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ORIGINAL ARTICLE

On generalized superposition operator acting of analytic function spaces

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Abstract In this paper we introduce a new integration operator $S^{(n)}_{g,\phi}$, where

$$S^{(n)}_{g,\phi} = \int_0^z \phi^{(n)}(f(\xi))g(\xi)d\xi.$$

We characterize all entire functions that transform a Bloch-type space into another by this new integration operator. Also, we prove that all generalized superposition operators induced by such entire functions are bounded.

AMS: 46E15; 47B33; 47B38; 54C35

1. Introduction

Let $H(\mathbb{D})$ denote the space of all analytic functions on the unit disk $\mathbb{D}$ of $\mathbb{C}$. Let ϕ be analytic self-map of $\mathbb{D}$, n be a positive integer and $g \in H(\mathbb{D})$. Let X and Y be two metric spaces of analytic functions on the unit disk and ϕ denotes a complex-valued function of the plan $\mathbb{C}$. The superposition operator S_ϕ on X is defined by

$$S_\phi(f) = \phi \circ f, \quad f \in X.$$

If $S_\phi f \in Y$ for $f \in X$, we say that ϕ acts by superposition from X into Y. We see that if X contains linear functions, ϕ must be an entire function. Let $H(\mathbb{D})$ be the class of all analytic function on $\mathbb{D}$, then for $g \in H(\mathbb{D})$, we define a new nonlinear superposition operator as follows:

$$(S^{(n)}_{g,\phi} f)(z) = \int_0^z \phi^{(n)}(f(\xi))g(\xi)d\xi.$$

The operator $S^{(n)}_{g,\phi}$ is called the generalized superposition operator. When $g = f'$ and $n = 1$, we see that this operator is essentially superposition operator, since the following difference $S^{(n)}_{g,\phi} - S_\phi$ is a constant. Therefore, $S^{(n)}_{g,\phi}$ is a generalization of the superposition operator. To the best of our Knowledge, the operator $S^{(n)}_{g,\phi}$ is introduced in the present paper for the first time. The graph of $S^{(n)}_{g,\phi}$ is usually closed but, since the operator is nonlinear, this is not enough to assure its boundedness. Nonetheless, for a number of important spaces X, Y, such as Hardy, Bergman, Dirichlet, and Bloch, the mere action $S^{(n)}_{g,\phi} : X \to Y$ implies that ϕ must belong to a very special class of entire functions, which in turn implies boundedness.

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Wen Xu studied superposition operators on Bloch-type spaces in [1].

In this paper we give a complete description of the generalized superpositions on Bloch-type spaces in terms of the order and type of ϕ and the degree of polynomials.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Recall that the well known Bloch space (cf. [2]) is defined as follows:

$$\mathcal{B} = \{f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty\};$$

the little Bloch space $\mathcal{B}_0$ (cf. [2]) is a subspace of $\mathcal{B}$ consisting of all $f \in \mathcal{B}$ such that

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0.$$

Definition 1.1 [3]. Let f be an analytic function in $\mathbb{D}$ and $0 < \alpha < \infty$. The α-Bloch space $\mathcal{B}^\alpha$ is defined by

$$\mathcal{B}^\alpha = \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty\},$$

the little α-Bloch space $\mathcal{B}^\alpha_0$ is given as follows

$$\mathcal{B}^\alpha_0 = \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}^\alpha_0} = \lim_{|z|\to 1^-}(1-|z|^2)^\alpha|f'(z)| = 0\}.$$

The spaces $\mathcal{B}^1$ and $\mathcal{B}^1_0$ are called the Bloch space and denoted by $\mathcal{B}$ and $\mathcal{B}_0$ respectively (see [4]).

As a simple example one can get that the function $f(z) = \log(1-z)$ is a Bloch function but $f(z) = \log^2(1-z)$ is not a Bloch function.

