ORIGINAL RESEARCH

Open Access

Bivariate general exponential models with stress-strength reliability application

S. K. khames[1]* and N. A. Mokhlis[2]

* Correspondence: sohair_smz@
women.asu.edu.eg
[1]Department of Mathematics,
Faculty of Women, Ain Shams
University, Cairo, Egypt
Full list of author information is
available at the end of the article

Abstract

In this paper, we introduce two families of general bivariate distributions. We refer to these families as general bivariate exponential family and general bivariate inverse exponential family. Many bivariate distributions in the literature are members of the proposed families. Some properties of the proposed families are discussed, as well as a characterization associated with the stress-strength reliability parameter, R, is presented. Concerning R, the maximum likelihood estimators and a simple estimator with an explicit form depending on some marginal distributions are obtained in case of complete sampling. When the stress is censored at the strength, an explicit estimator of R is also obtained. The results obtained can be applied to a variety of bivariate distributions in the literature. A numerical illustration is applied on some well-known distributions. Finally a real data example is presented to fit one of the proposed models.

Keywords: Stress-strength reliability, Exponential distribution model, Inverse exponential distribution model, Maximum likelihood estimator

Mathematics Subject Classifications: 62N05, 62E10, 62F10, 62G05, 62N02

Introduction

Mokhlis et al. [1] presented two forms of survival functions, given by

$$\overline{F}(u; \theta, c) = e^{-\theta g_1(u;c)}, \tag{1}$$

$$\overline{F}(u; \beta, c) = 1 - e^{-\beta g_2(u;c)}, \tag{2}$$

where $g_1(u; c)$ does not contain θ, $\theta \in \Theta$, and $g_2(u; c)$ does not contain $\beta \in \boldsymbol{\beta}$, $c \in \mathbf{C}$, $\{\Theta$, $\boldsymbol{\beta}$, and $\mathbf{C}\}$ are the parametric spaces, where $g_1(u; c)$ is a continuous, monotone increasing, and differential function such that $g_1(u; c) \to 0$ as $u \to 0$ and $g_1(u; c) \to \infty$ as $u \to \infty$, while $g_2(u; c)$ is continuous, monotone decreasing and differential function such that $g_2(u; c) \to 0$ as $u \to \infty$ and $g_2(u; c) \to \infty$ as $u \to 0$. With appropriate choices of $g_i(u; c)$, $i = 1$, 2, in (1) and (2), many distributions in the literature can be obtained, such as exponential distribution, Weibull distribution, Rayleigh distribution, Pareto, Lomax, and others from the first form (1), and inverse exponential distribution, inverse Weibull distribution, inverse Rayleigh, Burr type III distribution and others from the second form (2), see Mokhlis et al. [1]. For facilitation we will denote the forms (1) and (2) by $EF(\theta, c)$ and $IEF(\beta; c)$ and denote its survival functions and probability density function by $F_{EF}(u; \theta, c)$, $f_{EF}(u; \theta, c)$ and $F_{IEF}(u; \beta, c)$, $f_{IEF}(u; \beta, c)$, respectively.

In the area of stress-strength models, there have been a large amount of work regarding estimation of the reliability parameter, $R = P(Y < X)$, when X and Y independent random variables belonging to the same univariate family, see for example Mokhlis [2], Kundu and Gupta [3], Singh et al. [4] and others. Recently, Mokhlis et al. [1] discussed R, when the variables are independent with survival functions having forms (1) and (2), respectively. Indeed, many real situations entail that X and Y are related in some way. However, some authors have studied the stress-strength reliability parameter, R, for some specified bivariate distributions, see for example Kotz et al. [5], Mokhlis [6], Nadarajah and Kotz [7], Nguimkeu et al. [8], Pak et al. [9] and, Abdel-Hamid [10].

There are many methods in the literature for obtaining bivariate distribution. Some of the popular methods are the copula type, the bivariate pseudo type and the Marshall-Olkin type. Recently many attempts of obtaining generalized bivariate distributions using these types are presented in the literature. Among those are Kolesarova et al. [11], Arnold and Arvanitis [12], El-Bassiouny et al. [13] and Sarhan [14].

In the present paper, we introduce two bivariate models of distributions which are types of the bivariate Marshall-Olkin distribution. We call these models bivariate exponential and bivariate inverse exponential models. Some properties of the proposed models are discussed. Many bivariate distributions in the literature can be considered as special cases or members of our models, for example, Marshall and Olkin (M-O) bivariate exponential distribution and M-O bivariate Weibull introduced by Marshall and Olkin [15] and the bivariate Rayleigh distribution introduced by Pak et al. [9].

An explicit expression of the stress-strength parameter R is obtained showing that it is not a function of the parameter c (c could be a vector parameter). The maximum likelihood estimator of R is obtained as well as simple estimators of R are obtained in a closed form depending on the marginal distribution of X and the distribution of $\min\{X, Y\}$ or depending on the marginal distribution of Y and the distribution of $\max\{X, Y\}$. Since many bivariate distributions in the literature belong to the proposed families, the results obtained could be applicable to a variety of bivariate distributions.

The remaining part of the paper is organized as follows: In the "Proposed families of bivariate distributions" section, we introduce two new families (models) of bivariate distributions. Some characterization of the proposed models such as marginals and the distribution of $\min\{X, Y\}$ and $\max\{X, Y\}$ are also discussed. The stress-strength reliability parameter, R, concerning the new models is considered in the "Stress-strength reliability" section. In the "Point estimation of R" section, we obtain maximum likelihood estimators of R as well as simple estimators of R depending on some marginal distributions in case of complete sampling. When the stress is censored at the strength, an explicit estimator of R is also obtained. Some bivariate members of the proposed family are presented in the "Special cases" section. In the "Numerical illustrations" section, a numerical illustration using some well-known distributions is performed to highlight the theoretical results. Also an application is introduced using real data example. Finally conclusions of the results obtained are introduced in the "Conclusions" section.

Proposed families of bivariate distributions

In this section, we introduce two new families of bivariate distributions with marginals having distributions with forms (1) or (2). We apply a similar technique of that proposed by Marshall and Olkin [15], for obtaining these families.

The construction of the families (models)

Lifetime model

Suppose that a system consists of two subsystems, say A and B. Subsystem A contains two components, say A_1, and C, connected in series (parallel) with lifetimes U_1 and U_0, respectively. Subsystem B contains the two components, say B_1 and C, connected in series (parallel), where the lifetime of component B_1 is U_2.

Suppose that U_i, $i = 0, 1, 2$. , are independent random variables following $EF(\theta_i, c)$, $i = 0, 1, 2$ for the series case and $IEF(\beta_i; c)$, $i = 0, 1, 2$. , for the parallel case, i.e.,

$$\overline{F}_{U_i}(u) = \begin{cases} \overline{F}_{EF}(u; \theta, c) = e^{-\theta_i g_1(u;c)}, i = 0, 1, 2, \text{for the series case,} \\ \overline{F}_{IEF}(u; \beta, c) = 1 - e^{-\beta_i g_2(u;c)}, i = 0, 1, 2, \text{for the parallel case.} \end{cases} \tag{3}$$

If X and Y are the lifetimes of the two subsystems A and B, respectively, then we have $X = \min\{U_0, U_1\}$ and $Y = \min\{U_0, U_2\}$., for the series case, while $X = \max\{U_0, U_1\}$ and $Y = \max\{U_0, U_2\}$, for the parallel case.

