Prediction for Inverted Topp-Leone Distribution Based on Constant Stress-Partially Accelerated Life Testing

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Abstract

Prediction of future observations on the basis of the past and present information is a fundamental problem of statistics, arising in many contexts and producing varied solutions. The predictor can be either a point or an interval predictor. This paper focuses on predicting the future observations from the inverted Topp-Leone distribution for constant stress-partially accelerated life test based on Type II censored samples. The two-sample prediction is applied to obtain the conditional maximum likelihood, Bayesian and E-Bayesian prediction (point and interval) for future order statistics. The Bayes and E-Bayes predictors are considered under two different loss functions, the balanced squared error loss function; as a symmetric loss function and balanced linear exponential loss function; as an asymmetric loss function. The predictors are obtained based on gamma prior and uniform hyperprior distributions. A numerical example is provided to illustrate the theoretical results and an application using real data sets are used to demonstrate how the results can be used in practice.

Keywords: *Inverted Topp-Leone distribution; balanced loss functions; two-sample prediction; maximum likelihood, Bayesian and E-Bayesian prediction.*

1. Introduction

Rapid developments, improvements of the high technology, consumer's demands for highly reliable products and competitive markets have placed pressure on manufacturers to deliver products with high quality and reliability. In life testing, it is very difficult to estimate

the time of failure for modern high reliability products such as electronics, power cables, metal fatigue, insulating materials, laser, airplane parts, aerospace vehicles, etc.. Therefore, these types of products are not likely to fail under usual operating conditions in the relatively short time available for test. For this reason, *accelerated life testing* (ALT) or *partially accelerated life testing* (PALT) are preferred to be used in manufacturing industries to obtain enough failure data in a short period of time and necessary to study its relationship with external stress variables. Such testing could save much time, man power, material sources and money. The stress can be applied in different ways like constant stress, step stress and progressive stress among others [see Nelson (1990)].
For more details about ALT [see Bai and Chung (1989), Balakrishnan and Han (2008), AL-Dayian *et al.* (2014), Basak and Balakrishnan (2018) and Kumar *et al*. (2021)] among others.

In ALT the main assumption is that a life-stress relationship is known or can be assumed so that the data obtained from accelerated conditions can be extrapolated to usual conditions. In some cases, such relationship cannot be known or assumed so PALT are often used in such cases.

In a *constant stress-PALT* (CS-PALT) each test item is run at a constant stress under either usual use condition or accelerated condition only until the test is terminated and the analysis of PALT has been extensively studied in recent years [see Bai *et al.* (1993), Hyun and Lee (2015), EL-Sagheer (2018) and AL-Dayian *et al.* (2021)].

Han (2007) introduced the *expected Bayesian* (E-Bayesian) estimation method which is very simple and it is a special Bayesian method used in the area related for the life testing of products with high reliability, with small sample size or censored data. It is more popular now. Many researchers applied the E-Bayesian method to many distributions, such as, Yin and Liu (2010), Jaheen and Okasha (2011), Azimi *et al.* (2013), Reyad and Ahmed (2015), EL-Sagheer (2017), Han (2020) and Rabie and Li (2020). Also, few studies have considered E-Bayesian method assuming PALT such as Rabie (2021).

The general problem of prediction may be described as that of inferring the values of unknown observables (future observations; known as future sample) or functions of such variables, from current available observations; known as informative sample. Prediction has been applied in a variety of disciplines such as medicine (medical prognosis, antibiotic assays and pre-operative medical diagnosis), engineering (mechanical tool replacements, quality control and maximization of the yield of an industrial process), business (determining the difference in future mean performance of a specified number of systems), economic and other areas as well.

Prediction for order statistics of future observables from certain distributions has been studied by several authors, such as, Valiollahi *et al.* (2017) who obtained the *maximum likelihood* (ML) and Bayesian prediction (point and interval) of a future observation based on Type I, Type II and hybrid censored samples when the lifetime distribution of the experimental units is assumed to be a generalized exponential random variable. The one-sample, two-sample prediction and intervals of the future samples under Bayesian paradigm of a weighted exponential distribution under Type II progressive censoring were introduced by Dey *et al*. (2018). Also, Faizan and Sana (2018) considered prediction intervals for future observations of the two unknown parameters of Chen distribution based on upper record value. The one-sample Bayesian prediction and intervals of the generalized half-normal distribution under progressive Type II censoring were studied by Abd El-Raheem (2019). Arshad and Jamal (2019) predicted future record values using Bayesian approach of the Topp-Leone family of distributions. Okasha *et al.* (2020) derived the Bayesian and E-Bayesian prediction (point and interval) based on observed order statistics with two samples from two parameter Burr XII model based on Type II censored data. Moreover, they obtained the predictors under symmetric and asymmetric loss functions assuming gamma informative prior density. AL-Dayian *et al.* (2021) applied the two-sample prediction method to obtain the conditional ML, Bayesian and E-Bayesian prediction (point and interval) for future order statistics of the modified Topp Leone-Chen distribution based on progressive Type II censored samples.

Few studies have considered prediction assuming PALT such as Abushal and AL-Zaydi (2017), Prakash and Singh (2018), Behariy *et al.* (2019), AL-Dayian *et al.* (2021) and Lone *et al.* (2022).

This paper focuses on predicting the future observations from the *inverted Topp-Leone* (ITL) distribution for CS-PALT based on Type II censored samples. In Section 2, the ITL distribution, basic assumptions, model description and *balanced loss function* (BLF) are presented. In Section 3, the conditional ML, Bayesian and E-Bayesian prediction (point and interval) for a future observation of the ITL distribution for CS-PALT based on two-sample prediction are obtained. In Section 4, a numerical example is given to illustrate the theoretical results and an application using real data sets are used to demonstrate how the results can be used in practice. Finally, general conclusion is presented in Section 5.

2. Inverted Topp-Leone Distribution, Basic Assumptions, Model Description and Balanced Loss Function

2.1 Inverted Topp-Leone distribution

The inverted distributions have a great importance due to their applicability in many areas such as biological sciences, life test problems, medical, etc. Hassan *et al.* (2020) introduced the ITL distribution and obtained some statistical properties of the proposed distribution such as quantile function, mode, moments, probability weighted moments, incomplete moments, stress-strength model, moments of residual life function and Rényi entropy. They derived the ML estimator based on complete, Type I and Type II censored samples for the distribution parameter.

The *probability density function* (pdf) and *cumulative distribution function* (cdf) of the ITL distribution are, respectively, given by

$$f(x;\lambda) = 2\lambda x(1+x)^{-2\lambda-1}(1+2x)^{\lambda-1}, \qquad x>0;\ \lambda>0, \tag{1}$$

and

$$F(x;\lambda) = 1-\left\{\frac{(1+2x)^{\lambda}}{(1+x)^{2\lambda}}\right\}, \qquad x>0;\ \lambda>0, \tag{2}$$

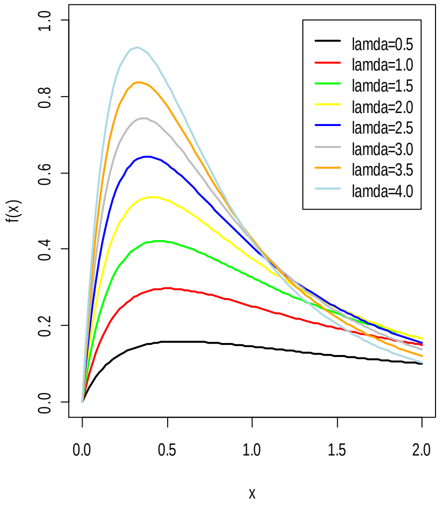
where λ is a shape parameter.

