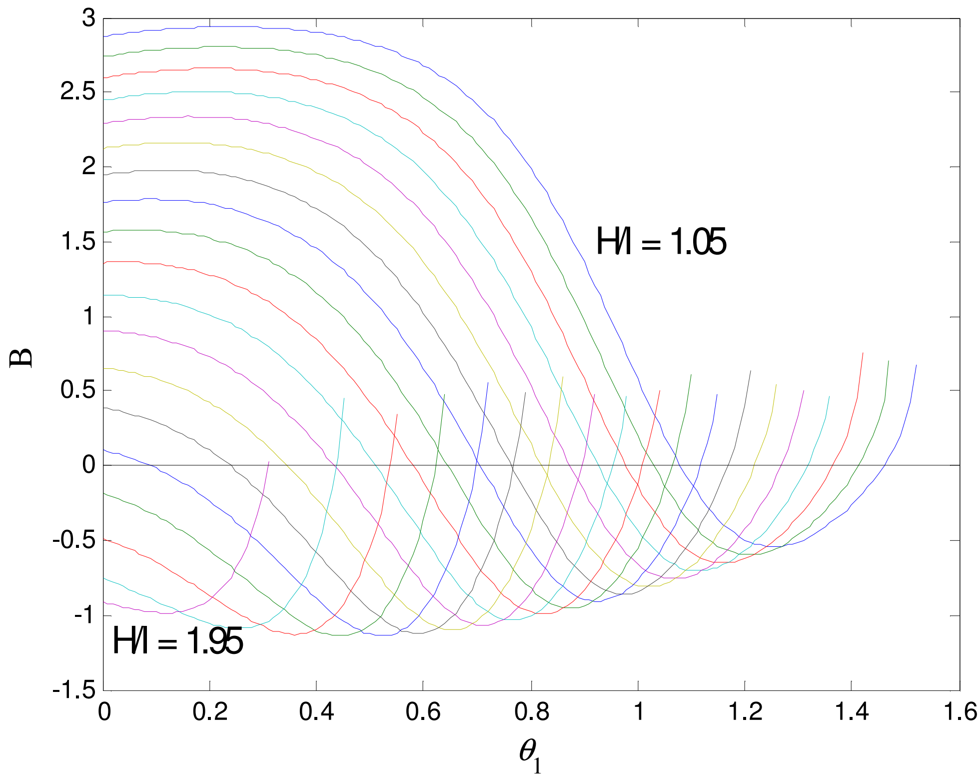

Fig. 7. The Angular Position θ_1 vs *B* for a Range of Values of H/l $(\mu = 4/3)$.