Stress model

Consider a two-component system and consider three independent stresses say U_0, U_1, and U_2. Each component is subject to an individual stress say U_1 and U_2, respectively, while U_0 is an overall stress transmitted to both the components equally. Then,

1. The observed stress on the two components is $X = \max\{U_0, U_1\}$ and $Y = \max\{U_0, U_2\}$., respectively.
2. If the stresses are always fatal, then the lifetime of the two components are $X = \min\{U_0, U_1\}$ and $Y = \min\{U_0, U_2\}$.

We can observe that in the two models there is the possibility of having $X = Y$; thus, the two models have both an absolute continuous part and a singular part, similar to M-O bivariate exponential model.

Theorems 1–3 present the survival functions and the probability density functions of the proposed bivariate families.

Theorem 1 Suppose U_i, $i = 0, 1, 2$., are independent random variables following $EF(\theta_i; c)$, $i = 0, 1, 2$., and let $X = \min\{U_0, U_1\}$ and $Y = \min\{U_0, U_2\}$; then, the bivariate vector (X, Y) will have the survival function

$$\overline{F}_{BEF}(X, Y) = \exp\{-\theta_1 g_1(x; c) - \theta_2 g_1(y; c) - \theta_0 g_1(\max\{x, y\}; c)\}. \tag{4}$$

Proof Obviously, from $\overline{F}_{X,Y}(x, y) = P(X > x, Y > y)$, we can write $\overline{F}_{BEF}(X, Y)$ as

$$P(\min\{U_0, U_1\} > x, \min\{U_0, U_2\} > y) = P(U_1 > x, U_2 > y, U_0 > \min(x, y)).$$

Since U_i are independent random variables following $EF(\theta_i; c)$, $i = 0, 1, 2$. Hence, (4) holds.

We will denote the bivariate distribution with survival function having the form (4) by BEF$(\theta_0, \theta_1, \theta_2; c)$. Clearly, X and Y are independent if and only if (iff) $\theta_0 = 0$. The joint survival function can also written as

$$\overline{F}_{\mathrm{BEF}}(X, Y) = \begin{cases} \exp\{-(\theta_0 + \theta_1)g_1(x; c) - \theta_2 g_1(y; c)\}, & \text{if } x \geq y \\ \exp\{-\theta_1 g_1(x; c) - (\theta_0 + \theta_2)g_1(y; c)\}, & \text{if } y > x \end{cases}$$

Theorem 2 Suppose U_i, $i = 0$, 1, 2. , are independent random variables following $EIF(\beta_i; c)$, $i = 0$, 1, 2. , and let $X = \max\{U_0, U_1\}$ and $Y = \max\{U_0, U_2\}$; then, the bivariate vector (X, Y) has the cumulative function

$$F_{\mathrm{BIEF}}(X, Y) = \exp\{-\beta_1 g_2(x; c) - \beta_2 g_2(y; c) - \beta_0 g_2(\min\{x, y\}; c)\}. \tag{5}$$

Proof Similarly as in Theorem 1, using $F_{X, Y}(x, y) = P(X < x, Y < y)$, we can show that (5) holds.

We will denote the bivariate distribution with cumulative function with the form (5) by BIEF$(\beta_0, \beta_1, \beta_2; c)$. Clearly, X and Y are independent iff $\beta_0 = 0$. The joint cumulative function can also be written as

$$F_{\mathrm{BIEF}}(X, Y) = \begin{cases} \exp\{-\beta_1 g_2(x; c) - (\beta_0 + \beta_2)g_2(y; c)\}, \text{if } x \geq y \\ \exp\{-(\beta_0 + \beta_1)g_2(x; c) - \beta_2 g_2(y; c)\}, \text{if } y > x \end{cases}$$

Theorem 3 If the vector (X, Y) has either BEF$(\theta_0, \theta_1, \theta_2; c)$ or BIEF$(\beta_0, \beta_1, \beta_2; c)$, then their joint pdf is given by

$$f_{X,Y}(x, y) = \begin{cases} f_1(x, y), \text{if } x > y \\ f_2(x, y), \text{if } x < y \\ f_0(x), \text{if } x = y \end{cases}$$

$$\tag{6}$$

where

$$f_1(x, y) = \begin{cases} \theta_2(\theta_0 + \theta_1)g_1'(x; c)g_1'(y; c) \, e^{-(\theta_0 + \theta_1)g_1(x;c) - \theta_2 g_1(y;c)}, \text{for BEF}(\theta_0, \theta_1, \theta_2; c) \\ \beta_1(\beta_0 + \beta_2)g_2'(x; c)g_2'(y; c)e^{-\beta_1 g_2(x;c) - (\beta_0 + \beta_2)g_2(y;c)}, \text{for BIEF}(\beta_0, \beta_1, \beta_2; c) \end{cases}$$

$$f_2(x, y) = \begin{cases} \theta_1(\theta_0 + \theta_2)g_1'(x; c)g_1'(y; c) \, e^{-\theta_1 g_1(x;c) - (\theta_0 + \theta_2)g_1(y;c)}, \text{for BEF}(\theta_0, \theta_1, \theta_2; c) \\ \beta_2(\beta_0 + \beta_1)g_2'(x; c)g_2'(y; c)e^{-(\beta_0 + \beta_1)g_2(x;c) - \beta_2 g_2(y;c)}, \text{for BIEF}(\beta_0, \beta_1, \beta_2; c) \end{cases}$$

and

$$f_0(x) = \begin{cases} \theta_0 g_1'(x; c)e^{-\theta g_1(x;c)}, \text{for BEF}(\theta_0, \theta_1, \theta_2; c) \\ -\beta_0 g_2'(x; c)e^{-\beta g_2(x;c)}, \text{for BIEF}(\beta_0, \beta_1, \beta_2; c) \end{cases}$$

With $\theta = \theta_0 + \theta_1 + \theta_2$, $\beta = \beta_0 + \beta_1 + \beta_2$ and $g_i'(t; c), i = 1, 2$, is the first derivative of $g_i(t; c)$ with respect to t.