The *reliability function* (rf) and *hazard rate function* (hrf) of the ITL distribution are, respectively, given by

$$R(x;\lambda) = \frac{(1+2x)^{\lambda}}{(1+x)^{2\lambda}}, \qquad x > 0;\ \lambda > 0, \qquad (3)$$

and

$$h(x;\lambda) = 2\lambda x[(1+x)(1+2x)]^{-1}, \qquad x > 0;\ \lambda > 0. \qquad (4)$$



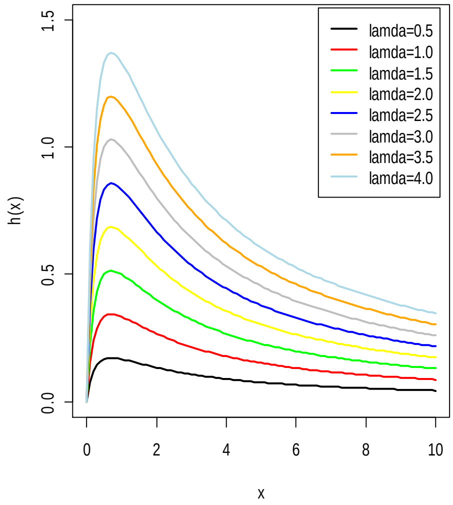


Figure 1. Different shapes for the pdf **Figure 2. Different shapes for the hrf**

One can see that the plots of the hrf of the ITL distribution are positive skewed, so the ITL distribution is a flexible reliability model and it is suitable for studying PALT model.

2.2 Basic assumptions and model description

Total items are divided randomly into two samples of size $n(1-\tau)$ and $n\tau$, respectively, where τ is the sample proportion. The first sample is allocated to usual conditions and the other is assigned to accelerated conditions. Each test item of every sample is run without changing the test condition until reaching the censoring number.

- **Assumptions**

1. The lifetimes X_i, i = 1, …, $n(1-\tau)$ of items allocated to usual conditions are *independent and identically distributed* (i.i.d) random variables.
2. The lifetimes Y_j, $j = 1, \dots, n\tau$ of items allocated to accelerated conditions are i.i.d random variables.
3. The lifetimes X_i and Y_j are mutually statistically independent.

In this study, the lifetimes of test items are assumed to have the ITL distribution. The pdf of an item at usual conditions is given by (1).

The pdf and cdf for an item tested at accelerated conditions are given by:

$$f(y;\lambda,\beta) = 2\lambda\beta y(1+\beta y)^{-2\lambda-1}(1+2\beta y)^{\lambda-1}, \qquad y>0;\ \lambda>0,\ \beta>1, \quad (5)$$

and

$$F(y;\lambda,\beta) = 1-\left\{\frac{(1+2\beta y)^{\lambda}}{(1+\beta y)^{2\lambda}}\right\}, \qquad y>0;\ \lambda>0,\ \beta>1, \quad (6)$$

where $Y=\beta^{-1}X$, β is the acceleration factor which is the ratio of the mean life at usual condition to that at accelerated condition and $\beta>1.$

The rf and hrf for an item tested at accelerated conditions are as follows:

$$R(y;\lambda,\beta) = \frac{(1+2\beta y)^{\lambda}}{(1+\beta y)^{2\lambda}}, \qquad y>0;\ \lambda>0,\ \beta>1, \quad (7)$$

and

$$h(y;\lambda,\beta) = 2\lambda\beta y[(1+\beta y)(1+2\beta y)]^{-1}, \quad y>0;\ \lambda>0,\ \beta>1. \quad (8)$$

2.3 Balanced loss function

Bayes estimator is an estimator that minimizes the posterior expected value of a loss function (i.e., the posterior expected loss). Equivalently, it maximizes the posterior expectation of a utility function. Loss function separated into two groups symmetric and asymmetric loss function. There are many types of symmetric and asymmetric loss function.

Ahmadi *et al.* (2009) suggested the use of the BLF which was originated by Zellner (1994), to be of the form

$$L^{*}(\theta,\theta^{*}) = \omega\, l(\theta_0,\ \theta^{*}) + (1-\omega)\, l(\theta,\theta^{*}), \tag{9}$$

where $l(\theta,\theta^{*})$ is an arbitrary loss function, θ_0 is a chosen target estimator of θ^{*} and the weight $\omega\epsilon[0,1]$. The BLF specializes to various choices of loss functions such as the absolute error loss, entropy, *linear exponential* (LINEX) and *squared error loss* (SEL) functions. The estimator of a function using BLF is a mixture of the ML estimator, least squares estimators or any other estimator and the Bayes estimator using any loss function.

The Bayes estimator of θ, using the *balanced SEL* (BSEL) function is given by

$$\tilde{\theta}_{\mathrm{BSE}} = \omega\, \hat{\theta}_{\mathrm{ML}} + (1-\omega)\, \tilde{\theta}_{\mathrm{SE}}, \tag{10}$$

where $\hat{\theta}_{\mathrm{ML}}$ is the ML estimator of θ and $\tilde{\theta}_{\mathrm{SE}}$ is its Bayes estimator using SEL function. Also, the Bayes estimator using the *balanced LINEX loss* (BLL) function of θ is obtained as follows:

$$\tilde{\theta}_{BL} = \frac{-1}{v}\ \ln\{\omega\, exp(-v\hat{\theta}_{ML}) + (1-\omega)\ E(exp(-v\theta)\,|\underline{x})\}, \tag{11}$$

where $v \neq 0$ is the shape parameter of BLL function.

3. Prediction for Inverted Topp-Leone Distribution

In this section, the conditional ML, Bayesian and E-Bayesian prediction (point and interval) for a future observation $T_{(s)}$, of the ITL distribution for CS-PALT based on Type II censored data under two-sample prediction method are derived.

Considering the failure times consist of r^{th} smallest lifetimes $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(r)}$ out of a random sample of $n(1-\tau)$ lifetimes $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{[n(1-\tau)]}$ under usual conditions and $y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(r)}$ out of a random sample of $n\tau$ lifetimes $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n\tau)}$ at accelerated conditions, respectively.

The *likelihood function* (LF) for $\{(x_{(i)}): i = 1, \dots, n(1-\tau)\}$ at usual conditions is given by

$$L_u(\lambda|\underline{x}) \propto [\prod_{i=1}^{r} f(x_{(i)}|\lambda)][R(x_{(r)}|\lambda)]^{n(1-\tau)-r}, \quad (12)$$

where $\underline{x} = x_{(1)}, x_{(2)}, \dots, x_{[n(1-\tau)-r]}$, $f(x_{(i)}|\lambda)$ and $R(x_{(r)}|\lambda)$ are given by (1) and (3), respectively.

The LF for $\{(y_{(j)}): j = 1, \dots, n\tau\}$ at accelerated conditions is given by

$$L_a(\underline{\theta}|\underline{y}) \propto [\prod_{j=1}^{r} f(y_{(j)}|\underline{\theta})][R(y_{(r)}|\underline{\theta})]^{n\tau-r}, \quad (13)$$

where $\underline{y} = y_{(1)}, y_{(2)}, \dots, y_{(n\tau-r)}$, $\underline{\theta} = (\lambda, \beta)'$, $f(y_{(j)}|\underline{\theta})$ and $R(y_{(r)}|\underline{\theta})$ are given by (5) and (7), respectively.