Definition 1.2 (*see [5]*). For $p \in (0,\infty)$ and $-1 < \alpha < \infty$, the Bergman-type spaces $\mathcal{A}^p_\alpha$ are defined by

$$\mathcal{A}^p_\alpha = \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{A}^p_\alpha} = \sup_{z\in\mathbb{D}}|f(z)|^p(1-|z|^2)^\alpha < \infty\}.$$

Moreover, $f \in \mathcal{A}_{0,\alpha}$; if and only if

$$\lim_{|z|\to 1^-}\sup_{z\in\mathbb{D}}|f(z)|(1-|z|^2)^\alpha = 0.$$

Conformally invariant spaces of the disk: It is a standard fact that the set of all disk automorphisms (i.e., of all one-to-one analytic maps φ of $\mathbb{D}$ onto itself\,), denoted $Aut(\mathbb{D})$, coincides with the set of all Möbius transformations of $\mathbb{D}$ onto itself:

$$Aut(\mathbb{D}) = \{\lambda\varphi_a : |\lambda| = 1; a \in \mathbb{D}\},$$

where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ are the automorphisms: $\varphi_a(\varphi_a(z)) \equiv z$.

A space X of analytic functions in $\mathbb{D}$, equipped with a semi-norm ρ, is said to be conformally invariant or Möbius invariant if whenever $f \in X$, then also $f \circ \varphi \in X$ for any $\varphi \in Aut(\mathbb{D})$ and, moreover, $\rho(f \circ \varphi) \leqslant C\rho(f)$ for some positive constant C and all $f \in X$.

Definition 1.3. In topology, a geometrical object or space is called simply connected (or 1-connected) if it is path-connected and every path between two points can be continuously transformed into every other while preserving the two endpoints in question.

Definition 1.4. A path from a point x to a point y in a topological space X is a continuous function f from the unit interval $[0,1]$ to X with $f(0) = x$ and $f(1) = y$. A path-component of X is an equivalence class of X under the equivalence relation defined by x is equivalent to y if there is a path from x to y. The space X is said to be path-connected (or path-wise connected or 0-connected) if there is only one path-component, i.e. if there is a path joining any two points in X.

Remark 1.1. Every path-connected space is connected. The converse is not always true.

In this section, we give some auxiliary results which are incorporated in the following lemmas.

Lemma 1.1. *Let and $f \in \mathcal{B}_\alpha$ and $0 < \alpha < \infty$. Suppose that*

$$I_\alpha = \int_0^1 \frac{|z|dt}{(1-t^2|z|^2)^\alpha} < \infty. \tag{1}$$

Then we have,

$$|f(z)| \leqslant |f(0)| + C\|f\|_{\mathcal{B}^\alpha},$$

for some $C > 0$ independent of f.

Proof. Let $|z| > \frac{1}{2}$, $z = r\xi$, and $\xi \in \partial\mathbb{D}$. We have

$$\left|f(z) - f\left(\frac{r\xi}{2}\right)\right| = \left|\int_{\frac{1}{2}}^1 zf'(tz)dt\right| \leqslant \int_{\frac{1}{2}}^1 |z||f'(tz)|dt$$
$$\leqslant 2\|f\|_{\mathcal{B}^\alpha}\int_0^1 \frac{|z|dt}{(1-t^2|z|^2)^\alpha} \leqslant C\|f\|_{\mathcal{B}^\alpha}.$$

Also, we have

$$|f(z)| \leqslant \max_{|z|\leqslant\frac{1}{2}}|f(z)| + C\|f\|_{\mathcal{B}^\alpha}. \tag{2}$$

Let $|z| \leqslant \frac{1}{2}$, then, by the mean value property of the function $f(z) - f(0)$ (see [6]) and Jensen's inequality, we obtain

$$\max_{|z|\leqslant\frac{1}{2}}|f(z)-f(0)| \leqslant 4^n\int_{|z|\leqslant\frac{3}{4}}|f(w)-f(0)|dA(w)$$
$$\leqslant 4^n\int_{|z|\leqslant\frac{3}{4}}|f'(w)|^2dA(w) \leqslant 3^n\max_{|z|\leqslant\frac{3}{4}}|f'(w)|^2.$$

The second inequality can be easily proved by using the homogeneous expansion of f.