Proof Clearly, for the two models, $f_1(x, y)$ and $f_2(x, y)$ can be easily obtained by using $\frac{\partial^2 \overline{F}_{X,Y}(x,y)}{\partial x \partial y}$ or $\frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$ for $x > y$ and $y > x$ respectively. For $f_0(x)$, we use the relation $\int_0^\infty \int_0^x f_1(x, y)dydx + \int_0^\infty \int_0^y f_2(x, y)dxdy + \int_0^\infty f_0(x)dx = 1$. So, for the BEF, we have

$$\int_0^\infty \int_0^x f_1(x, y)dydx = 1 - (\theta_0 + \theta_1)\int_0^\infty g_1'(t; c)e^{-\theta g_1(t;c)}dt$$

and

$$\int_0^\infty \int_0^y f_2(x,y)dxdy = 1 - (\theta_0 + \theta_2) \int_0^\infty g_1'(t;c)e^{-\theta g_1(t;c)}dt,$$

Thus,

$$\int_0^\infty f_0(x)dx = 1 - \left[2 - (\theta_0 + \theta)\int_0^\infty g_1'(t;c)e^{-\theta g_1(t;c)}dt\right] = \theta_0 \int_0^\infty g_1'(t;c)e^{-\theta g_1(t;c)}dt.$$

Similarly for the BIEF, we have $\int_0^\infty f_0(x)dx = 1 + (\beta_1 + \beta_2)\int_0^\infty g_2'(t;c)e^{-\beta g_2(t;c)}dt = -\beta_0 \int_0^\infty g_2'(t;c)e^{-\beta g_2(t;c)}dt$.

Hence, the proof is complete.

Notice that both distribution BEF(θ_0, θ_1, θ_2; c) and BIEF (β_0, β_1, β_2; c) are singular on the line $X = Y$, since $P(X = Y) \neq 0$. Thus the two models have a singular part and an absolute continuous part, similar to Marshall and Olkin's model. The following theorem provides explicitly the absolute continuous part and the singular part of BEF and BIEF.

Theorem 4 If the vector (X, Y) has BEF(θ_0, θ_1, θ_2; c) or BIEF(β_0, β_1, β_2; c), then

(i) The survival function for the BEF is

$$\overline{F}_{\text{BEF}}(x,y) = \frac{\theta_1 + \theta_2}{\theta}\overline{F}_{\text{BEF}(a)}(x,y) + \frac{\theta_0}{\theta}\overline{F}_{\text{BEF}(s)}(x,y), \tag{7}$$

Where, $\theta = \theta_0 + \theta_1 + \theta_2$, $\overline{F}_{\text{BEF}(s)}(x,y) = e^{-\theta g_1(\max\{x,y\};c)}$ is the singular part, and $\overline{F}_{\text{BEF}(a)}(x,y) = \frac{\theta}{\theta_1+\theta_2}e^{-\theta_1 g_1(x;c)-\theta_2 g_1(y;c)-\theta_0 g_1(\max\{x,y\};c)} - \frac{\theta_0}{\theta_1+\theta_2}e^{-\theta g_1(\max\{x,y\};c)}$ is the absolute continuous part.

(ii) The cumulative function for the BIEF is

$$F_{\text{BIEF}}(x,y) = \frac{\beta_1 + \beta_2}{\beta}F_{\text{BIEF}(a)}(x,y) + \frac{\beta_0}{\beta}F_{\text{BIEF}(s)}(x,y), \tag{8}$$

where, $\beta = \beta_0 + \beta_1 + \beta_2$, $F_{\text{BIEF}(s)}(x,y) = e^{-\beta g_2(\min\{x,y\};c)}$ is the singular part and $F_{\text{BIEF}(a)}(x,y) = \frac{\beta}{\beta_1+\beta_2}e^{-\beta_1 g_2(x;c)-\beta_2 g_2(y;c)-\beta_0 g_2(\min\{x,y\};c)} - \frac{\beta_0}{\beta_1+\beta_2}e^{-\beta g_2(\min\{x,y\};c)}$ is the absolute continuous part.

Proof (i) For the BEF, using the fact that $\overline{F}_{\text{BEF}}(x,y) = \alpha\overline{F}_{\text{BEF}(a)}(x,y) + (1-\alpha)\overline{F}_{\text{BEF}(s)}(x,y)$

$$\frac{\partial^2 \overline{F}_{\text{BEF}}(x,y)}{\partial x \partial y} = \alpha f_{\text{BEF}(a)}(x,y) = \begin{cases} f_{\text{EF}}(x;\theta_0+\theta_1,c)f_{\text{EF}}(y;\theta_2,c), & \text{if } x > y \\ f_{\text{EF}}(x;\theta_1,c)f_{\text{EF}}(y;\theta_0+\theta_2,c), & \text{if } x < y \end{cases}$$

Hence α may be obtained as

$$\alpha = \int_0^\infty \int_0^x f_{\text{EF}}(x;\theta_0+\theta_1,c)f_{\text{EF}}(y;\theta_2,c)dydx + \int_0^\infty \int_0^y f_{\text{EF}}(x;\theta_1,c)f_{\text{EF}}(y;\theta_0+\theta_2,c)dxdy = \frac{\theta_1 + \theta_2}{\theta},$$

and $\overline{F}_{\text{BEF}(a)}(x,y) = \int_y^\infty \int_x^\infty f_{\text{BEF}(a)}(u,v)dudv$; hence, with α and $\overline{F}_{\text{BEF}(a)}(x,y)$ known, the singular part $\overline{F}_{\text{BEF}(s)}(x,y)$ can be obtained by subtraction.

(ii) Similarly for the BIEF, $F_{\text{BIEF}(a)}(x,y)$ is computed by using $F_{\text{BIEF}}(x,y) = \gamma F_{\text{BIEF}(a)}(x,y) + (1-\gamma)F_{\text{BIEF}(s)}(x,y)$, $0 \leq \gamma \leq 1$. Using a similar manner as in part (i), we can show that (8) holds.

The marginal distributions of X and Y and the conditional distributions are given by Theorems 5 and 6, while the distributions of min{X, Y}, for the BEF, and max{X, Y}, for the BIEF, are given by Theorem 7.

Theorem 5 If the vector (X, Y) has either BEF$(\theta_0, \theta_1, \theta_2; c)$ or BIEF$(\beta_0, \beta_1, \beta_2; c)$, then the marginal distributions of X and Y are either EF$(\theta_0, \theta_i; c)$ or IEF $(\beta_0, \beta_i; c)$, $i=1,2$, respectively.

Proof If (X, Y) has BEF$(\theta_0, \theta_1, \theta_2; c)$, then from (6) we have

$$f_X(x) = \int_0^x f_1(x,y)dy + \int_x^\infty f_2(x,y)dy + f_0(x) = (\theta_0 + \theta_1)g_1'(x;c)\, e^{-(\theta_0+\theta_1)g_1(x;c)}.$$

Similarly we can derive $f_Y(y)$. In a similar manner, $f_X(x)$ and $f_Y(y)$ can be shown to have IEF $(\beta_0, \beta_i; c)$, $i=1,2$, respectively for the BIEF.

Notice that the marginal distributions of X and Y can also be obtained using the next lemma.

Lemma 1

(i) Let $X = \min\{U_0, U_1\}$, then $X \sim EF(\theta_0 + \theta_1; c)$ iff U_0 and U_1 are independent and $U_0 \sim EF(\theta_0; c)$, $U_1 \sim EF(\theta_1; c)$.