Let m_u and m_a be the number of censored items at usual and accelerated conditions, respectively, where

$$m_u = n(1-\tau) - r \quad \text{and} \quad m_a = n\tau - r. \quad (14)$$

Substituting (1) and (3) into (12) and substituting (5) and (7) into (13), hence the LF according to CS-PALT for $\{(x_{(i)}), (y_{(j)}): i = 1, \dots, n(1-\tau),\ j = 1, \dots, n\tau\}$ can be written as:

$$L\left(\underline{\theta}|\underline{x}, \underline{y}\right) \propto (2\lambda)^{2r} \beta^r \{\prod_{i=1}^{r} x_{(i)}(1+x_{(i)})^{-2\lambda-1}(1+2x_{(i)})^{\lambda-1}\}$$

$$\times \left\{\prod_{j=1}^{r} y_{(j)}(1+\beta y_{(j)})^{-2\lambda-1}(1+2\beta y_{(j)})^{\lambda-1}\right\}$$

$$\times\left[\frac{(1+2x_{(r)})^{\lambda}}{(1+x_{(r)})^{2\lambda}}\right]^{m_u}\left[\frac{(1+2\beta y_{(r)})^{\lambda}}{(1+\beta y_{(r)})^{2\lambda}}\right]^{m_a}. \tag{15}$$

The natural logarithm of LF in (15) is given by

$$\ell\propto 2r\log(\lambda)+r\log(\beta)+\sum_{i=1}^{r}\log[x_{(i)}]-(2\lambda+1)\sum_{i=1}^{r}\log[1+x_{(i)}]$$
$$+(\lambda-1)\sum_{i=1}^{r}\log[1+2x_{(i)}]+m_u\log\left[\frac{(1+2x_{(r)})^{\lambda}}{(1+x_{(r)})^{2\lambda}}\right]+\sum_{j=1}^{r}\log[y_{(j)}]$$
$$-(2\lambda+1)\sum_{j=1}^{r}\log[1+\beta y_{(j)}]+(\lambda-1)\sum_{j=1}^{r}\log[1+2\beta y_{(j)}]$$
$$+m_a\log\left[\frac{(1+2\beta y_{(r)})^{\lambda}}{(1+\beta y_{(r)})^{2\lambda}}\right]. \tag{16}$$

The ML estimators of the parameters, $\underline{\theta}=(\lambda,\beta)'$, can be obtained by differentiating (16) with respect to λ,β and then setting to zero. Hence

$$\frac{\partial\ell}{\partial\lambda}=\frac{2r}{\lambda}-2\sum_{i=1}^{r}\log[1+x_{(i)}]+\sum_{i=1}^{r}\log[1+2x_{(i)}]$$
$$-2\sum_{j=1}^{r}\log[1+\beta y_{(j)}]+\sum_{j=1}^{r}\log[1+2\beta y_{(j)}]$$
$$+m_u\log\left[\frac{(1+2x_{(r)})}{(1+x_{(r)})^{2}}\right]+m_a\log\left[\frac{(1+2\beta y_{(r)})}{(1+\beta y_{(r)})^{2}}\right], \tag{17}$$

and

$$\frac{\partial\ell}{\partial\beta}=\frac{r}{\beta}-(2\lambda+1)\sum_{j=1}^{r}\frac{y_{(j)}}{(1+\beta y_{(j)})}+(\lambda-1)\sum_{j=1}^{r}\frac{2y_{(j)}}{(1+2\beta y_{(j)})}$$
$$-\frac{2\lambda\beta\, m_a y_{(r)}^{2}}{(1+\beta y_{(r)})(1+2\beta y_{(r)})}. \tag{18}$$

The ML estimators are obtained by equating the derivatives (17) and (18) to zeros. The system of non-linear equations can be solved numerically using Newton-Raphson method to obtain the ML estimates of the parameters λ and β.

Considering the prior knowledge of the vector of parameters, $\underline{\theta}=(\lambda,\beta)'$, is adequately represented by informative prior which is gamma distribution with parameters a_j and b_j and pdf as follows:

$$\pi(\theta_j; a_j, b_j) = \frac{b_j^{a_j}}{\Gamma(a_j)} \theta_j^{a_j-1} exp(-b_j\theta_j), \quad (a_j, b_j > 0), \quad j = 1,2, \qquad (19)$$

where $\theta_1 = \lambda$ and $\theta_2 = \beta$, a_j and b_j are the hyper-parameters of the prior distribution.

Assuming that the parameters, $\underline{\theta} = (\lambda, \beta)'$, are unknown and independent. Then the joint prior distribution of the unknown parameters has a joint pdf given by

$$\pi(\underline{\theta};\ \underline{a}, \underline{b}) \propto \lambda^{a_1-1} \beta^{a_2-1} exp[-(b_1\lambda + b_2\beta)],$$
$$\lambda > 0, \beta > 1; (\underline{a}, \underline{b} > \underline{0}). \qquad (20)$$

Combining the LF in (15) and the joint prior distribution given by (20), then the joint posterior distribution of the parameters, $\underline{\theta} = (\lambda, \beta)'$, can be obtained as follows:

$$\pi\left(\underline{\theta}\middle|\underline{x}, \underline{y}\right) \propto L\left(\underline{\theta}\middle|\underline{x}, \underline{y}\right) \pi(\underline{\theta};\ \underline{a}, \underline{b})$$
$$\propto \lambda^{2r+a_1-1}\beta^{r+a_2-1} exp[-(b_1\lambda + b_2\beta)]$$
$$\times \left\{\prod_{i=1}^{r}(1 + x_{(i)})^{-2\lambda}(1 + 2x_{(i)})^{\lambda}\right\}\left[\frac{(1+2x_{(r)})^{\lambda}}{(1+x_{(r)})^{2\lambda}}\right]^{m_u}$$
$$\times \left\{\prod_{j=1}^{r}(1 + \beta y_{(j)})^{-2\lambda-1}(1 + 2\beta y_{(j)})^{\lambda-1}\right\}\left[\frac{(1+2\beta y_{(r)})^{\lambda}}{(1+\beta y_{(r)})^{2\lambda}}\right]^{m_a}. \qquad (21)$$

The joint posterior distribution given by (21) can be written as follows:

$$\pi\left(\underline{\theta}\middle|\underline{x}, \underline{y}\right) = A\ \lambda^{2r+a_1-1}\beta^{r+a_2-1} exp[-(b_1\lambda + b_2\beta)]$$
$$\times \left\{\prod_{i=1}^{r}(1 + x_{(i)})^{-2\lambda}(1 + 2x_{(i)})^{\lambda}\right\}\left[\frac{(1+2x_{(r)})^{\lambda}}{(1+x_{(r)})^{2\lambda}}\right]^{m_u}$$
$$\times \left\{\prod_{j=1}^{r}(1 + \beta y_{(j)})^{-2\lambda-1}(1 + 2\beta y_{(j)})^{\lambda-1}\right\}\left[\frac{(1+2\beta y_{(r)})^{\lambda}}{(1+\beta y_{(r)})^{2\lambda}}\right]^{m_a},$$

(22)

where $A^{-1} = \int_{\underline{\theta}} \lambda^{2r+a_1-1}\beta^{r+a_2-1}\, exp[-(b_1\lambda + b_2\beta)]$

$$\times\left\{\prod_{i=1}^{r}\left(1+x_{(i)}\right)^{-2\lambda}\left(1+2x_{(i)}\right)^{\lambda}\right\}\left[\frac{\left(1+2x_{(r)}\right)^{\lambda}}{\left(1+x_{(r)}\right)^{2\lambda}}\right]^{m_u}$$
$$\times\left\{\prod_{j=1}^{r}\left(1+\beta y_{(j)}\right)^{-2\lambda-1}\left(1+2\beta y_{(j)}\right)^{\lambda-1}\right\}$$
$$\times\left[\frac{\left(1+2\beta y_{(r)}\right)^{\lambda}}{\left(1+\beta y_{(r)}\right)^{2\lambda}}\right]^{m_a} d\underline{\theta}, \tag{23}$$

and

$\int_{\underline{\theta}} = \int_{\lambda}\int_{\beta}$ and $d\underline{\theta} = d\beta\, d\lambda$. (24)

Considering that $Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(r)}$ are the first r ordered life times in a random sample of n components (Type II censoring) whose failure times are i.i.d as random variables $Y's$ having the pdf for an item tested at accelerated conditions is given by (5) which is an informative sample and that $T_{(1)}, T_{(2)}, \dots, T_{(k)}$ is a future independent random sample (of size k) from the same distribution. Our aim is to predict a statistic in the future sample based on the informative sample.

