Journal of the Egyptian Mathematical Society (2015) 23, 161–166

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

On generalized ϕ-recurrent LP-Sasakian Manifolds

Absos Ali Shaikh [a,*], D.G. Prakasha [b], Helaluddin Ahmad [a]

[a] *Department of Mathematics, University of Burdwan, Burdwan, 713 104 West Bengal, India*
[b] *Department of Mathematics, Karnatak University, Dharwad, 580 003 Karnataka, India*

Received 9 February 2013; revised 27 September 2013; accepted 25 December 2013
Available online 22 February 2014

KEYWORDS

Recurrent manifold;
Generalized recurrent manifold;
LP-Sasakian manifold;
ϕ-recurrent LP-Sasakian manifold;
Generalized ϕ-recurrent LP-Sasakian manifold

Abstract The object of the present paper was to introduce the notion of *generalized ϕ-recurrent LP-Sasakian manifold* and study its various geometric properties with the existence by an interesting example. Among others it is shown that a generalized ϕ-recurrent LP-Sasakian manifold is an Einstein manifold. Also it is proved that there exists no generalized projectively ϕ-recurrent LP-Sasakian manifold.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 53C15; 53C25

1. Introduction

On the analogy of Sasakian manifolds, in 1989 Matsumoto [1] introduced the notion of LP-Sasakian manifolds. Again the same notion is introduced by Mihai and Rosca [2] and obtained many interesting results. LP-Sasakian manifolds are also studied by Aqeel et al. [3], Bagewadi et al. [4], De et al. [5], Mihai et al. [6], Murathan et al. [7], Shaikh et al. [8–12], Venkatesha and Bagewadi [13] and others. The study of Riemann symmetric manifolds began with the work of Cartan [14]. The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, Takahashi [15] introduced the notion of local ϕ-symmetry on a Sasakian manifold. Generalizing the notion of local ϕ-symmetry of Takahashi [15], De et al. [16] introduced the notion of ϕ-recurrent Sasakian manifolds. Also De et al. [17] studied the ϕ-recurrent Kenmotsu manifolds. In this connection, it may be mentioned that Shaikh and Hui [18] studied the locally ϕ-symmetric β-Kenmotsu manifolds.

In the context of Lorentzian geometry, the notion of local ϕ-symmetry on an LP-Sasakian manifold is introduced and studied by Shaikh and Baishya [9] with several examples. Also Shaikh et al. [11] studied ϕ-recurrent LP-Sasakian manifolds and proved the existence of such notion by several non-trivial examples. Generalizing all these notions of local ϕ-symmetry, in the present paper we introduce *generalized ϕ-recurrent LP-Sasakian manifolds*.

The notion of generalized recurrent manifolds introduced by Dubey [19] and studied by De and Guha [20]. A Riemannian manifold (M^n, g), $n > 2$, is called generalized recurrent [20] if its curvature tensor R of type $(1, 3)$ satisfies the condition

$$\nabla R = A \otimes R + B \otimes G, \tag{1.1}$$

* Corresponding author. Tel.: +91 9434546184.
E-mail addresses: aashaikh@math.buruniv.ac.in, aask2003@yahoo.co.in (A.A. Shaikh), prakashadg@gmail.com (D.G. Prakasha).

Peer review under responsibility of Egyptian Mathematical Society.

ELSEVIER Production and hosting by Elsevier

1110-256X © 2014 Production and hosting by Elsevier B.V. on behalf of Egyptian Mathematical Society.
http://dx.doi.org/10.1016/j.joems.2013.12.019

where A and B are non-vanishing 1-forms defined by $A(\cdot) = g(\cdot, \rho_1)$, $B(\cdot) = g(\cdot, \rho_2)$ and the tensor G of type $(1,3)$ is given by

$$G(X,Y)Z = g(Y,Z)X - g(X,Z)Y \tag{1.2}$$

for all $X, Y, Z \in \chi(M)$; $\chi(M)$ being the Lie algebra of smooth vector fields on M and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. The 1-forms A and B are called the associated 1-forms of the manifold.

Specially, if the 1-form B vanishes, then (1.1) turns into the notion of recurrent manifold introduced by Walker [21].

A Riemannian manifold (M^n, g), $n > 2$, is called a generalized Ricci-recurrent manifold [22] if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$\nabla S = A \otimes S + B \otimes g, \tag{1.3}$$

where A and B are non-vanishing 1-forms defined in (1.1).