(ii) Let $X = \max\{U_0, U_1\}$, then $X \sim IEF(\beta_0 + \beta_1; c)$ iff U_0 and U_1 are independent and $U_0 \sim IEF(\beta_0; c)$, $U_1 \sim IEF(\beta_1; c)$.

Here "~" means follows or has the distribution.

Proof (i) for $X = \min\{U_0, U_1\}$, we have

$$P(X > x) = P(\min\{U_0, U_1\} > x) = P(U_0 > x, U_1 > x).$$

If U_0 and U_1 are independent and $U_0 \sim EF(\theta_0; c)$ and $U_1 \sim EF(\theta_1; c)$ $U_1 \sim EF(\theta_1; c)$, then

$$P(X > x) = P(U_0 > x)P(U_1 > x) = e^{-(\theta_0+\theta_1)g_1(x;c)}.$$

Conversely, if $X \sim EF(\theta_0 + \theta_1; c)$, then

$$P(X > x) = e^{-(\theta_0+\theta_1)g_1(x;c)} = e^{-\theta_0 g_1(x;c)}e^{-\theta_1 g_1(x;c)}.$$

Then, U_0 and U_1 are independent and $\overline{F}_{U_0}(x) = e^{-\theta_0 g_1(x;c)}$ and $\overline{F}_{U_1}(x) = e^{-\theta_1 g_1(x;c)}$, i.e. $U_0 \sim EF(\theta_0; c)$ and $U_1 \sim EF(\theta_1; c)$.

(ii) Similarly for the BIEF.

Consequently, from Theorems 1 and 2 and Lemma 1, we have the following lemma, Lemma 2.

Lemma 2

(i) $(X, Y) \sim BEF(\theta_0, \theta_1, \theta_2; c)$ iff there exist independent EF random variables U_i, $i = 0, 1, 2$, such that $X = \min\{U_0, U_1\}$ and $Y = \min\{U_0, U_2\}$.

(ii) $(X, Y) \sim BIEF(\beta_0, \beta_1, \beta_2; c)$ if and only if there exist independent IEF random variables U_i, $i = 0, 1, 2$, such that $X = \max\{U_0, U_1\}$ and $Y = \max\{U_0, U_2\}$.

Theorem 6 The conditional distribution of X given $Y = y$, is given by

$$
f_{X|Y}(x|y) = \begin{cases} \dfrac{\theta_2(\theta_0 + \theta_1)}{\theta_0 + \theta_2}\, g_1'(x;c)\, e^{-(\theta_0+\theta_1)g_1(x;c)+\theta_0 g_1(y;c)}, \text{if } x > y \\ \theta_1 g_1'(x;c)\, e^{-\theta_1 g_1(x;c)}, \text{if } x < y \\ \dfrac{\theta_0}{\theta_0 + \theta_2}\, e^{-\theta_1 g_1(x;c)}, \text{if } x = y \end{cases} \tag{9}
$$

for the BEF, while for the BIEF is given by

$$
f_{X|Y}(x|y) = \begin{cases} -\beta_1 g_2'(x;c) e^{-\beta_1 g_2(x;c)}, \text{if } x > y \\ \dfrac{-\beta_2(\beta_0 + \beta_1)}{(\beta_0 + \beta_2)}\, g_2'(x;c) e^{-(\beta_0+\beta_1)g_2(x;c)+\beta_0 g_2(y;c)}, \text{if } x < y \\ \dfrac{\beta_0}{\beta_0 + \beta_2}\, e^{-\beta_1 g_2(x;c)}, \text{if } x = y \end{cases} \tag{10}
$$

Proof The proof is trivial so it is omitted.

Theorem 7 If (X, Y) is a bivariate vector of continuous random variables, then

(i) $\min\{X, Y\} \sim \mathrm{EF}(\theta; c)$, if $(X, Y) \sim \mathrm{BEF}(\theta_0, \theta_1, \theta_2; c)$,

(ii) $\max\{X, Y\} \sim \mathrm{IEF}(\beta; c)$, if $(X, Y) \sim \mathrm{BIEF}(\beta_0, \beta_1, \beta_2; c)$.

Proof (i) if $(X, Y) \sim \mathrm{BEF}(\theta_0, \theta_1, \theta_2; c)$, then using (4), we have

$$
P(\min\{X, Y\} > t) = P(X > t, Y > t) = e^{-\theta_1 g_1(t;c) - \theta_2 g_1(t;c) - \theta_0 g_1(t;c)} = e^{-\theta g_1(t;c)}.
$$

Similarly by using (5) for the BIEF, we can show that $\max\{X, Y\} \sim \mathrm{IEF}(\beta; c)$.

Stress-strength reliability

In this section, we present the stress-strength reliability of the two bivariate models. Many bivariate distributions in the literature have forms of the proposed models, for example, M-O bivariate exponential distribution, Marshal and Olkin [15], and the bivariate Rayleigh distribution introduced by Pak et al. [9] for the BEF and bivariate inverse Weibull and bivariate Burr type III for the BIEF. So the following theorem can be applied to many distributions possessing BEF or BIEF.

Theorem 8 Let (X, Y) be a bivariate vector. Then, the stress-strength reliability function, R, is given by

(i) $R = P(Y < X) = \dfrac{\theta_2}{\theta}$, $\tag{11}$

iff $(X, Y) \quad \mathrm{BEF}(\theta_0, \theta_1, \theta_2; c)$, where $\theta = \theta_0 + \theta_1 + \theta_2$.

(ii) $R = P(Y < X) = \dfrac{\beta_1}{\beta}$, $\tag{12}$

iff $(X, Y) \quad \mathrm{BIEF}(\beta_0, \beta_1, \beta_2; c)$, where $\beta = \beta_0 + \beta_1 + \beta_2$.

Proof (i) First, suppose that $(X, Y) \sim \mathrm{BEF}(\theta_0, \theta_1, \theta_2; c)$, then using (6),

$$
R = \int_0^\infty \int_0^x f_1(x, y)\, dy\, dx = \frac{\theta_2}{\theta}.
$$

Conversely, suppose that Eq. (11) holds. From Mokhlis et al. ([1], Theorem 1), since $R = \frac{\theta_2}{\theta_0 + \theta_1 + \theta_2}$, we have two independent random variables, say, X and U_2 where $X \sim EF(\theta_0 + \theta_1; c)$ and $U_2 \sim EF(\theta_2; c)$.

From Lemma 1, since $X \sim EF(\theta_0 + \theta_1; c)$, then $X = \min\{U_0, U_1\}$, where $U_0 \sim EF(\theta_0; c)$ and $U_1 \sim EF(\theta_1; c)$. Then,

$$P(U_2 < x) = P(U_2 < \min\{U_0, U_1\}) \equiv P(\min\{U_0, U_2\} < \min\{U_0, U_1\}).$$

Let $Y = \min\{U_0, U_2\}$. Thus, using Lemma 2, the proof is completed.