For the future sample of size k, let $T_{(s)}$ denotes the s^{th} order statistic, $1 \le s \le k$. The conditional density function of $T_{(s)}$, given the vector of the parameters $\underline{\theta}$, is given by

$$h(t_{(s)}|\underline{\theta}) = D(s)f(t_{(s)}|\underline{\theta})[F(t_{(s)}|\underline{\theta})]^{s-1}[1-F(t_{(s)}|\underline{\theta})]^{k-s},$$
$$t_{(s)} > 0, \tag{25}$$

where s is the order statistic of the predicted future observation in the future sample,

$$D(s) = s\binom{k}{s} = \frac{k!}{(s-1)!(k-s)!} = \frac{1}{B(s,\ k-s+1)} \text{ and } s = 1,2,3,\dots,k. \tag{26}$$

Using the binomial expansion theorem for $[1-F(t_{(s)}|\underline{\theta})]^{k-s}$, yields

$$h\left(t_{(s)}\middle|\underline{\theta}\right) = D(s)f\left(t_{(s)}\middle|\underline{\theta}\right)\sum_{l_1=0}^{k-s}(-1)^{l_1}\binom{k-s}{l_1}\left[F\left(t_{(s)}\middle|\underline{\theta}\right)\right]^{s+l_1-1}. \quad (27)$$

Substituting (5) and (6) into (27), then one can obtain the pdf of s^{th} order statistic for an item tested at accelerated conditions:

$$h\left(t_{(s)}\middle|\underline{\theta}\right) = D(s)\sum_{l_1=0}^{k-s}\sum_{l_2=0}^{s+l_1-1}\varphi_{l_1,l_2}(\lambda,\beta)\ t_{(s)}\left[1+\beta t_{(s)}\right]^{-2\lambda(l_2+1)-1}$$

$$\times\left[1+2\beta t_{(s)}\right]^{\lambda(l_2+1)-1}, \qquad t_{(s)} > 0;\ \lambda > 0, \beta > 1, \quad (28)$$

where

$$\varphi_{l_1,l_2}(\lambda,\beta) = 2\lambda\beta(-1)^{l_1+l_2}\binom{k-s}{l_1}\binom{s+l_1-1}{l_2}. \quad (29)$$

3.1 Conditional maximum likelihood prediction

In this subsection, the conditional ML prediction (point and interval) for a future observation $T_{(s)}$, of the ITL distribution for CS-PALT based on two-sample prediction are derived.

Assuming that the parameters $\underline{\theta}$ are unknown and independent, then the conditional *ML prediction density* (MLPD) of $T_{(s)}$ given $\underline{\hat{\theta}}_{ML} = (\hat{\lambda},\hat{\beta})'$ can be obtained using the conditional pdf of the s^{th} order statistic which is given by (28) after replacing the vector of parameters $\underline{\theta}$ by their conditional ML estimators $\underline{\hat{\theta}}_{ML}$ as follows:

$$h_1\left(t_{(s)};\underline{\hat{\theta}}_{ML}\right) = D(s)\sum_{l_1=0}^{k-s}\sum_{l_2=0}^{s+l_1-1}\varphi_{l_1,l_2}(\hat{\lambda},\hat{\beta})\ t_{(s)}$$

$$\times\left[1+\hat{\beta}t_{(s)}\right]^{-2\hat{\lambda}(l_2+1)-1}\left[1+2\hat{\beta}t_{(s)}\right]^{\hat{\lambda}(l_2+1)-1},$$

$$t_{(s)} > 0; (\hat{\lambda} > 0,\ \hat{\beta} > 1), \quad (30)$$

where $D(s)$ and $\varphi_{l_1,l_2}(\lambda,\beta)$ are given by (26) and (29), respectively.

3.1.1 Point prediction

The conditional *ML predictor* (MLP) for the future observation $T_{(s)}$, based on Type II censoring can be derived using (30) as follows:

$$\begin{aligned}\hat{t}_{(s)(ML)} &= E\left(t_{(s)};\hat{\underline{\theta}}_{ML}\right)\\ &= \int_{t_{(s)}} t_{(s)}\, h_1\left(t_{(s)};\hat{\underline{\theta}}_{ML}\right) dt_{(s)}\\ &= D(s) \sum_{l_1=0}^{k-s} \sum_{l_2=0}^{s+l_1-1} \varphi_{l_1,l_2}(\hat{\lambda},\hat{\beta}) \int_0^{\infty} t_{(s)}^2 \left[1+\hat{\beta}t_{(s)}\right]^{-2\hat{\lambda}(l_2+1)-1}\\ &\times \left[1+2\hat{\beta}t_{(s)}\right]^{\hat{\lambda}(l_2+1)-1} dt_{(s)}. \qquad (31)\end{aligned}$$

3.1.2 Interval prediction

A 100(1- α)% conditional *ML predictive bounds* (MLPB) for the future observation $T_{(s)}$, such that $\left(L_{(s)}(\underline{y}) < T_{(s)} < U_{(s)}(\underline{y})\middle|\underline{y}\right) = 1-\alpha,$ are

$$P\left(T_{(s)} > L_{(s)}(\underline{y})\middle|\underline{y}\right) = \int_{L_{(s)}(\underline{y})}^{\infty} h_1\left(t_{(s)};\hat{\underline{\theta}}_{ML}\right) dt_{(s)} = 1-\frac{\alpha}{2}, \qquad (32)$$

and

$$P\left(T_{(s)} > U_{(s)}(\underline{y})\middle|\underline{y}\right) = \int_{U_{(s)}(\underline{y})}^{\infty} h_1\left(t_{(s)};\hat{\underline{\theta}}_{ML}\right) dt_{(s)} = \frac{\alpha}{2}. \qquad (33)$$

Substituting (30) in (32) and (33), then the conditional MLPB are obtained as follows:

$$\begin{aligned}P\left(T_{(s)} > L_{(s)}(\underline{y})\middle|\underline{y}\right) &= D(s) \sum_{l_1=0}^{k-s} \sum_{l_2=0}^{s+l_1-1} \varphi_{l_1,l_2}(\hat{\lambda},\hat{\beta}) \int_{L_{(s)}(\underline{y})}^{\infty} t_{(s)}\\ &\times \left[1+\hat{\beta}t_{(s)}\right]^{-2\hat{\lambda}(l_2+1)-1}\left[1+2\hat{\beta}t_{(s)}\right]^{\hat{\lambda}(l_2+1)-1} dt_{(s)}\\ &= 1-\frac{\alpha}{2}, \qquad (34)\end{aligned}$$

and

$$P\left(T_{(s)} > U_{(s)}(\underline{y})\middle|\underline{y}\right) = D(s) \sum_{l_1=0}^{k-s} \sum_{l_2=0}^{s+l_1-1} \varphi_{l_1,l_2}(\hat{\lambda},\hat{\beta}) \int_{U_{(s)}(\underline{y})}^{\infty} t_{(s)}$$

$$\times \left[1+\hat{\beta}t_{(s)}\right]^{-2\hat{\lambda}(l_2+1)-1}\left[1+2\hat{\beta}t_{(s)}\right]^{\hat{\lambda}(l_2+1)-1} dt_{(s)}$$
$$=\frac{\alpha}{2}, \qquad (35)$$

where s = 1, 2, 3, …, . k

3.2 Bayesian prediction

In this subsection, Bayesian prediction (point and interval) for a future observation $T_{(s)}$, of the ITL distribution for CS-PALT based on two-sample prediction are considered.