Hence,

$$\max_{|z|\leqslant\frac{1}{2}}|f(z)| \leqslant |f(0)| + (\sqrt{3})^n\max_{|z|\leqslant\frac{3}{4}}|f'(z)|$$
$$\leqslant |f(0)| + \frac{2^{4\alpha}(\sqrt{3})^n}{7^\alpha}\|f\|_{\mathcal{B}^\alpha}. \tag{3}$$

From (2) and (3), the result follows easily when $\alpha \neq 1$. If $\alpha = 1$, then we have

$$|f(z)| \leqslant |f(0)| + \frac{16(\sqrt{3})^n}{7}\|f\|_{\mathcal{B}^1} + C\|f\|_{\mathcal{B}^1}$$
$$\leqslant |f(0)| + \left(\frac{16(\sqrt{3})^n}{7} + C\right)\|f\|_{\mathcal{B}^1}.$$

This complete the proof. □

Throughout this work, the letter Ω will be used to denote a planar domain and $\partial\Omega$ its boundary.

A univalent function in $\mathbb{D}$ is an analytic function which is one-to-one in the unit disk. By the Riemann mapping theorem [6], for any given simply connected domain Ω (other than the plane itself\,) there is such a function f (called a Riemann map) that takes $\mathbb{D}$ onto Ω and the origin to a prescribed point. Denoting by $dist(w, \partial\Omega)$ the Euclidean distance of the point w to the boundary of the domain Ω, the Riemann map f has the following property:

$$\frac{1}{4}(1-|z|^2)|f'(z)| \leqslant dist(f(z), \partial\Omega) \leqslant (1-|z|^2)|f'(z)| \text{ for all } z \in \mathbb{D}. \quad (4)$$

This estimate plays an important role in the geometric theory of functions. In particular, (4) tells us that a function f univalent in $\mathbb{D}$ belongs to $\mathcal{B}$ if and only if the image domain $f(\mathbb{D})$ does not contain arbitrarily large disks.

The auxiliary construction of a conformal map onto a specific Bloch domain with the maximal growth along a certain polygonal line displayed below might be of some independent interest. Thus, we state it separately as a lemma. Loosely speaking, such a domain can be imagined as a ''highway from the origin to infinity'' of width 2δ. Somewhat similar constructions of simply connected domains as the images of functions in various function spaces can be found in the recent papers [7,8].

Lemma 1.2. *For each positive number δ and for every sequence w_n of complex number such that $w_0 = 0$, $|w_1| \geqslant 5\delta$, $|\arg w_1 - \theta_0| < \frac{\pi}{4}$, $\arg w_n \searrow \theta_0$, or $\arg w_n \nearrow \theta_0$ and*

$$|w_n| \geqslant \max\left\{3|w_{n-1}|, \sum_{k=1}^{n-1}|w_k - w_{k-1}|\right\} \quad \text{for all} \quad n \geqslant 2, \quad (5)$$

there exists a domain Ω with the following properties:

(i) *Ω is simply connected;*
(ii) *Ω contains the infinite polygonal line $L = \bigcup_{n=1}^{\infty}[w_{n-1}, w_n]$, where $[w_{n-1}, w_n]$ denotes the line segment from w_{n-1} to w_n;*
(iii) *there exists a conformal mapping f of $\mathbb{D}$ onto Ω which takes the origin to a prescribed point belongs to $\mathcal{B}$;*
(iv) *$dist(w, \partial\mathbb{D}) = \delta$ for each point w on L, where $\nearrow$ denotes the increasing functions and $\searrow$ denotes the decreasing functions.*

Proof. It is clear from (5) that $|w_n| \nearrow \infty$, as $n \to \infty$. We construct the domain Ω as follows. First connect the points w_n by a polygonal line L as indicated in the statement. Let $D(z, \delta) = \{w : |z - w| < \delta\}$ and define

$$\Omega = \bigcup\{D(z, \delta) : z \in L\},$$

i.e. let Ω be a δ-thickening of L. In other words, Ω is the union of simply connected cigar-shaped domains

$$C_n = \bigcup\{D(z, \delta) : z \in [w_{n-1}, w_n]\}.$$

By our choice of w_n, it is easy to check inductively that $|w_n - w_k| \geqslant 5\delta$ whenever $n > k$. Since our construction implies that

$$C_n \subset \{w : |w_{n-1}| - \delta < |w| < |w_n| + \delta\},$$

we see immediately that

(a) for all m and n, $C_m \cap C_n \neq \emptyset$, if and only if $|m - n| \leqslant l$;
(b) for all n, $C_n \cap C_{n+1}$ is either $D(w_n, \delta)$ or the interior of the convex hull of $D(w_n, \delta) \cup \{a_n\}$ for some point a_n outside of $\overline{D}(w_n, \delta)$, where $\overline{D}(w_n, \delta)$ is the closure of $D(w_n, \delta)$. Thus, each $\Omega_N = \bigcup_{n=1}^{N} C_n$ is also simply connected. Since

$$\Omega = \bigcup_{N=1}^{\infty}\Omega_N \quad \text{and} \quad \Omega_N \subset \Omega_{N+1} \text{ for all } N,$$

we conclude that Ω is also simply connected (like in [8]). By construction, $dist(w, \partial\Omega) \leqslant \delta$ for all w in Ω, hence any Riemann map onto Ω will belong to $\mathcal{B}$. It is also clear that (iv) holds. $\square$