(ii) Similarly, suppose that $(X, Y) \sim \text{BIEF}(\beta_0, \beta_1, \beta_2; c)$, then using (6),

$$R = P(Y < X) = \frac{\beta_1}{\beta}.$$

Conversely, suppose that Eq. (12) holds. From Mokhlis et al. ([1], Theorem 2), since $R = \frac{\beta_1}{\beta_0 + \beta_1 + \beta_2}$, then we have two independent random variables, say U_1 and Y, where U_1 must be distributed as $\text{IEF}(\beta_1; c)$ and Y must be distributed as $\text{IEF}(\beta_0 + \beta_2; c)$. From Lemma 1, since $Y = \max\{U_0, U_2\}$, then $U_0 \sim \text{IEF}(\beta_0; c)$ and $U_2 \sim \text{IEF}(\beta_2; c)$. Thus, we have

$$R = P(\max\{U_0, U_2\} < U_1) \equiv P(\max\{U_0, U_2\} < \max\{U_0, U_1\}).$$

Let $X = \max\{U_0, U_1\}$. Using Lemma 2, the proof is completed.

Point estimation of R

Let (X_1, Y_1), ..., (X_n, Y_n) be a random sample of size n from either $\text{BEF}(\theta_0, \theta_1, \theta_2; c)$ or $\text{BIEF}(\beta_0, \beta_1, \beta_2; c)$, assuming c is known. Let n_1 be the number of observations having $y_i > x_i$ and n_2 be the number of observations having $y_i < x_i$ and n_0 be the number of observations having $y_i = x_i$ in the sample of size n, where $n = n_0 + n_1 + n_2$. Then, the non-parametric estimator of R is given by $\check{R} = \frac{n_2}{n}$, where n_2 is binomial (n, R). Thus, $E(\check{R}) = R$ and variance $V(\check{R}) = \frac{R}{n}(1-R)$.

Maximum likelihood estimators of R

Let (X_1, Y_1), ..., (X_n, Y_n) be a random sample of size n from either $\text{BEF}(\theta_0, \theta_1, \theta_2; c)$ or $\text{BIEF}(\beta_0, \beta_1, \beta_2; c)$, then the maximum likelihood estimator (MLE), $\hat{R}$, of R is given by

$$\hat{R} = \begin{cases} \dfrac{\hat{\theta}_2}{\hat{\theta}_0 + \hat{\theta}_1 + \hat{\theta}_2}, \text{for BEF}(\theta_0, \theta_1, \theta_2; c) \\ \dfrac{\hat{\beta}_1}{\hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2}, \text{for BIEF}(\beta_0, \beta_1, \beta_2; c) \end{cases} \tag{13}$$

Where $\hat{\theta}_i, \hat{\beta}_i$ are the maximum likelihood estimators of θ_i, β_i, i, = 0, 1, 2, respectively.

First, suppose that (X_1, Y_1), ..., (X_n, Y_n) is a random sample of size n from $\text{BEF}(\theta_0, \theta_1, \theta_2; c)$, then the MLE's $\hat{\theta}_i$ of θ_i, $i = 0, 1, 2$, can be obtained by writing the log-likelihood function $\log L = \sum_{i=0}^{2} n_i \log \theta_i + \sum_{i=1}^{2} n_i \log(\theta_0 + \theta_{3-i}) + \sum_{i=1}^{n} \log g_1'(x_i; c) +$

$\sum_{i=1, x_i \neq y_i}^{n} \log g_1'(y_i; c) - \theta_1 \sum_{i=1}^{n} g_1(x_i; c) - \theta_2 \sum_{i=1}^{n} g_1(y_i; c) - \theta_0 \sum_{i=1}^{n} g_1(\max\{x_i, y_i\}; c).$

and solving the likelihood system of equations w.r.t. θ_i, $i = 0, 1, 2$.

$$\frac{n_1}{\theta_1} + \frac{n_2}{\theta_0 + \theta_1} - \sum_{i=1}^{n} g_1(x_i; c) = 0,$$
$$\frac{n_2}{\theta_2} + \frac{n_1}{\theta_0 + \theta_2} - \sum_{i=1}^{n} g_1(y_i; c) = 0,$$
$$\frac{n_0}{\theta_0} + \frac{n_1}{\theta_0 + \theta_2} + \frac{n_2}{\theta_0 + \theta_1} - \sum_{i=1}^{n} g_1(\max\{x_i, y_i\}; c) = 0. \tag{14}$$

Similarly for the $BIEF(\beta_0, \beta_1, \beta_2; c)$, the MLE's $\hat{\beta}_i$ of β_i, $i = 0, 1, 2$, can be obtained by writing the log-likelihood function

$$\log L = n_0 \log \beta_0 + \sum_{i=1}^{2} n_i \log(\beta_{3-i}) + \sum_{i=1}^{2} n_i \log(\beta_0 + \beta_i) + \sum_{i=1}^{n} \log(-g_2'(x_i; c))$$
$$\sum_{i=1, x_i \neq y_i}^{n} \log(-g_2'(y_i; c)) - \beta_1 \sum_{i=1}^{n} g_2(x_i; c) - \beta_2 \sum_{i=1}^{n} g_2(y_i; c) - \beta_0 \sum_{i=1}^{n} g_2(\min\{x_i, y_i\}; c),$$

and solving the likelihood system of equations w.r.t. β_i, $i = 0, 1, 2$.

$$\frac{n_2}{\beta_1} + \frac{n_1}{\beta_0 + \beta_1} - \sum_{i=1}^{n} g_2(x_i; c) = 0,$$
$$\frac{n_1}{\beta_2} + \frac{n_2}{\beta_0 + \beta_2} - \sum_{i=1}^{n} g_2(y_i; c) = 0,$$
$$\frac{n_0}{\beta_0} + \frac{n_1}{\beta_0 + \beta_1} + \frac{n_2}{\beta_0 + \beta_2} - \sum_{i=1}^{n} g_2(\min\{x_i, y_i\}; c) = 0. \tag{15}$$

However, the previews likelihood systems of equations generated by either BEF(θ_0, θ_1, θ_2; c) or BIEF($\beta_0, \beta_1, \beta_2$; c) are computational inconvenient and can be solved numerically by using a Newton Raphson procedure or by Fisher's method of scoring.

Now, we introduce a simple estimator of R, depending on the marginal distributions of X and min{X, Y} for the BEF and depending on the marginal distributions of Y and max{X, Y} for the BIEF.