Assuming that the parameters, $\underline{\theta}=(\lambda,\beta)'$, are unknown and independent, then the *Bayesian predictive density* (BPD) of $T_{(s)}$, given $\underline{x}$,$\underline{y}$ based on the informative prior can be obtained as follows:

$$h_2\left(t_{(s)}\middle|\underline{x},\underline{y}\right)=\int_{\underline{\theta}} h\left(t_{(s)}\middle|\underline{\theta}\right)\pi\left(\underline{\theta}\middle|\underline{x},\underline{y}\right)d\underline{\theta}, \qquad (36)$$

where $\pi\left(\underline{\theta}\middle|\underline{x},\underline{y}\right)$ is given by (22), $\int_{\underline{\theta}}$ and $d\underline{\theta}$ are given by (24) and $h\left(t_{(s)}\middle|\underline{\theta}\right)$ is defined in (28).

Substituting (22) and (28) into (36), then the BPD of $T_{(s)}$, given $\underline{x}$,$\underline{y}$ is given by

$$h_2\left(t_{(s)}\middle|\underline{x},\underline{y}\right)=D(s)\,A\sum_{l_1=0}^{k-s}\sum_{l_2=0}^{s+l_1-1}\int_{\underline{\theta}}\varphi^{*}_{l_1,l_2}(\lambda,\beta)\,t_{(s)}$$
$$\times\left\{\prod_{i=1}^{r}\left(1+x_{(i)}\right)^{-2\lambda}\left(1+2x_{(i)}\right)^{\lambda}\right\}\left[\frac{\left(1+2x_{(r)}\right)^{\lambda}}{\left(1+x_{(r)}\right)^{2\lambda}}\right]^{m_u}$$
$$\times\left\{\prod_{j=1}^{r}\left(1+\beta y_{(j)}\right)^{-2\lambda-1}\left(1+2\beta y_{(j)}\right)^{\lambda-1}\right\}$$
$$\times exp[-(b_1\lambda+b_2\beta)]\left[\frac{\left(1+2\beta y_{(r)}\right)^{\lambda}}{\left(1+\beta y_{(r)}\right)^{2\lambda}}\right]^{m_a}$$
$$\times\left[1+\beta t_{(s)}\right]^{-2\lambda(l_2+1)-1}\left[1+2\beta t_{(s)}\right]^{\lambda(l_2+1)-1}d\underline{\theta}, \qquad (37)$$

where A^{-1} is defined in (23), $\int_{\underline{\theta}}$ and $d\underline{\theta}$ are given by (24) and

$$\varphi^*_{l_1,l_2}(\lambda,\beta) = 2\,\lambda^{2r+a_1}\,\beta^{r+a_2}(-1)^{l_1+l_2}\binom{k-s}{l_1}\binom{s+l_1-1}{l_2}. \qquad (38)$$

3.2.1 Point prediction

Based on Type II censoring, the Bayesian prediction is considered under two types of loss functions, the BSEL function; as a symmetric loss function and BLL function; as an asymmetric loss function.

I. Balanced squared error loss function

The *Bayes predictor* (BP) for the future observation $T_{(s)}$, under BSEL function can be derived using (10) and (37) as given below

$$\hat{t}_{(s)BBSE} = \omega \hat{t}_{(s)ML} + (1-\omega)\int_{t_{(s)}} t_{(s)}\, h_2\left(t_{(s)}\middle|\underline{x},\underline{y}\right) dt_{(s)}$$

$$= \omega \hat{t}_{(s)ML} + (1-\omega) D(s)\, A \sum_{l_1=0}^{k-s}\sum_{l_2=0}^{s+l_1-1}\int_{\underline{\theta}_*}\varphi^*_{l_1,l_2}(\lambda,\beta)\, t^2_{(s)}$$

$$\times\, exp[-(b_1\lambda + b_2\beta)]\left\{\prod_{i=1}^{r}(1+x_{(i)})^{-2\lambda}(1+2x_{(i)})^{\lambda}\right\}$$

$$\times\left\{\prod_{j=1}^{r}\left(1+\beta y_{(j)}\right)^{-2\lambda-1}\left(1+2\beta y_{(j)}\right)^{\lambda-1}\right\}$$

$$\times\left[\frac{(1+2x_{(r)})^{\lambda}}{(1+x_{(r)})^{2\lambda}}\right]^{m_u}\left[\frac{(1+2\beta y_{(r)})^{\lambda}}{(1+\beta y_{(r)})^{2\lambda}}\right]^{m_a}\left[1+\beta t_{(s)}\right]^{-2\lambda(l_2+1)-1}$$

$$\times\left[1+2\beta t_{(s)}\right]^{\lambda(l_2+1)-1}\, d\underline{\theta}_*, \qquad (39)$$

where $\hat{t}_{(s)ML}$ is the conditional ML prediction for the future observation of $t_{(s)}$, $D(s)$ is given by (26), $\varphi^*_{l_1,l_2}(\lambda,\beta)$ is defined in (38),

$$\int_{\underline{\theta}_*} = \int_{t_{(s)}}\int_{\lambda}\int_{\beta} \quad \text{and} \quad d\underline{\theta}_* = d\beta\, d\lambda\, dt_{(s)}. \qquad (40)$$

II. Balanced linear exponential loss function

The BP for the future observation $T_{(s)}$, under BLL function can be derived using (11) and (37) as follows:

$$\hat{t}_{(s)BBL} = \frac{-1}{v}\ln\left\{\omega\, exp(-v\,\hat{t}_{(s)ML}) + (1-\omega)\int_{t_{(s)}} exp(-v\,t_{(s)})\, h_2\left(t_{(s)}\middle|\underline{x},\underline{y}\right) dt_{(s)}\right\}$$

$$= \frac{-1}{v}\ln\{\omega\, exp(-v\,\hat{t}_{(s)ML}) + (1-\omega) D(s)\; A \sum_{l_1=0}^{k-s} \sum_{l_2=0}^{s+l_1-1} \int_{\underline{\theta}_*} t_{(s)}$$

$$\times \{\prod_{i=1}^{r}(1+x_{(i)})^{-2\lambda}\,(1+2x_{(i)})^{\lambda}\} \left[\frac{(1+2x_{(r)})^{\lambda}}{(1+x_{(r)})^{2\lambda}}\right]^{m_u}$$

$$\times \{\prod_{j=1}^{r}(1+\beta y_{(j)})^{-2\lambda-1}\,(1+2\beta y_{(j)})^{\lambda-1}\} \left[\frac{(1+2\beta y_{(r)})^{\lambda}}{(1+\beta y_{(r)})^{2\lambda}}\right]^{m_a}$$

$$\times \varphi^{*}_{l_1,l_2}(\lambda,\beta)\,[1+\beta t_{(s)}]^{-2\lambda(l_2+1)-1}\,[1+2\beta t_{(s)}]^{\lambda(l_2+1)-1}$$

$$\times\; exp[-(v\,t_{(s)} + b_1\lambda + b_2\beta)]\; d\underline{\theta}_*\}, \qquad (41)$$

where $\hat{t}_{(s)ML}$ is the conditional ML prediction for the future observation of $t_{(s)}$, $D(s)$ is given by (26), $\varphi^{*}_{l_1,l_2}(\lambda,\beta)$ is defined in (38), $\int_{\underline{\theta}_*}$ and $d\underline{\theta}_*$ are given by (40).