In particular, if $B = 0$, then (1.3) reduces to the notion of Ricci-recurrent manifold introduced by Patterson [23].

A Riemannian manifold (M^n, g), $n > 2$, is called a super generalized Ricci-recurrent manifold [24] if its Ricci tensor S of type $(0,2)$ satisfies the condition

$$\nabla S = \alpha \otimes S + \beta \otimes g + \gamma \otimes \eta \otimes \eta, \tag{1.4}$$

where α, β and γ are non-vanishing unique 1-forms.

In particular, if $\beta = \gamma$, then (1.4) reduces to the notion of quasi-generalized Ricci-recurrent manifold introduced by Shaikh and Roy [25].

The paper is organized as follows. Section 2 is concerned with some preliminaries about LP-Sasakian manifolds. Section 3 deals with *generalized ϕ-recurrent LP-Sasakian manifolds* and obtained a necessary and sufficient condition for an LP-Sasakian manifold to be a generalized ϕ-recurrent LP-Sasakian manifold (see, Theorem 3.3). It is shown that a generalized ϕ-recurrent LP-Sasakian manifold is Einstein manifold and in such a manifold the 1-forms A and B are related by $A + B = 0$. We also study *generalized concircularly (resp., projectively) ϕ-recurrent LP-Sasakian manifolds*. It is proved that a generalized concircularly ϕ-recurrent LP-Sasakian manifold is super generalized Ricci-recurrent, but it can not be quasi-generalized Ricci-recurrent. Finally, the last section deals with an interesting example, which ensures the existence of generalized ϕ-recurrent LP-Sasakian manifold.

2. LP-Sasakian manifolds

An n-dimensional smooth manifold M is said to be an LP-Sasakian manifold [2,8] if it admits a $(1,1)$ tensor field ϕ, a unit timelike vector field ξ, an 1-form η and a Lorentzian metric g, which satisfy

$$\eta(\xi) = -1,\ g(X,\xi) = \eta(X),\ \phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y),\ \nabla_X \xi = \phi X, \tag{2.2}$$

$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.3}$$

for all X, $Y \in \chi(M)$, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. It can be easily seen that in an LP-Sasakian manifold, the following relations hold:

$$\phi\xi = 0,\ \eta(\phi X) = 0, \quad \text{rank}\ \phi = n-1. \tag{2.4}$$

Again, if we put

$$\Omega(X,Y) = g(X, \phi Y)$$

for all X, $Y \in \chi(M)$, then the tensor field $\Omega(X,Y)$ is a symmetric $(0,2)$ tensor field [1]. Also, since the 1-form η is closed in an LP-Sasakian manifold, we have [1,5]

$$(\nabla_X \eta)(Y) = \Omega(X,Y), \Omega(X,\xi) = 0 \tag{2.5}$$

for all vector fields X, $Y \in \chi(M)$.

Let M be an n-dimensional LP-Sasakian manifold with structure (ϕ, ξ, η, g). Then the following relations hold [1,8]:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.6}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{2.7}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.8}$$

$$\begin{aligned}(\nabla_W R)(X,Y)\xi = {} & 2[\Omega(Y,W)X - \Omega(X,W)Y] - \phi R(X,Y)W \\ & - g(Y,W)\phi X + g(X,W)\phi Y \\ & - 2[\Omega(X,W)\eta(Y) - \Omega(Y,W)\eta(X)]\xi \\ & - 2[\eta(Y)\phi X - \eta(X)\phi Y]\eta(W).\end{aligned} \tag{2.9}$$

$$\begin{aligned}(\nabla_W R)(X,\xi)Z = {} & \Omega(W,Z)X - g(X,Z)\phi W \\ & - R(X,\phi W)Z\end{aligned} \tag{2.10}$$

for all vector fields X, Y, $Z \in \chi(M)$, where R is the curvature tensor of the manifold.

The above results will be used in the later sections.