Let (X_1, Y_1), ..., (X_n, Y_n) be a random sample of size n from either BEF($\theta_0, \theta_1, \theta_2$; c) or BIEF($\beta_0, \beta_1, \beta_2$; c) then a simple estimator, $\tilde{R}$, of R is given by

$$\tilde{R} = \begin{cases} 1 - \dfrac{\sum_{i=1}^{n} g_1(\min\{x_i, y_i\}; c)}{\sum_{i=1}^{n} g_1(x_i; c)}, \text{ for BEF}(\theta_0, \theta_1, \theta_2; c) \\[3em] 1 - \dfrac{\sum_{i=1}^{n} g_2(\max\{x_i, y_i\}; c)}{\sum_{i=1}^{n} g_2(y_i; c)}, \text{ for BIEF}(\beta_0, \beta_1, \beta_2; c) \end{cases} \tag{16}$$

For the BEF, we have $X \sim EF(\theta_0 + \theta_1; c)$; thus, the MLE of $(\theta_0 + \theta_1)$, is given by (see Mokhlis et al. [1])

$$(\widehat{\theta_0 + \theta_1}) = \frac{n}{\sum_{i=1}^{n} g_1(x_i; c)}. \tag{17}$$

Similarly, since min{X, Y} $\sim EF(\theta; c)$, hence the MLE of θ is given by

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} g_1(\min\{x_i, y_i\}; c)}. \tag{18}$$

Thus,

$$\theta_2 = \frac{n}{\sum_{i=1}^{n} g_1(\min\{x_i, y_i\}; c)} - \frac{n}{\sum_{i=1}^{n} g_1(x_i; c)}. \tag{19}$$

Replacing the parameters in (11) by their estimators in (18) and (19) we get the simple estimator of R for the BEF.

Similarly for the BIEF model, $Y \sim \text{IEF}(\beta_0 + \beta_2; c)$; thus, the MLE of $(\beta_0 + \beta_2)$ is given by (see Mokhlis et al. [1])

$$\left(\widehat{\beta_0 + \beta_2}\right) = \frac{n}{\sum_{i=1}^{n} g_2(y_i; c)}, \tag{20}$$

and since $\max\{X, Y\} \sim \text{IEF}(\beta; c)$, then the MLE of β is given by

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n} g_2(\max\{x_i, y_i\}; c)} \tag{21}$$

Thus,

$$\hat{\beta}_1 = \frac{n}{\sum_{i=1}^{n} g_2(\max\{x_i, y_i\}; c)} - \frac{n}{\sum_{i=1}^{n} g_2(x_i; c)}. \tag{22}$$

Again replacing the parameters in (12) by their estimators given by (21) and (22), we obtain the simple estimator of R for the BIEF.

Estimation of R when the stress is censored at the strength

Sometimes, obtaining the estimate of R based on complete sample is neither possible nor desirable on account of lack of time or minimization of the experiment cost. Thus, there are some situations where the stress is censored at the strength (see Hanagel [16]).

Let (X_1, Y_1), ..., (X_n, Y_n) be a random sample of size n from $\text{BEF}(\theta_0, \theta_1, \theta_2; c)$; then, the strength and stress associated with the ith pair of sample is

$$(X_i, Y_i) = (x_i, x_i) \text{ if } x_i \le y_i$$
$$= (x_i, y_i) \text{ if } x_i > y_i$$

and the likelihood function can be written as

$$L = (\theta_0 + \theta_1)^n \theta_2^{n_2} \prod_{i=1}^{n} g_1'(x_i; c) \prod_{i=1}^{n_2} g_1'(y_i; c) e^{-(\theta_0 + \theta_1)\sum_{i=1}^{n} g_1(x_i; c) - \theta_2 \sum_{i=1}^{n} g_1(\min\{x_i, y_i\}; c)}$$

Easily, the likelihood equations are

$$\frac{n}{\theta_0 + \theta_1} - \sum_{i=1}^{n} g_1(x_i; c) = 0,$$

$$\frac{n_2}{\theta_2} - \sum_{i=1}^{n} g_1(\min\{x_i, y_i\}; c) = 0.$$

Thus, the MLE's $\overline{\theta_0 + \theta_1}, \overline{\theta_2}$ of $\theta_0 + \theta_1$ and θ_2, respectively, are

$$\overline{\theta_0 + \theta_1} = \frac{n}{\sum_{i=1}^{n} g_1(x_i; c)}, \text{ and } \overline{\theta_2} = \frac{n_2}{\sum_{i=1}^{n} g_1(\min\{x_i, y_i\}; c)},$$

then, the MLE, R, of R when the stress is censored at the strength is given by

$$\overline{R} = \left(1 + \frac{n \sum_{i=1}^{n} g_1(\min\{x_i, y_i\}; c)}{n_2 \sum_{i=1}^{n} g_1(x_i; c)}\right)^{-1}. \tag{23}$$

Special cases

Table 1 present some well-known bivariate distributions as members of the BEF(θ_0, θ_1, θ_2; c) or BIEF(β_0, β_1, β_2; c) and some other distributions for some choices of θ_i, β_i, $i = 0$, 1, 2, $g_1(x; c)$ and $g_2(x; c)$.

Clearly putting $c = 1$ and 2 in the bivariate inverse Weibull, we get bivariate inverse exponential and bivariate inverse Rayleigh distributions, respectively.

Notice that the bivariate modified Weibull distribution proposed by El-Bassiouny [13] is a special case of BEF(θ_0, θ_1, θ_2; c) where $\theta_1 = \alpha_1$, $\theta_2 = \alpha_2$, $\theta_0 = \alpha_3$, and $c = (\beta, \lambda)$. Also, the bivariate generalized Rayleigh distribution introduced by Sarhan [14] with shape parameter equals 1 is a special case of BIEF(β_0, β_1, β_2; c), where $\beta_0 = \beta_1 = \beta_2 = \lambda^2$ and $c = 1$.

Numerical illustrations

For illustrations of the results obtained in the previous sections numerically, a simulation study is performed; 1000 samples each of size, 10, 20, 30, and 50, are generated from some BEF and BIEF distributions. The reliability, R, is computed for the following cases.

Table 1 Summary of some special cases of the BEF($\boldsymbol{\theta_0}, \boldsymbol{\theta_1}, \boldsymbol{\theta_2}; \boldsymbol{c}$) or BIEF($\boldsymbol{\beta_0}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2}; \boldsymbol{c}$)

Distribution	BEF($\theta_0, \theta_1, \theta_2; c$)		BIEF($\beta_0, \beta_1, \beta_2; c$)		R
	θ_i	$g_1(x; c)$	β_i	$g_2(x; c)$	
M-O bivariate exponential	θ_i	x	–	–	$\frac{\theta_2}{\theta_0 + \theta_1 + \theta_2}$, see [7]
Bivariate Rayleigh	λ_i	x^2	–	–	$\frac{\lambda_2}{\lambda_0 + \lambda_1 + \lambda_2}$, see [9]
M-O bivariate Weibull	$\frac{1}{\theta_i^c}$	x^c	–	–	$\frac{\theta_0^c \theta_1^c}{\theta_0^c + \theta_1^c + \theta_2^c}$
M-O Pareto	a_i	$\ln\left(\frac{x}{c}\right)$	–	–	$\frac{a_2}{a_0 + a_1 + a_2}$
Bivariate inverse Weibull	–	–	$\frac{1}{\theta_i^c}$	x^{-c}	$\frac{\theta_0^c \theta_2^c}{\theta_0^c + \theta_1^c + \theta_2^c}$
Bivariate Burr type III	–	–	b_i	$\ln\left(\frac{1 + x^c}{x^c}\right)$	$\frac{b_1}{b_0 + b_1 + b_2}$

Case 1 (X, Y) has M-O bivariate exponential distribution with parameters $\theta_0 = 0.15$, $\theta_1 = 0.2$ and $\theta_2 = 0.5$.