3.2.2 Interval prediction

A 100(1- α)% *Bayesian prediction bounds* (BPB) for the future observation $T_{(s)}$, such that $P\left(L_{(s)}(\underline{x},\underline{y}) < T_{(s)} < U_{(s)}(\underline{x},\underline{y})\middle|\underline{x},\underline{y}\right) = 1-\alpha$, can be obtained from (37) as given below

$$P(T_{(s)} > L_{(s)}(\underline{x},\underline{y})\,|\underline{x},\underline{y}) = 1-\frac{\alpha}{2}, \qquad (42)$$

and

$$P(\;T_{(s)} > U_{(s)}(\underline{x})|\underline{x}) = \frac{\alpha}{2}. \qquad (43)$$

Then the BPB for $T_{(s)}$ can be derived as follows:

$$P\left(T_{(s)} > q_s\middle|\underline{x},\underline{y}\right) = \int_{q_s}^{\infty} h_2\left(t_{(s)}\middle|\underline{x},\underline{y}\right) dt_{(s)}. \qquad (44)$$

Using (37) and substituting q_s in (44) by $L_{(s)}(\underline{x},\underline{y})$ and $U_{(s)}(\underline{x},\underline{y})$, then solving two nonlinear equations one obtains

$$P\left(T_{(s)} > q_s \middle| \underline{x}, \underline{y}\right) = D(s)\, A \sum_{l_1=0}^{k-s} \sum_{l_2=0}^{s+l_1-1} \int_{q_s}^{\infty} \int_{\underline{\theta}} exp[-(b_1\lambda + b_2\beta)]$$
$$\times \{\prod_{i=1}^{r}(1+x_{(i)})^{-2\lambda}\,(1+2x_{(i)})^{\lambda}\} \left[\frac{(1+2x_{(r)})^{\lambda}}{(1+x_{(r)})^{2\lambda}}\right]^{m_u}$$
$$\times \{\prod_{j=1}^{r}(1+\beta y_{(j)})^{-2\lambda-1}\,(1+2\beta y_{(j)})^{\lambda-1}\}$$
$$\times\; t_{(s)}[1+\beta t_{(s)}]^{-2\lambda(l_2+1)-1}[1+2\beta t_{(s)}]^{\lambda(l_2+1)-1}$$
$$\times\; \varphi^{*}_{l_1,l_2}(\lambda,\beta)\left[\frac{(1+2\beta y_{(r)})^{\lambda}}{(1+\beta y_{(r)})^{2\lambda}}\right]^{m_a} d\underline{\theta}\; dt_{(s)}, \qquad (45)$$

where $s = 1, 2, 3, \dots, k$.

3.3 E-Bayesian prediction

In this subsection, the E-Bayesian prediction (point and interval) for a future observation $T_{(s)}$, of the ITL distribution for CS-PALT based on two-sample prediction are obtained.

According to Han (2007), the hyper-parameters a_j and b_j should be selected to guarantee that $\pi(\theta_j; a_j, b_j)$, given in (19), can be decreasing functions of θ_j ($j = 1,2$).

The derivative of $\pi(\theta_j; a_j, b_j)$ with respect to θ_j is given below

$$\frac{d\,\pi(\theta_j; a_j, b_j)}{d\theta_j} = \frac{b_j^{\,a_j}}{\Gamma(a_j)}\, \theta_j^{\,a_j-2} exp(-b_j\theta_j)\,[(a_j-1) - b_j\theta_j],\; j = 1,2, \quad (46)$$

for $0 < a_j < 1$ and $b_j > 0$, then $\frac{d\,\pi(\theta_j; a_j, b_j)}{d\theta_j} < 0$, which means that $\pi(\theta_j; a_j, b_j)$ can be decreasing functions of θ_j.

The E-Bayes estimators of the parameters are obtained based on three different distributions of the hyper-parameters a_j and b_j. These distributions are used to investigate the effect of different prior distributions on the E-Bayesian estimation of θ_j.

Assuming that the hyper-parameters a_j and b_j are independent with bivariate density functions

$$\pi_\hbar(a_j,b_j) = \pi_\hbar(a_j)\,\pi_\hbar(b_j), \qquad j = 1,2,\ \hbar = 1,2,\dots,6. \tag{47}$$

Then, the bivariate uniform hyperprior distributions are:

$$\pi_\hbar(a_j,b_j) = \frac{2(c_j-b_j)}{c_j^2}, \qquad 0 < a_j < 1, 0 < b_j < c_j, \tag{48}$$

$$\pi_\hbar(a_j,b_j) = \frac{1}{c_j}, \qquad 0 < a_j < 1,\ 0 < b_j < c_j, \tag{49}$$

$$\pi_\hbar(a_j,b_j) = \frac{2b_j}{c_j^2}, \qquad 0 < a_j < 1,\ 0 < b_j < c_j. \tag{50}$$

The E-Bayes estimators of θ_j (expectation of the Bayes estimators of θ_j) can be derived as follows:

$$\tilde{\theta}_{jEB} = E_{\pi_\hbar}\left(\tilde{\theta}_{jB}(a_j,b_j)\right) = \iint_D \tilde{\theta}_{jB}(a_j,b_j)\,\pi_\hbar(a_j,b_j)\,da_j\,db_j,$$

$$j = 1,2,\ \hbar = 1,2,\dots,6, \tag{51}$$

where $E_{\pi_\hbar}\,(\hbar = 1,2,\dots,6)$ stands for the expectation of the bivariate hyperprior distributions, D is the domain of the function $\pi_\hbar(a_j,b_j)$ and $\tilde{\theta}_{jB}(a_j,b_j)$ are the Bayes estimators of the parameters θ_j based on BSEL and BLL functions.

3.3.1 Point prediction

Based on Type II censoring, the E-Bayesian prediction is considered under two types of loss functions, the BSEL function; as a symmetric loss function and BLL function; as an asymmetric loss function.

I. Balanced squared error loss function

The three *E-Bayes predictors* (EBPs) for the future observation $T_{(s)}$, under BSEL function can be obtained by substituting (39) and (48)-(50) in (51) as given below

$$\hat{t}_{(s)EBBS\hbar} = E_{\pi_{\hbar}}\left(\hat{t}_{(s)BBSE}\right) = \int_0^c \int_0^1 \hat{t}_{(s)BBSE}\, \pi_{\hbar}(a,b)\, da\, db, \quad \hbar = 1,2,3. \quad (52)$$

II. Balanced linear exponential loss function

The EBPs for the future observation $T_{(s)}$, under BLL function can be derived by substituting (41) and (48)-(50) in (51) as follows:

$$\hat{t}_{(s)EBBL\hbar} = E_{\pi_{\hbar}}\left(\hat{t}_{(s)BBL}\right) = \int_0^c \int_0^1 \hat{t}_{(s)BBL}\, \pi_{\hbar}(a,b)\, da\, db, \quad \hbar = 1,2,3. \quad (53)$$

3.3.2 Interval prediction

A 100(1- α)% *E-Bayesian prediction bounds* (EBPB) for the future observation $T_{(s)}$, such that

$P\left(L_{(s)}(\underline{x},\underline{y}) < T_{(s)} < U_{(s)}(\underline{x},\underline{y}) \middle| \underline{x},\underline{y}\right) = 1-\alpha,$ can be obtained by substituting (45) and (48)-(50) in (51).

Remark:

- If $s = 1$, in (31), (39), (41), (52) and (53), one can predict the minimum observable, $T_{(1)}$, which represents the first failure time in a future sample of size $\hbar$.
- If $s = \hbar$, in (31), (39), (41), (52) and (53), one can predict the maximum observable, $T_{(\hbar)}$, which represents the largest failure time in a future sample of size $\hbar$.
- If $s = \frac{\hbar+1}{2}$, in (31), (39), (41), (52) and (53), one can predict the median observable if $\hbar$ is odd, $T_{\left(\frac{\hbar+1}{2}\right)}$, which represents the median failure time in a future sample of size $\hbar$.

4. Numerical Illustration

This section aims to investigate the precision of the theoretical results of prediction on the basis of simulated and real data sets.

4.1 Simulation algorithm

In this subsection, the conditional ML, Bayes and E-Bayes predictors (point and interval) for a future observation from the ITL distribution for CS-PALT distribution based on Type II censored data are computed. All simulation studies are performed using Mathematica 9 and R programming language.