3. Generalized ϕ-recurrent LP-Sasakian manifolds

Definition 3.1. A tensor field K of type $(1,3)$ on an LP-Sasakian manifold (M^n, g), $n > 2$, is said to be *generalized K-ϕ-recurrent* if it satisfies the relation

$$\begin{aligned}\phi^2((\nabla_W K)(X,Y)Z) = {} & A(W)\phi^2(K(X,Y)Z) \\ & + B(W)\phi^2(G(X,Y)Z)\end{aligned} \tag{3.1}$$

for all X, Y, Z, $W \in \chi(M)$, where A and B are non-vanishing 1-forms such that $A(X) = g(X, \rho_1)$, $B(X) = g(X, \rho_2)$. The 1-forms A and B are called the associated 1-forms of the manifold.

In particular, if $K = R$ (resp., $\widetilde{C}$, P) then the LP-Sasakian manifold (M^n, g), $n > 2$, is said to be generalized ϕ-recurrent (resp., generalized concircularly ϕ-recurrent, generalized projectively ϕ-recurrent), where $\widetilde{C}$ and P denote the concircular and projective curvature tensor of type $(1,3)$ and are respectively given by

$$\widetilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)} G(X,Y)Z \tag{3.2}$$

and

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]. \tag{3.3}$$

We consider a generalized ϕ-recurrent LP-Sasakian manifold. Then by virtue of (2.1), we have from (3.1) that

$$\begin{aligned}&(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi \\ &\quad = A(W)[R(X,Y)Z + \eta(R(X,Y)Z)\xi] \\ &\qquad + B(W)[G(X,Y)Z + \eta(G(X,Y)Z)\xi],\end{aligned} \tag{3.4}$$

from which it follows that

$$g((\nabla_W R)(X,Y)Z,U)+\eta((\nabla_W R)(X,Y)Z)\eta(U)$$
$$=A(W)[g(R(X,Y)Z,U)+\eta(R(X,Y)Z)\eta(U)]$$
$$+B(W)[g(G(X,Y)Z,U)+\eta(G(X,Y)Z)\eta(U)]. \quad (3.5)$$

Taking an orthonormal frame field and then contracting (3.5) over X and U and then using (1.2), we get

$$(\nabla_W S)(Y,Z)=-g((\nabla_W R)(\xi,Y)Z,\xi)+A(W)S(Y,Z)+\{(n-2)B(W)$$
$$-A(W)\}g(Y,Z)-\{A(W)+B(W)\}\eta(Y)\eta(Z). \quad (3.6)$$

Using (2.6), (2.9) and the relation $g((\nabla_W R)(X,Y)Z,U) = -g((\nabla_W R)(X,Y)U,Z)$, we have

$$g((\nabla_W R)(\xi,Y)Z,\xi)=0. \quad (3.7)$$

By virtue of (3.7), it follows from (3.6) that

$$(\nabla_W S)(Y,Z)=A(W)S(Y,Z)+[(n-2)B(W)-A(W)]g(Y,Z)$$
$$-\{A(W)+B(W)\}\eta(Y)\eta(Z). \quad (3.8)$$

Setting $Z=\xi$ in (3.8) and using (2.7), we obtain

$$(\nabla_W S)(Y,\xi)=(n-1)\{A(W)+B(W)\}\eta(Y). \quad (3.9)$$

Also we have

$$(\nabla_W S)(Y,\xi)=\nabla_W S(Y,\xi)-S(\nabla_W Y,\xi)-S(Y,\nabla_W\xi) \quad (3.10)$$
$$=(n-1)g(Y,\phi W)-S(Y,\phi W), \text{using (2.5) and (2.7)}.$$

From (3.9) and (3.10), we obtain

$$(n-1)g(Y,\phi W)-S(Y,\phi W)$$
$$=(n-1)\{A(W)+B(W)\}\eta(Y). \quad (3.11)$$

Again plugging $Y=\xi$ in (3.11), we get

$$A(W)+B(W)=0 \text{ for all } W. \quad (3.12)$$

By virtue of (3.12), it follows from (3.11) that

$$S(Y,\phi W)=(n-1)g(Y,\phi W). \quad (3.13)$$

Substituting Y by ϕY in (3.13) and then using (2.2) and (2.8), we get

$$S(Y,W)=(n-1)g(Y,W).$$

This leads to the following:

Theorem 3.1. *A generalized ϕ-recurrent LP-Sasakian manifold is an Einstein manifold and moreover the associated 1-forms A and B are related by $A+B=0$.*

Corollary 3.1. *[8] An LP-Sasakian manifold is Ricci-semisymmetric if and only if it is Einstein.*