2. $S_{g,\phi}^{(n)}$ on Bloch space

First we will show that if $0 < \beta < \alpha$, then $S_{g,\phi}^{(n)}$ maps $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$ unless ϕ is a constant.

Theorem 2.1. *Let $0 < \beta < \alpha$ and ϕ be an entire function. Then the generalized superposition operator $S_{g,\phi}^{(n)}$ maps $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$ if and only if ϕ is a constant function.*

Proof. If ϕ is a constant, it is obvious that $S_{g,\phi}^{(n)}(\mathcal{B}^\alpha) \subset \mathcal{B}^\beta$. Now we assume that ϕ is not a constant. We distinguish three cases to prove that $S_{g,\phi}^{(n)}(\mathcal{B}^\alpha) \not\subset \mathcal{B}^\beta$.

(i) When $\alpha < 1$. Since ϕ is not a constant, there exists a disk $|w - w_0| < r$ on which

$$|\phi^{(n)}(w)| > \delta > 0.$$

Let $|g(z)| = |1 - z|^{-\alpha} \in \mathcal{B}^\alpha$. Then, for $z \in \mathbb{D}$, we have

$$(1-|z|^2)^\beta |(S_{g,\phi}^{(n)} f)(z)| = (1-|z|^2)^\beta |\phi^{(n)}(f(z))||g(z)| \geqslant \frac{\delta(1-|z|^2)^\beta}{|1-z|^\alpha}.$$

The right side of the above inequality tends to infinity as $z \to 1$ This shows that $S_{g,\phi}^{(n)}(f) \notin \mathcal{B}^\beta$ and $S_{g,\phi}^{(n)}(\mathcal{B}^\alpha) \not\subset \mathcal{B}^\beta$.

(ii) When $\alpha = 1$. Since ϕ is unbounded, there exists a complex sequence $w_n \to \infty$ such that $|\phi(w_n)| \to \infty$ as $n \to \infty$. Without loss of generality, we may assume that w_n satisfies the conditions in Lemma 1.2 with some $\delta > 0$ by adding $w_0 = 0$ and choosing a subsequence if necessary. By Lemma 1.2, there exists a domain Ω and a conformal mapping f of $\mathbb{D}$ onto Ω such that $w_n \in \Omega$ for $n = 0, 1, \ldots$ and $f \in \mathcal{B}$. By Lemma 1.1, any function in $\mathcal{B}^\beta$ is bounded and, hence, $S_{g,\phi}^{(n)}(f) \notin \mathcal{B}^\beta$ and $S_{g,\phi}^{(n)}(\mathcal{B}^\alpha) \not\subset \mathcal{B}^\beta$, since $S_{g,\phi}^{(n)}$ is unbounded.

(iii) When $\alpha > 1$. Since ϕ is not a constant, there is a sequence $w_n \to \infty$ such that $|\phi^{(n)}(f(z_n))| \geqslant \delta$ for some fixed $\delta > 0$ and $n \in \mathbb{N} = \{1, 2, \ldots\}$. We may assume that $|\arg w_n| < \min\{\frac{(\alpha-1)\pi}{4}, \frac{\pi}{2}\}$ and $|w_n| > 1$ for $n \in \mathbb{N}$, by rotating and choosing a subsequence if necessary.