4.1.1 Maximum likelihood prediction

The steps of the simulation procedure based on Type II censored data are as follows:

Step 1: For given values of λ, random samples of size n are generated from the ITL (λ) distribution.

- The transformation between the uniform distribution and ITL (λ) distribution is obtained from Hassan *et al.* (2020) as follows:

$$x_u = \frac{-2\left[(1-u)^{\frac{1}{\lambda}}-1\right]+\sqrt{4\left[(1-u)^{\frac{1}{\lambda}}-1\right]^2-4(1-u)^{\frac{1}{\lambda}}\left[(1-u)^{\frac{1}{\lambda}}-1\right]}}{2(1-u)^{\frac{1}{\lambda}}}, \quad 0<u<1.$$

Step 2: For each sample size x_i`s, are sorted such that $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{[n(1-\tau)]}$.

Step 3: For each sample size y_j`s, are sorted such that $y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(n\tau)}$.

- The number of failures r are chosen to be less than $n(1-\tau)$ and $n\tau$.

Step 4: The ML estimates for the parameters λ and β are computed based on Type II censored scheme.

Step 5: Substituting the conditional ML estimates of the parameters in the equation of $\hat{t}_{(s)(ML)}$ and for given values for s, the conditional MLP for the future observation $T_{(s)}$ can be computed under Type II censored sample.

Step 6: Using the ML estimates for the parameters and a certain value of s, the conditional MLPB for the future observation $T_{(s)}$, can be computed under Type II censored sample.

Step 7: Repeat all the previous steps N=2000 times.

4.1.2 Bayesian and E-Bayesian prediction

Step 1: Generate a_j and b_j from the bivariate uniform hyperprior distributions; $\pi_h(a_j, b_j)$, $j = 1,2, h = 1,2, \dots, 6,$ given in (48)-(50).

Step 2: For given values of a_j and b_j, generate λ and β from the gamma prior distributions.

Step 3: Applying the previous generation steps, Type II censored sample can be generated from the ITL distribution.

Step 4: Calculate the joint posterior distribution for the parameters based on Type II censored sample from the ITL distribution.

Step 5: The BPD of the future observation $T_{(s)}$, can be obtained.

Step 6: The BP is calculated based on BSEL and BLL functions. Also, the BPB is evaluated.

Step 7: Using the BP, the EBPs for a future observation from the ITL distribution based on BSEL and BLL functions are calculated. Similarly, using the BPB, the EBPB are evaluated.

Step 8: Repeat all the previous steps N=10000 times.

The conditional ML two-sample predictors are presented in Table 1. Also, the Bayes and E-Bayes two-sample predictors are presented in Tables 3 and 4 based on BSEL and BLL functions.

4.2 Some applications

The main aim of this subsection is to demonstrate how the proposed methods can be used in practice. Two real lifetime data sets are used for this purpose. The ITL distribution is fitted to the two real data using

Kolmogorov-Smirnov goodness of fit test through R programming language.

Application 1

The first data introduced by Liu *et al.* (2021). The data refer to the survival times of patients suffering from the COVID-19 epidemic in China. The considered data set representing the survival times of patients from the time admitted to the hospital until death. Among them, a group of fifty-three (53) COVID-19 patients were found in critical condition in hospital from January to February 2020. The data set can be retrieved from https://www.worldometers.info/coronavirus/ and is given by: 0.054, 0.064, 0.704, 0.816, 0.235, 0.976, 0.865, 0.364, 0.479, 0.568, 0.352, 0.978, 0.787, 0.976, 0.087, 0.548, 0.796, 0.458, 0.087, 0.437, 0.421, 1.978, 1.756, 2.089, 2.643, 2.869, 3.867, 3.890, 3.543, 3.079, 3.646, 3.348, 4.093, 4.092, 4.190, 4.237, 5.028, 5.083, 6.174, 6.743, 7.274, 7.058, 8.273, 9.324, 10.827, 11.282, 13.324, 14.278, 15.287, 16.978, 17.209, 19.092 and 20.083.

Application 2

The second data represents a COVID-19 mortality rates data belongs to Italy of 59 days, that is recorded from 27 February to 27 April 2020, used by Almongy *et al.* (2021). The data are as follows: 4.571, 7.201, 3.606, 8.479, 11.410, 8.961, 10.919, 10.908, 6.503, 18.474, 11.010, 17.337, 16.561, 13.226, 15.137, 8.697, 15.787, 13.333, 11.822, 14.242, 11.273, 14.330, 16.046, 11.950, 10.282, 11.775, 10.138, 9.037, 12.396, 10.644, 8.646, 8.905, 8.906, 7.407, 7.445, 7.214, 6.194, 4.640, 5.452, 5.073, 4.416, 4.859, 4.408, 4.639, 3.148, 4.040, 4.253, 4.011, 3.564, 3.827, 3.134, 2.780, 2.881, 3.341, 2.686, 2.814, 2.508, 2.450 and 1.518. [See https://covid19.who.int/].

The Kolmogorov–Smirnov goodness of fit test is applied to check the validity of the fitted model. The p values are given, respectively, 0.7444 and 0.2582. The p value given in each case showed that the model fits the data very well.

Table 2 presents the conditional ML two-sample predictors of the real data sets. Also, Tables 5 and 6 display the Bayes and E-Bayes two-sample predictors of the real data sets based on BSEL and BLL functions.

4.3 Concluding remarks

- The results in Tables 1-6 indicate that the length of the interval of the first future order statistic is smaller than the length of the interval of the last future order statistic.
- The conditional ML, Bayes and E-Bayes intervals include the predictive values [between the *lower limit* (LL) and *upper limit* (UL)].
- The lengths of the intervals of the E-Bayes predictors are less than the lengths of the intervals of the Bayes predictors, so the E-Bayesian prediction method is better than the Bayesian prediction method.
- In most cases, the lengths of the intervals of the Bayes and E-Bayes predictors under BLL function are less than the lengths of the intervals of the Bayes and E-Bayes predictors under BSEL function.
- The lengths of the intervals of the conditional ML, Bayes and E-Bayes predictors increase when s increases.

5. General Conclusion

For products having a high reliability, the test of product life under usual conditions often requires a long period of time. So, ALT or PALT is used to facilitate estimating the reliability of the unit in a short period of time. In ALT test items are run only at accelerated conditions, in some cases, such relationship cannot be known or assumed. So, PALT are often used in such cases, in PALT they are run at both usual and accelerated conditions. In this research, the two-sample prediction method is applied to obtain the conditional ML, Bayesian and E-Bayesian prediction (point and interval) for future order statistics of the ITL distribution for CS-PALT based on Type II censored samples. The predictors are considered under two different loss functions, the BSEL function; as a symmetric loss function and BLL function; as an asymmetric loss function. The predictors are obtained based on gamma prior and uniform hyperprior distributions. A numerical example is given to illustrate the theoretical results and two applications using real data sets are used to demonstrate how the results can be used in practice. In general, numerical computations showed that the length of the interval of the first future order statistic is smaller than the length of the interval of the last future order statistic. The conditional ML, Bayes and E-Bayes intervals include the predictive values. Also, the lengths of the interval of the E-Bayes predictors are less than the lengths of the interval of the Bayes predictors, so the E-Bayes prediction technique is better than the Bayes prediction technique. The Bayesian and E-Bayesian prediction (point and interval) for future order statistics of the ITL distribution for CS-PALT under different type of loss functions such as general entropy and precautionary loss functions would be useful as a basis for further researches in distribution theory.