Thus from Theorem 3.1 and Corollary 3.1, we can state the following:

Theorem 3.2. *A generalized ϕ-recurrent LP-Sasakian manifold is Ricci-semisymmetric.*

Using (2.9) and the relation $g((\nabla_W R)(X,Y)Z,U)=-g((\nabla_W R)(X,Y)U,Z)$ in (3.4), we have

$$(\nabla_W R)(X,Y)Z$$
$$=[g(X,W)g(\phi Y,Z)-g(Y,W)g(\phi X,Z)+2\{g(Y,\phi W)g(X,Z)$$
$$-g(X,\phi W)g(Y,Z)\}-2\{g(X,\phi W)\eta(Y)-g(Y,\phi W)\eta(X)\}\eta(Z)$$
$$-2\{\eta(Y)g(\phi X,Z)-\eta(X)g(\phi Y,Z)\}\eta(W)-g(\phi R(X,Y)W,Z)]\xi$$
$$+A(W)[R(X,Y)Z+\eta(R(X,Y)Z)\xi]+B(W)[g(Y,Z)X$$
$$+g(Y,Z)\eta(X)\xi-g(X,Z)Y-g(X,Z)\eta(Y)\xi]. \quad (3.14)$$

Applying ϕ^2 on both sides, we get the relation (3.1). Hence we can state the following:

Theorem 3.3. *An LP-Sasakian manifold is generalized ϕ-recurrent if and only if the relation (3.14) holds.*

Changing W, X, Y cyclically in (3.5) and adding them, we get by virtue of Bianchi identity and (3.12) that

$$A(W)[g(R(X,Y)Z,U)-g(G(X,Y)Z,U)$$
$$+\{\eta(R(X,Y)Z)-\eta(G(X,Y)Z)\}\eta(U)]$$
$$+A(X)[g(R(Y,W)Z,U)-g(G(Y,W)Z,U)$$
$$+\{\eta(R(Y,W)Z)-\eta(G(Y,W)Z)\}\eta(U)]$$
$$+A(Y)[g(R(W,X)Z,U)-g(G(W,X)Z,U)$$
$$+\{\eta(R(W,X)Z)-\eta(G(W,X)Z)\}\eta(U)]=0.$$

Contracting the above relation over Y and Z, we get

$$A(W)[S(X,U)-(n-1)g(X,U)]-A(X)[S(W,U)-(n-1)g(W,U)]$$
$$A(R(W,X)U)-A(R(W,X)\xi)\eta(U)-A(X)g(W,U)+A(W)g(X,U)$$
$$-\{A(X)\eta(W)-A(W)\eta(X)\}\eta(U)=0. \quad (3.15)$$

Again contracting (3.15) over X and U and using (2.6), we get

$$S(W,\rho_1)=\frac{r-(n-1)(n-2)}{2}g(Y,\rho_1). \quad (3.16)$$

This leads to the following:

Theorem 3.4. *In a generalized ϕ-recurrent LP-Sasakian manifold, $\frac{r-(n-1)(n-2)}{2}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ_1.*

We now consider a generalized concircularly ϕ-recurrent LP-Sasakian manifold (M^n,g), $n>2$. Then from (3.1) we have

$$\phi^2\left(\left(\nabla_W\widetilde{C}\right)(X,Y)Z\right)=A(W)\phi^2\left(\widetilde{C}(X,Y)Z\right)$$
$$+B(W)\phi^2(G(X,Y)Z), \quad (3.17)$$

where A and B are defined as in (3.1).

Let us consider a generalized concircularly ϕ-recurrent LP-Sasakian manifold. Then by virtue of (2.1), it follows from (3.17) that

$$\left(\nabla_W\widetilde{C}\right)(X,Y)Z=-\eta\left(\left(\nabla_W\widetilde{C}\right)(X,Y)Z\right)\xi$$
$$+A(W)\left[\widetilde{C}(X,Y)Z+\eta\left(\widetilde{C}(X,Y)Z\right)\xi\right]$$
$$+B(W)[G(X,Y)Z+\eta(G(X,Y)Z)\xi], \quad (3.18)$$

from which it follows that

$$g\left(\left(\nabla_W\widetilde{C}\right)(X,Y)Z,U\right)=-\eta\left(\left(\nabla_W\widetilde{C}\right)(X,Y)Z\right)\eta(U)$$
$$+A(W)\left[g\left(\widetilde{C}(X,Y)Z,U\right)+\eta\left(\widetilde{C}(X,Y)Z\right)\eta(U)\right]$$
$$+B(W)[g(G(X,Y)Z,U)+\eta(G(X,Y)Z)\eta(U)]. \quad (3.19)$$