Case 2 (X, Y) has bivariate Rayleigh distribution with parameters $\theta_0 = 0.2$, $\theta_1 = 0.25$ and $\theta_2 = 0.8$.

Case 3 (X, Y) has bivariate inverse exponential distribution with parameters $\beta_0 = 0.25$, $\beta_1 = 1$, and $\beta_2 = 0.35$.

Case 4 (X, Y) has bivariate inverse Rayleigh distribution with parameters $\beta_0 = 0.2$, $\beta_1 = 1.2$, and $\beta_2 = 0.1$.

It is to be noted that these values are chosen arbitrary just for illustrating the results obtained.

Tables 2 and 3 show the true value of R and its corresponding estimate by using maximum likelihood method, ($R^M = \hat{R}$), our simple method of estimation ($R^S = \tilde{R}$), and when the stress is censored at the strength, the estimate of R is denoted by ($R^C = \bar{R}$) and non-parametric estimate ($R^N = \breve{R}$). The values $R^{(M)}$, $R^{(S)}$, $R^{(C)}$ and $R^{(N)}$ that appear in Tables 2 and 3 are the mean of the 1000 replicates of the corresponding estimates. For comparison, we calculate the bias (b) and mean square error (MSE), of the different estimates in each case considered. Where bias (b) is the difference of the mean of the 1000 replicates estimates from the true values of R and MSE is the mean of the squares of the differences of the 1000 replicates estimates from the true values of R. The calculations are performed by applying the Maple program.

From Tables 2 and 3, we see that all estimates converge to R, when n increases and MSE decreases. In Table 2, we see $\text{MSE}^{(M)} < \text{MSE}^{(C)} < \text{MSE}^{(S)} < \text{MSE}^{(N)}$, while in Table 3, $\text{MSE}^{(M)} < \text{MSE}^{(S)} < \text{MSE}^{(N)}$. However, the $R^{(C)}$, when the stress is censored at the strength for the BEF, and $R^{(s)}$, for the BEF and BIEF, both estimators are simple, easier in computation and provide sufficient results for biasedness and mean square error.

Real data example

In real life, there are many situations where we have $X < Y$, $Y < X$ or $X = Y$, such as nuclear reactor safety, competing risks, (see Kotz et al. [17]). In the medical field X and Y

Table 2 Reliability of special cases of BEF

N	Case 1, $R = 0.5882$				Case 2, $R = 0.6400$			
	10	20	30	50	10	20	30	50
$R^{(M)}$	0.5475	0.5860	0.5869	0.5886	0.6102	0.6334	0.6395	0.6391
$R^{(C)}$	0.5429	0.5809	0.5840	0.5862	0.6069	0.6319	0.6377	0.6389
$R^{(S)}$	0.5353	0.5774	0.5809	0.5831	0.6096	0.6292	0.6355	0.6371
$R^{(N)}$	0.5458	0.5859	0.5886	0.5905	0.5956	0.6347	0.6416	0.6419
$\text{MSE}^{(M)}$	0.0158	0.0079	0.0048	0.0031	0.0135	0.0062	0.0041	0.0025
$\text{MSE}^{(C)}$	0.0179	0.0079	0.0051	0.0034	0.0147	0.0068	0.0045	0.0026
$\text{MSE}^{(S)}$	0.0259	0.0099	0.0067	0.0043	0.0177	0.0091	0.0054	0.0033
$\text{MSE}^{(N)}$	0.0280	0.0119	0.0081	0.0050	0.0185	0.0105	0.0076	0.0043
$b^{(M)}$	0.0408	0.0022	0.0012	$-.0003$	0.0298	0.0066	0.0004	0.0009
$b^{(C)}$	0.0453	0.0073	0.0042	0.0020	0.0331	0.0081	0.0023	0.0011
$b^{(S)}$	0.0529	0.0108	0.0073	0.0051	0.0304	0.0108	0.0045	0.0029
$b^{(N)}$	0.0424	0.0023	$-.0004$	$-.0022$	0.0444	0.0052	-0.0016	-0.0019

Table 3 Reliability of special cases of BIEF

N	Case 3, $R = 0.625$				Case 4, $R = 0.8000$			
	10	20	30	50	10	20	30	50
$R^{(M)}$	0.5946	0.6172	0.6246	0.6231	0.7494	0.7875	0.7952	0.7988
$R^{(S)}$	0.5881	0.6114	0.6200	0.6202	0.7607	0.7888	0.7957	0.7982
$R^{(N)}$	0.5908	0.6211	0.6264	0.6258	0.6986	0.7790	0.7914	0.7994
$MSE^{(M)}$	0.0135	0.0061	0.0041	0.0025	0.0097	0.0028	0.0017	0.0010
$MSE^{(S)}$	0.0193	0.0087	0.0061	0.0033	0.0106	0.0034	0.0020	0.0012
$MSE^{(N)}$	0.0197	0.0105	0.0065	0.0049	0.0202	0.0061	0.0047	0.0031
$b^{(M)}$	0.0304	0.0078	0.0004	0.0019	0.0506	0.0125	0.0048	0.0012
$b^{(S)}$	0.0369	0.0136	0.0050	0.0048	0.0393	0.0112	0.0043	0.0018
$b^{(N)}$	0.0342	0.0039	− .0014	− 0.0008	0.1014	0.0210	0.0086	0.0006

can represent the blood pressure or count of the white blood cells for patients before and after a certain operation.

The following data set is from the American Football (National Football League) matches for three consecutive weekends in 1986. The data was first published in the "Washington Post" and available in Csörgő and Welsh [18] Table 4.

The bivariate variables X and Y are as follows: X represents the game time to the first points scored by kicking the ball between goal posts and Y represents the game time to the first points scored by moving the ball into the end zone. This data was first analyzed by Csörgő and Welsh [18], by converting the seconds to decimal minutes. Also Kundu and Gupta [19] and Jamalizadeh and Kundu [20] analyzed this data.

We consider BEF and BIEF for fitting this data set. First, we fit each EF and IEF to X and Y separately. The data fit two cases, namely exponential which is special case of the EF and inverse exponential which is special case of the IEF, respectively. In case of exponential distribution, the MLEs of the scale parameters of X and Y are 0.1102 and 0.0745, respectively, while for the inverse exponential the MLEs of the scale parameters are 4.4000 and 5.0214, respectively.