Consider the function $|g(z)| = |1-z|^{-\alpha} \in \mathcal{B}^\alpha$. The pre-image of w_n under the function $|g|$ is $z_n = 1 - \left(\frac{w_n}{\omega(1-|z|)}\right)^{\frac{-1}{\alpha}}$, which belongs to the domain

$$S = \left\{ z \in \mathbb{D} : |1-z| < 1, |\arg(1-z)| < \frac{\pi}{4} \right\}.$$

Therefore, there exists a constant M such that $|1-z_n| \leqslant M(1-|z_n|)$ for $n \in \mathbb{N}$

$$(1-|z_n|^2)^\beta |\phi^{(n)}(f(z_n))||g(z_n)| \geqslant \delta(1-|z_n|^2)^\beta |g(z_n)|$$
$$\geqslant \frac{\delta(1-|z_n|^2)^\beta}{|1-z_n|^\alpha} \geqslant \frac{\delta}{M^\alpha(1-|z_n|)^{\alpha-\beta}} \to \infty \quad \text{as } n \to \infty.$$

This shows that $S^{(n)}_{g,\phi} \notin \mathcal{B}^\beta$ and $S^{(n)}_{g,\phi}(\mathcal{B}^\alpha) \not\subset \mathcal{B}^\beta$. The proof is completed. □

Now, we will study generalized superposition operators from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ $(\alpha \leqslant \beta)$.

An operator acting between two metric spaces is said to be bounded if it maps bounded sets into bounded sets.

Theorem 2.2. *Let $0 < \alpha < 1$,; and $\alpha \leqslant \beta$. Then for any entire function ϕ, $S^{(n)}_{g,\phi}$ is a bounded operator of $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$.*

Proof. Let $0 < \alpha < 1$, $\alpha \leqslant \beta$, and ϕ be an entire function. Let $M > 0$. For a function g with $\|g\|_{\mathcal{B}^\alpha} \leqslant M$, by Lemma 1.1, $|f(z)| < C_\alpha M$ (where C_α is a constant depending only on α), then

$$\phi(f(0)) \leqslant M_1 = \max_{|w|=C_\alpha M} |\phi(w)|,$$

$$\phi^{(n)}(f(z)) \leqslant M_2 = \max_{|w|=C_\alpha M} |\phi^{(n)}(w)|, \quad \text{for } z \in \mathbb{D}.$$

Thus

$$\|S^{(n)}_{g,\phi} f\|_{\mathcal{B}^\beta} \leqslant |S^{(n)}_{g,\phi}(f)(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |(S^{(n)}_{g,\phi} f)(z)|$$
$$\leqslant |S^{(n)}_{g,\phi}(f)(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |\phi^{(n)}(f(z))||g(z)|$$
$$\leqslant M_2 \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |g(z)| \leqslant M_1 + M_2\|g\|_{\mathcal{B}^\alpha}$$
$$\leqslant M_1 + MM_2.$$

where M_2 depend on α, ϕ and M only. This completes the proof. □

Theorem 2.3. *Let be a nondecreasing function, $1 < \alpha \leqslant \beta$ and ϕ be an entire function. Then the following statements are equivalent:*

(i) $S^{(n)}_{g,\phi}$ *maps $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$;*
(ii) $S^{(n)}_{g,\phi}$ *is a bounded operator of $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$;*
(iii) *ϕ is a polynomial of degree at most $\frac{\beta-1}{\alpha-1}$.*

Proof. We need to show $(i) \Rightarrow (iii) \Rightarrow (ii)$. First, assume that (iii) holds. It suffices to verify the statement (ii) for $\phi(z) = z^r$ with positive integer $n < r \leqslant \frac{\beta-1}{\alpha-1}$. For $f \in \mathcal{B}^\alpha$, by Lemma 1.1, we have

$$(1-|z|^2)^\beta |(S^{(n)}_{g,\phi} f)(z)| \leqslant (r-n)!(1-|z|^2)^\beta |f(z)|^{r-n+1}|g(z)|$$
$$= (r-n)!(1-|z|^2)^\alpha |f(z)|^{r-n+1}(1-|z|^2)^{\beta-\alpha}|g(z)|$$
$$\leqslant C\|g\|_{\mathcal{B}^\alpha} \cdot \|f\|_{\mathcal{A}^{r-n+1}_{\beta-\alpha}}.$$

Thus,

$$\|S^{(n)}_{g,\phi} f(z)\|_{\mathcal{B}^\beta} \leqslant C\|g\|_{\mathcal{B}^\alpha} \cdot \|f\|_{\mathcal{A}^{r-n+1}_{\beta-\alpha}}.$$

This shows that the generalized superposition operator $S^{(n)}_{g,\phi}$ is a bounded operator from $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$. The implication $(iii) \Rightarrow (ii)$ is proved.