Contracting (3.19) over X and U and using (2.10) and (3.2), we get

$$(\nabla_W S)(Y,Z)=A(W)S(Y,Z)$$
$$+(n-2)\left[B(W)+\frac{dr(W)}{n(n-1)}-\left\{\frac{r}{n(n-1)}+\frac{1}{n-2}\right\}A(W)\right]$$
$$g(Y,Z)-\left[\frac{dr(W)}{n(n-1)}+\left\{1-\frac{r}{n(n-1)}\right\}A(W)+B(W)\right]\eta(Y)\eta(Z), \quad (3.20)$$

which can be written as

$$\nabla S = A \otimes S + \psi \otimes g + H \otimes \eta \otimes \eta, \tag{3.21}$$

where

$$\psi(W) = (n-2)\left[B(W) + \frac{dr(W)}{n(n-1)} - \left\{\frac{r}{n(n-1)} + \frac{1}{n-2}\right\}A(W)\right]$$

and

$$H(W) = -\left[\frac{dr(W)}{n(n-1)} + \left\{1 - \frac{r}{n(n-1)}\right\}A(W) + B(W)\right].$$

Hence we can state the following:

Theorem 3.5. *A generalized concircularly ϕ-recurrent LP-Sasakian manifold is a super generalized Ricci-recurrent manifold.*

Setting $Y = Z = \xi$ in (3.20) and using (2.7), we get

$$dr(W) = \{r - n(n-1)\}A(W) - n(n-1)B(W). \tag{3.22}$$

This leads to the following:

Theorem 3.6. *In a generalized concircularly ϕ-recurrent LP-Sasakian manifold, the 1-forms A and B are related by the equation (3.22).*

Corollary 3.2. *In a generalized concircularly ϕ-recurrent LP-Sasakian manifold with non-zero constant scalar curvature, the associated 1-forms A and B are related by*

$$\{r - n(n-1)\}A - n(n-1)B = 0.$$

We now consider a generalized concircularly ϕ-recurrent LP-Sasakian manifold which is quasi-generalized Ricci-recurrent [25]. Then $\psi(W) = H(W)$, where ψ and H are 1-forms given in (3.21). The equality of ψ and H gives us

$$dr(W) = rA(W) - n(n-1)B(W). \tag{3.23}$$

From (3.22) and (3.23), we obtain $A = 0$, which is inadmissible. Hence we can state the following:

Theorem 3.7. *A generalized concircularly ϕ-recurrent LP-Sasakian manifold can not be a quasi-generalized Ricci-recurrent manifold.*

We now consider a generalized projectively ϕ-recurrent LP-Sasakian manifold (M^n, g), $n > 2$. Then from (3.1) we have

$$\phi^2((\nabla_W P)(X, Y)Z) = A(W)\phi^2(P(X, Y)Z) + B(W)\phi^2(G(X, Y)Z), \tag{3.24}$$

where A and B are defined as in (3.1).

Let us consider a generalized projectively ϕ-recurrent LP-Sasakian manifold. Then by virtue of (2.1), it follows from (3.24) that

$$\begin{aligned}(\nabla_W P)(X, Y)Z = &-\eta((\nabla_W P)(X, Y)Z)\xi \\ &+ A(W)[P(X, Y)Z + \eta(P(X, Y)Z)\xi] \\ &+ B(W)[G(X, Y)Z + \eta(G(X, Y)Z)\xi],\end{aligned} \tag{3.25}$$

from which it follows that

$$\begin{aligned}g((\nabla_W P)(X, Y)Z, U) = &-\eta((\nabla_W P)(X, Y)Z)\eta(U) \\ &+ A(W)[g(P(X, Y)Z, U) + \eta(P(X, Y)Z)\eta(U)] \\ &+ B(W)[g(G(X, Y)Z, U) + \eta(G(X, Y)Z)\eta(U)].\end{aligned} \tag{3.26}$$