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Appendix

Table 1: ML predictors and bounds of the future observation based on Type II censoring under two-sample prediction (N=2000, n =30, $r = 0.1n$, $\hat{\lambda} = 1.2$ and $\hat{\beta} = 1.3$)

$\tau = 0.20$					$\tau = 0.50$				
s	$\hat{t}_{(s)(ML)}$	LL	UL	length	s	$\hat{t}_{(s)(ML)}$	LL	UL	length
1	0.0200	1.1293E-08	0.1035	0.1035	1	0.0205	0.0000	0.1042	0.1042
15	0.1737	0.0100	0.6235	0.6135	15	0.2277	0.0010	0.7386	0.7376
25	0.4796	0.0100	1.8620	1.8520	25	0.6407	0.0010	2.2681	2.2671

Table 2: ML predictors and bounds of the future observation for real data sets based on Type II censoring under two-sample prediction ($\tau = 0.20$)

Application I					Application II				
s	$\hat{t}_{(s)(ML)}$	LL	UL	length	s	$\hat{t}_{(s)(ML)}$	LL	UL	length
1	0.0436	1.8783 E-08	0.1772	0.1772	1	0.1943	0.0777	0.3978	0.3201
25	0.2317	0.1000	0.6260	0.5260	30	1.4982	0.1000	4.3629	4.2629
45	0.8195	0.0001	10.055	10.0549	50	5.5625	0.1000	7.3192	7.2192

Table 3: Bayes and E-Bayes predictors and bounds of the future observation under BSEL function based on CS-PALT under two-sample prediction (N=10000, $n = 30$, $\hat{\lambda} = 1.5$, $\hat{\beta} = 1.1$ and $\omega = 0.3$)

		Bayesian				E-Bayesian			
τ	s	$\hat{t}_{(s)(BBS)}$	LL	UL	Length	$\hat{t}_{(s)(EBBS)}$	LL	UL	Length
0.30	1	0.0782	0.0733	0.0819	0.0085	0.0739 0.0760 0.0735	0.0713 0.0736 0.0703	0.0764 0.0775 0.0753	0.0051 0.0039 0.0050
	15	0.0799	0.0748	0.08381	0.0090	0.0770 0.0789 0.0752	0.0728 0.0749 0.0702	0.0809 0.0819 0.0781	0.0082 0.0071 0.0079
	25	0.0940	0.0860	0.0996	0.0136	0.0833 0.0828 0.0905	0.0793 0.0793 0.0870	0.0888 0.0875 0.0953	0.0095 0.0082 0.0082

Table 3 (continued)

0.60	1	0.0821	0.0769	0.0856	0.0087	0.0777 0.0763 0.0742	0.0730 0.0735 0.0712	0.0801 0.0781 0.0767	0.0071 0.0045 0.0055
	15	0.0875	0.0799	0.0935	0.0136	0.0819 0.0789 0.0752	0.0764 0.0749 0.0702	0.0896 0.0819 0.0781	0.0133 0.0071 0.0079
	25	0.1015	0.0932	0.1114	0.0182	0.0954 0.0871 0.0919	0.0886 0.0777 0.0862	0.1013 0.0918 0.0973	0.0126 0.0140 0.0111

Table 4: Bayes and E-Bayes predictors and bounds of the future observation under BLL function based on CS-PALT under two-sample prediction (N=10000, $n = 30$, $\hat{\lambda} = 1.5$, $\hat{\beta} = 1.1$, $\nu = -2$ and $\omega = 0.3$)

		Bayesian				E-Bayesian			
τ	s	$\hat{t}_{(s)(BBL)}$	LL	UL	Length	$\hat{t}_{(s)(EBBL)}$	LL	UL	Length
0.30	1	0.0786	0.0739	0.0805	0.0066	0.0766 0.0741 0.0762	0.0736 0.0720 0.0733	0.0782 0.0760 0.0779	0.0047 0.0040 0.0046
	15	0.0849	0.0783	0.0875	0.0091	0.0794 0.0844 0.0831	0.0769 0.0811 0.0797	0.0819 0.0873 0.0859	0.0050 0.0062 0.0062
	25	0.0929	0.0871	0.0968	0.0097	0.0914 0.0852 0.0844	0.0879 0.0815 0.0803	0.0937 0.0890 0.0890	0.0058 0.0075 0.0087
0.60	1	0.0799	0.0751	0.0830	0.0079	0.0780 0.0784 0.0772	0.0743 0.0760 0.0744	0.0813 0.0802 0.0798	0.0070 0.0042 0.0055
	15	0.0930	0.0854	0.0990	0.0136	0.0925 0.0927 0.0846	0.0866 0.0890 0.0759	0.0968 0.0973 0.0920	0.0102 0.0083 0.0160
	25	0.1030	0.0941	0.1110	0.0168	0.0990 0.0933 0.1011	0.0927 0.0829 0.0903	0.1062 0.0967 0.1055	0.0135 0.0137 0.0152

Table 5: Bayes and E-Bayes predictors and bounds for real data sets of the future observation under BSEL function based on CS-PALT under two-sample prediction ($\tau = 0.30$)

		Bayesian				E-Bayesian			
	s	$\hat{t}_{(s)(BBS)}$	LL	UL	Length	$\hat{t}_{(s)(EBBS)}$	LL	UL	Length
Application I	1	0.0438	0.0398	0.0466	0.0068	0.0421 0.0389 0.0412	0.0405 0.0359 0.0388	0.0445 0.0409 0.0431	0.0040 0.0049 0.0043
	25	0.8899	0.8848	0.8938	0.0085	0.8878 0.8898 0.8880	0.8853 0.8866 0.8849	0.8897 0.8920 0.8906	0.0043 0.0054 0.0057
	45	1.2552	1.2486	1.2613	0.0127	1.2432 1.2488 1.2550	1.2396 1.2449 1.2492	1.2448 1.2522 1.2617	0.0051 0.0073 0.0126
Application II	1	1.5504	1.5471	1.5525	0.0054	1.5470 1.5499 1.5499	1.5457 1.5479 1.5482	1.5477 1.5513 1.5509	0.0020 0.0034 0.0027
	30	2.5541	2.5494	2.5584	0.0090	2.5536 2.5528 2.5539	2.5522 2.5508 2.5515	2.5549 2.5544 2.5554	0.0027 0.0036 0.0039
	50	3.5484	3.5386	3.5529	0.0143	3.5423 3.5418 3.5422	3.5390 3.5382 3.5374	3.5445 3.5439 3.5470	0.0055 0.0058 0.0096

Table 6: Bayes and E-Bayes predictors and bounds for real data sets of the future observation under BLL function based on CS-PALT under two-sample prediction ($\tau = 0.30$ and $v = -2$)

	Bayesian					E-Bayesian			
	s	$\hat{t}_{(s)(BBL)}$	LL	UL	Length	$\hat{t}_{(s)(EBBL)}$	LL	UL	Length
Application I	1	0.0417	0.0390	0.0445	0.0056	0.0399 0.0393 0.0406	0.0388 0.0376 0.0387	0.0407 0.0406 0.0417	0.0019 0.0029 0.0029
	25	0.8932	0.8884	0.8961	0.0077	0.8877 0.8875 0.8845	0.8851 0.8839 0.8806	0.8890 0.8904 0.8856	0.0039 0.0065 0.0051
	45	1.2522	1.2438	1.2594	0.0156	1.2454 1.2492 1.2447	1.2423 1.2451 1.2416	1.2476 1.2521 1.2469	0.0053 0.0070 0.0053
Application II	1	1.5526	1.5500	1.5543	0.0042	1.5495 1.5489 1.5487	1.5483 1.5480 1.5476	1.5508 1.5498 1.5493	0.0025 0.0018 0.0017
	30	2.5556	2.5499	2.5591	0.0092	2.5462 2.5511 2.5532	2.5443 2.5497 2.5504	2.5481 2.5521 2.5555	0.0038 0.0024 0.0051
	50	3.5549	3.5468	3.5654	0.0187	3.5498 3.5543 3.5521	3.5449 3.5488 3.5466	3.5542 3.5578 3.5554	0.0093 0.0090 0.0088