Table 4 American Football League data

X	Y	X	Y	X	Y
2.05	3.98	5.78	25.98	10.40	14.25
9.05	9.05	13.80	49.75	2.98	2.98
0.85	0.85	7.25	7.25	3.88	6.43
3.43	3.43	4.25	4.25	0.75	0.75
7.78	7.78	1.65	1.65	11.63	17.37
10.57	14.28	6.42	15.08	1.38	1.38
7.05	7.05	4.22	9.48	10.35	10.35
2.58	2.58	15.53	15.53	12.13	12.13
7.23	9.68	2.90	2.90	14.58	14.58
6.85	34.58	7.02	7.02	11.82	11.82
32.45	42.35	6.42	6.42	5.52	11.27
8.53	14.57	8.98	8.98	19.65	10.70
31.13	49.88	10.15	10.15	17.83	17.83
14.58	20.57	8.87	8.87	10.85	38.07

The Kolmogorov-Smirnov distances between the fitted distribution and the empirical distribution function for X and Y are 0.14997 and 0.1182 respectively for the exponential case, while those for the inverse exponential case are 0.1530 and 0.1955. The above values are less than the critical value $D_{0.05} \cong 0.2099$, for n = 42, so that each of exponential distribution and inverse exponential distribution is an appropriate fit for the given data. This means that there may exist three independent random variables, say U_i, i = 1, 2, 3, with EF or IEF thus X = min $\{U_0, U_1\}$ or max$\{U_0, U_1\}$ and Y = min $\{U_0, U_2\}$ or max$\{U_0, U_2\}$.

Now, we try to test whether M-O bivariate exponential distribution or bivariate inverse exponential distribution provides better fit to the above data set. We use the Akaike information criterion (AIC) to check the model validity. Based on the above data, the MLEs of parameters for the M-O bivariate exponential distribution θ_0 = 0.0715, θ_1 = 0.0456, and θ_2 = 0.0030, and the MLEs of parameters for the bivariate inverse exponential distribution are β_0 = 4.2769, β_1 = 0.1746, and β_2 = 2.0715. Thus, for the case of M-O bivariate exponential the log-likelihood value is − 227.9347 and the corresponding AIC is 461.8694, while for bivariate inverse exponential distribution the log-likelihood value is − 249.6874 and the AIC is 505.3748. Therefore, M-O bivariate exponential provides better fit than bivariate inverse exponential distribution. We estimate the reliability parameter R using the corresponding MLEs $\hat{\theta}_i$ θ_i, i = 0, 1, 2 for the M-O bivariate exponential distribution is R = 0.0248, while using the proposed simple estimators, we have R^S = 0.0235 and R^C = 0.0238 and the non-parametric estimator R^N = 0.0238.

Conclusions

In this paper, we have suggested two forms of bivariate distributions, BEF and BIEF, with marginal distributions having a general exponential form or inverse exponential form. Some distributions in the literature belong to these families, such as the M-O bivariate exponential distribution, Marshall and Olkin [15], and bivariate Rayleigh distribution, Pak et al. [9]. Other bivariate distributions could belong to these families such as bivariate Weibull and bivariate Burr type III and others according to the form of $g_1(x; c)$ or $g_2(x; c)$. We discussed some properties of the proposed families and studied the stress-strength reliability parameter, $R = P(Y < X)$. The MLEs of the distribution parameters are derived and simple estimators of R based on some marginal distributions are introduced in case of complete sampling. When the stress is censored at the strength, an explicit estimator of R is also obtained for the BEF distribution. Some bivariate members of the proposed families are presented. A simulation study is performed showing that the proposed simple estimators of R are easier in computation and provide sufficient results with respect to biasedness and mean square error. An example of a real data of bivariate variables (X, Y) belonging to the proposed family is also introduced.

Author details
[1]Department of Mathematics, Faculty of Women, Ain Shams University, Cairo, Egypt. [2]Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt.

References
1. Mokhlis, N.A., Ibrahim, E.J., Gharieb, M.D.: Stress-strength reliability with general form distributions. Commun. Stat. Theory. Methods. **46**(3), 1230–1246 (2017)
2. Mokhlis, N.A.: Reliability of a stress-strength model with Burr type III distributions. Commun. Stat. Theory. Methods. **34**(7), 1643–1657 (2005)
3. Kundu, D., Gupta, R.D.: Estimation of P(Y<X) for Weibull distribution. IEEE. Trans. Reliability. **55**, 270–280 (2006)
4. Singh, S.K., Singh, U., Singh Yadav, A., Vishwkarma, P.K.: On the estimation of stress-strength reliability parameter of inverted exponential distribution. Int. J. Sci. World. **3**, 98–112 (2015)
5. Kotz, S., Lumelskii, Y., Pensky, M.: The Stress-Strength Model and Its Generalizations: Theory and Applications. World Scientific, Singapore (2003)
6. Mokhlis, N.M.: Reliability of strength model with a bivariate exponential distribution. J. Egypt. Math. Soc. **14**, 69–78 (2006)
7. Nadarjah, S., Kotz, S.: Reliability for some bivariate exponential distributions. Math. Probl. Eng. **2006**, 1–14 (2006)
8. Nguimkeu, P., Rekkas, M., Wong, A.: Interval estimation for the stress-strength reliability with bivariate normal variables. Open J. Stat. **4**, 630–640 (2014)
9. Pak, A., Khoolenjani, N.B., Jafari, A.A.: Inference on P (Y< X) in bivariate Rayleigh distribution. Commun. Stat. Theory. Methods. **43**(22), 4881–4892 (2014)
10. Abdel-Hamid, A.H.: Stress-strength reliability for general bivariate distributions. J. Egypt. Math. Soc. **5**, 617–621 (2016)
11. Kolesárová, A., Mesiar, R., Saminger-Platz, S.: Generalized Farlie-Gumbel-Morgenstern Copulas, International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, pp. 244–252. Springer, Cham (2018)
12. Arnold, B.C., Arvanitis, M.A.: On bivariate pseudo-exponential distributions. J. Appl. Stat. 1–13 (2019). https://doi.org/10.1080/02664763.2019.1686132
13. El-Bassiouny, A.H., Shahen, H.S., Abouhawwash, M.: A new bivariate modified Weibull distribution and its extended distribution. J. Stat. Appl. Probability. **7**(2), 217–231 (2018)
14. Sarhan, A.: The bivariate generalized Rayleigh distribution. J. Math. Sci. Mod. **2**(2), 99–111 (2019)
15. Marshall, A.W., Olkin, I.: A multivariate exponential distribution. J. Am. Stat. Assoc. **62**, 30–44 (1967)
16. Hanagel, D.: Estimation of reliability when stress is censored at strength. Commun. Stat. Theory. Methods. **26**(4), 911–919 (1997)
17. Kotz, S., Balakrishnan, N., Johnson, N.L.: Continuous Multivariate Distributions, vol. 1. Models and Applications, 1, 2nd ed., John Wiley & Sons, New York (2004)
18. Csörgő, S., Welsh, A.H.: Testing for exponential and Marshall–Olkin distributions. J. Stat. Plann. Inference. **23**(3), 287–300 (1989)
19. Kundu, D., Gupta, R.D.: Modified Sarhan-Balakrishnan singular bivariate distribution. J. Stat. Plann. Inference. **140**(2), 526–538 (2010)
20. Jamalizadeh, A., Kundu, D.: Weighted Marshall–Olkin bivariate exponential distribution. Statistics. **47**(5), 917–928 (2013)

Publisher's Note