Contracting (3.26) over X and U and using (2.9), (3.10) and (3.3), we get

$$\begin{aligned}(\nabla_W S)(Y,Z) = &A(W)S(Y,Z) + (n-1)[(n-2)B(W) - A(W)]g(Y,Z) \\ &+ [S(Z,\phi W) - (n-1)B(W)\eta(Z)]\eta(Y) \\ &+ (n-1)\eta(R(Y,\phi W)Z).\end{aligned} \tag{3.27}$$

Setting $Y = Z = \xi$ in (3.27), we obtain $B = 0$, which is not possible. Hence we can state the theorem:

Theorem 3.8. *There exists no generalized projectively ϕ-recurrent LP-Sasakian manifold (M^n, g), $n > 3$.*

4. An Example of generalized ϕ-recurrent LP-Sasakian manifold

Example 4.1. We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be a linearly independent global frame on M given by [10]

$$E_1 = e^z\frac{\partial}{\partial x}, \quad E_2 = e^{z-ax}\frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z},$$

where a is a non-zero constant such that $a \neq 1$. Let g be the Lorentzian metric defined by $g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0$, $g(E_1, E_1) = g(E_2, E_2) = 1$, $g(E_3, E_3) = -1$. Let η be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi E_1 = -E_1$, $\phi E_2 = -E_2$ and $\phi E_3 = 0$. Then using the linearity of ϕ and g we have $\eta(E_3) = -1$, $\phi^2 U = U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[E_1, E_2] = -ae^z E_2, \quad [E_1, E_3] = -E_1, \quad [E_2, E_3] = -E_2.$$

Using Koszul formula for the Lorentzian metric g, we can easily calculate

$$\begin{aligned}&\nabla_{E_1}E_1 = -E_3,\ \nabla_{E_1}E_2 = 0,\ \nabla_{E_1}E_3 = -E_1, \\ &\nabla_{E_2}E_1 = ae^z E_2,\ \nabla_{E_2}E_2 = -ae^z E_1 - E_3,\ \nabla_{E_2}E_3 = -E_2, \\ &\nabla_{E_3}E_1 = 0,\ \nabla_{E_3}E_2 = 0,\ \nabla_{E_3}E_3 = 0.\end{aligned}$$

From the above it can be easily seen that for $E_3 = \xi$, (ϕ, ξ, η, g) is an LP-Sasakian structure on M. Consequently $M^3(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor R as follows:

$$\begin{aligned}&R(E_1, E_2)E_1 = -(1 - a^2e^{2z})E_2, \quad R(E_1, E_2)E_2 = (1 - a^2e^{2z})E_1, \\ &R(E_1, E_3)E_1 = -E_3, \quad R(E_1, E_3)E_3 = -E_1, \\ &R(E_2, E_3)E_2 = -E_3, \quad R(E_2, E_3)E_3 = -E_2\end{aligned}$$

and the components which can be obtained from these by the symmetry properties.

Since $\{E_1, E_2, E_3\}$ forms a basis of a 3-dimensional LP-Sasakian manifold, any vector field $X,\ Y,\ Z \in \chi(M)$ can be written as

$$X = a_1E_1 + b_1E_2 + c_1E_3,$$
$$Y = a_2E_1 + b_2E_2 + c_2E_3,$$
$$Z = a_3E_1 + b_3E_2 + c_3E_3,$$

where $a_i,\ b_i,\ c_i \in \mathbb{R}^+$ (the set of all positive real numbers), $i = 1,\ 2,\ 3$ such that $a,\ b,\ c$ are not proportional. Then

$$R(X, Y)Z = \{b_3(a_1b_2 - a_2b_1)(1 - a^2e^{2z}) - c_3(a_1c_2 - a_2c_1)\}E_1 - \{a_3(a_1b_2 - a_2b_1)(1 - a^2e^{2z}) + c_3(b_1c_2 - b_2c_1)\}E_2 - \{b_3(b_1c_2 - b_2c_1) + a_3(a_1c_2 - a_2c_1)\}E_3 \quad (4.1)$$

and

$$G(X, Y)Z = (a_2a_3 + b_2b_3 - c_2c_3)(a_1E_1 + b_1E_2 + c_1E_3) - (a_1a_3 + b_1b_3 - c_1c_3)(a_2E_1 + b_2E_2 + c_2E_3). \quad (4.2)$$

By virtue of (4.1), we have the following:

$$(\nabla_{E_1}R)(X, Y)Z = a^2e^{2z}[(a_1b_2 - a_2b_1)(b_3E_3 + c_3E_2) + (b_1c_2 - b_2c_1)(b_3E_1 - a_3E_2)], \quad (4.3)$$

$$(\nabla_{E_2}R)(X, Y)Z = (a_1b_2 - a_2b_1)\{a^2e^{2z}(c_3E_1 - a_3E_3) - (b_3 + c_3)E_1\} - (a_1c_2 - a_2c_1)a^2e^{2z}(b_3E_1 - a_3E_2) + (b_1c_2 - b_2c_1)(c_3 - b_3)ae^zE_1, \quad (4.4)$$

$$(\nabla_{E_3}R)(X, Y)Z = 0. \quad (4.5)$$

From (4.1) and (4.2), we get

$$\phi^2(R(X, Y)Z) = u_1E_1 + u_2E_2 \text{ and } \phi^2(G(X, Y)Z) = v_1E_1 + v_2E_2, \quad (4.6)$$

where

$$u_1 = b_3(a_1b_2 - a_2b_1)(1 - a^2e^{2z}) - c_3(a_1c_2 - a_2c_1),$$
$$u_2 = -\{a_3(a_1b_2 - a_2b_1)(1 - a^2e^{2z}) + c_3(b_1c_2 - b_2c_1)\},$$
$$v_1 = a_1(b_2b_3 - c_2c_3) - a_2(b_1b_3 - c_1c_3),$$
$$v_2 = b_1(a_2a_3 - c_2c_3) - b_2(a_1a_3 - c_1c_3).$$

Also from (4.3)–(4.5), we obtain

$$\phi^2((\nabla_{E_i}R)(X, Y)Z) = p_iE_1 + q_iE_2, \quad \text{for } i = 1, 2, 3, \quad (4.7)$$

where

$$p_1 = b_3(b_1c_2 - b_2c_1)a^2e^{2z},$$
$$q_1 = \{c_3(a_1b_2 - a_2b_1) - a_3(b_1c_2 - b_2c_1)\}a^2e^{2z},$$
$$p_2 = (a_1b_2 - a_2b_1)\{c_3(a^2e^{2z} - 1) - b_3\} - b_3(a_1c_2 - a_2c_1)a^2e^{2z} + (c_3 - b_3)(b_1c_2 - b_2c_1)ae^z, q_2 = a_3(a_1c_2 - a_2c_1)a^2e^{2z},\ p_3 = 0,\ q_3 = 0.$$

Let us now consider the 1-forms as

$$A(E_1) = \frac{v_2p_1 - v_1q_1}{u_1v_2 - u_2v_1}, \quad B(E_1) = \frac{u_1q_1 - u_2p_1}{u_1v_2 - u_2v_1}, \quad (4.8)$$
$$A(E_2) = \frac{v_2p_2 - v_1q_2}{u_1v_2 - u_2v_1}, \quad B(E_2) = \frac{u_1q_2 - u_2p_2}{u_1v_2 - u_2v_1},$$
$$A(E_3) = 0, \quad B(E_3) = 0,$$

where $v_2p_1 - v_1q_1 \neq 0,\ u_1q_1 - u_2p_1 \neq 0,\ v_2p_2 - v_1q_2 \neq 0,\ u_1q_2 - u_2p_2 \neq 0,\ u_1v_2 - u_2v_1 \neq 0$. From (3.4), we have

$$\phi^2((\nabla_{E_i}R)(X, Y)Z) = A(E_i)\phi^2(R(X, Y)Z) + B(E_i)\phi^2(G(X, Y)Z), \quad i = 1, 2, 3. \quad (4.9)$$

By virtue of (4.6)–(4.8), it can be easily shown that the manifold satisfies the relation (4.9). Hence the manifold under consideration is a 3-dimensional generalized ϕ-recurrent LP-Sasakian manifold, which is not ϕ-recurrent. This leads to the following:

Theorem 4.1. *There exists a 3-dimensional generalized ϕ-recurrent LP-Sasakian manifold, which is neither ϕ-symmetric nor ϕ-recurrent.*

Acknowledgement

The authors convey their sincere thanks to the referee for his valuable comments toward the improvement of the paper.

References

[1] K. Matsumoto, On Lorentzian almost paracontact manifolds, Bull. Yamagata Univ. Nat. Sci. 12 (1989) 151–156.
[2] I. Mihai, R. Rosca, On Lorentzian Para-Sasakian Manifolds, Classical Analysis, World Scientific Publ., Singapore, 1992, pp. 155–169.
[3] A.A. Aqeel, U.C. De, G.C. Ghosh, On Lorentzian para-Sasakian manifolds, Kuwait J. Sci. Eng. 31 (2) (2004) 1–13.
[4] C.S. Bagewadi, Venkatesha, N.S. Basavarajappa, On LP-Sasakian manifolds, Sci. Ser. A: Math. Sci. 16 (2008) 1–8.
[5] U.C. De, K. Matsumoto, A.A. Shaikh, On Lorentzian para-Sasakian manifolds, Rendiconti del Seminario Mat. de Messina (3) (1999) 149–156.
[6] I. Mihai, U.C. De, A.A. Shaikh, On Lorentzian para-Sasakian manifolds, Korean J. Math. Sci. 6 (1999) 1–13.
[7] C. Murathan, A. Yildiz, K. Arslan, U.C. De, On a class of Lorentzian para-Sasakian manifolds, Proc. Estonian Acad. Sci. Phys. Math. 55 (4) (2006) 210–219.
[8] A.A. Shaikh, K.K. Baishya, Some results on LP-Sasakian manifolds, Bull. Math. Soc. Sci. Math. Roumanie Tome 49 (97) (2) (2006) 193–205..
[9] A.A. Shaikh, K.K. Baishya, On ϕ-symmetric *LP*-Sasakian manifolds, Yokohama Math. J. 52 (2005) 97–112.
[10] A.A. Shaikh, K.K. Baishya, S. Eyasmin, On the existence of some types of *LP*-Sasakian manifolds, Commun. Korean Math. Soc. 23 (1) (2008) 95–110.
[11] A.A. Shaikh, T. Basu, K.K. Baishya, On the existence of locally ϕ-recurrent *LP*-Sasakian manifolds, Bull. Allahabad Math. Soc. 24 (2) (2009) 281–295.
[12] A.A. Shaikh, S. Biswas, Some results on LP-Sasakian manifolds, Bull. Malaysian Math. Sci. Soc. 27 (2004) 17–26.
[13] Venkatesha, C.S. Bagewadi, On concircular ϕ-recurrent LP-Sasakian manifolds, Diff. Geom.-Dyn. Syst. 10 (2008) 312–319.
[14] E. Cartan, Sur une classe remarquable d'espaces de Riemannian, Bull. Soc. Math. France 54 (1926) 214–264.
[15] T. Takahashi, Sasakian ϕ-symmetric spaces, Tohoku Math. J. 29 (1977) 91–113.
[16] U.C. De, A.A. Shaikh, S. Biswas, On ϕ-recurrent Sasakian manifolds, Novi Sad J. Math. 33 (2003) 13–48.
[17] U.C. De, A. Yildiz, A.F. Yaliniz, On ϕ-recurrent Kenmotsu manifolds, Turkish J. Math. 33 (2009) 17–25.
[18] A.A. Shaikh, S.K. Hui, On locally ϕ-symmetric β-Kenmotsu manifolds, Extr. Math. 24 (3) (2009) 301–316.

[19] R.S.D. Dubey, Generalized recurrent spaces, Indian J. Pure Appl. Math. 10 (12) (1979) 1508–1513.
[20] U.C. De, N. Guha, On generalized recurrent manifolds, J. Nat. Acad. Math., India 9 (1991) 85–92.
[21] A.G. Walker, On Ruses spaces of recurrent curvature, Proc. London Math. Soc. 52 (1950) 36–64.
[22] U.C. De, N. Guha, D. Kamilya, On generalized Ricci-recurrent manifolds, Tensor N.S. 56 (1995) 312–317.
[23] E.M. Patterson, Some theorems on Ricci-recurrent spaces, J. London Math. Soc. 27 (1952) 287–295.
[24] A.A. Shaikh, H. Ahmad, On generalized ϕ-recurrent Sasakian manifolds, Appl. Math. 2 (2011) 1317–1322.
[25] A.A. Shaikh, I. Roy, On quasi-generalized recurrent manifols, Math. Pannonica 21 (2) (2010) 251–263.