Now suppose that ϕ is not a polynomial of degree at most $\frac{\beta-1}{\alpha-1}$, or equivalently that the Taylor expansion of ϕ about zero has a non-zero coefficient of order $m > \frac{\beta-1}{\alpha-1}$. Then, there exists a constant $\delta > 0$ and a sequence $w_n \to \infty$ such that

$$|\phi^{(n)}(f(w_n))| \geqslant \delta \quad \text{for } n \in \mathbb{N}.$$

We want to find a function $f \in \mathcal{B}^\alpha$ with $S^{(n)}_{g,\phi}(\mathcal{B}^\alpha) \not\subset \mathcal{B}^\beta$. Without loss of generality $|\arg w_n| < \min\{\frac{(\alpha-1)\pi}{4}, \frac{\pi}{2}\}$ for $n \in \mathbb{N}$, by rotating and choosing a subsequence if necessary. Let $|g(z)| = |1-z|^{-\alpha} \in \mathcal{B}^\alpha$. As in the proof of Theorem 2.1, the point $z_n = 1 - (w_n)^{\frac{-1}{\alpha}}$, satisfies that $|1-z_n| < 1$ and $|\arg(1-z_n)| < \frac{\pi}{4}$ and consequently, that $|1-z_n| \leqslant M(1-|z_n|)$ for $n \in \mathbb{N}$. Thus,

$$(1-|z_n|^2)^\beta |(S^{(n)}_{g,\phi} f)(z_n)| = (1-|z_n|^2)^\beta |\phi^{(n)}(f(z_n))|g(z_n)|$$
$$\geqslant \frac{\delta(1-|z_n|^2)^\beta}{|1-z_n|^{(\alpha-1)(m-1)+\alpha}} \geqslant \frac{\delta}{M^{m(\alpha-1)+1}(1-|z_n|)^{m(\alpha-1)+1-\beta}}.$$

Since $m > \frac{\beta-1}{\alpha-1}$, we have $m(\alpha-1)+1-\beta > 0$. Thus,

$$(1-|z_n|^2)^\beta |(S^{(n)}_{g,\phi} f)(z_n)| \to \infty \quad \text{as } n \to \infty,$$

which implies that $S^{(n)}_{g,\phi}(\mathcal{B}^\alpha) \not\subset \mathcal{B}^\beta$. This shows that $(i) \Rightarrow (iii)$. The theorem is proved. □

The notion of the order and type of the entire function ϕ is involved in the investigation when $1 = \alpha < \beta$. Let $M(r) = \max_{|z|=r} |\phi(z)|$ for $r \geqslant 0$. The logarithmic order (log-order) of the function $M(r)$ is defined as

$$\rho = \limsup_{r\to\infty} \frac{\ln^+ \ln^+ M(r)}{\ln r},$$

where $\ln^+ x = \max\{\ln x, 0\}$. If $0 < \rho < \infty$, the logarithmic type (log-type) of the function $M(r)$ is defined as

$$\tau = \limsup_{r\to\infty} \frac{\ln^+ M(r)}{r^\rho}.$$

Note that if f is an entire function, then the growth order of f is just the log-order of $M(r)$, the maximum modulus function of f.

In the earlier papers [7,9], the cut usually occurred at the level of functions of infinite type. The appearance of the functions of given order and type zero in the result below seems to be a novelty in this context.

Theorem 2.4. *Let $\beta > 1$.*

(i) *If ϕ is an entire function of order less than one, or of order one type zero, and*

$$|f(z)| \leqslant M(1+\log\frac{1}{1-|z|}) \quad \text{for} \quad z \in \mathbb{D}.$$

Then,

$$\|S^{(n)}_{g,\phi}\|_{\mathcal{B}^\beta} \leqslant C\|g\|_{\mathcal{B}^\beta}.$$

(ii) *If ϕ is an entire function of order greater than one, or of order one and positive type and suppose that*

$$\int_0^{|z_n|} \frac{dt}{(1-t^2)} < \infty \quad \text{and} \quad (1-|z|^2)|f'(z)| \geqslant \delta; \quad \delta > 0.$$

Then,

$$\|S^{(n)}_{g,\phi}\|_{\mathcal{B}^\beta} \geqslant C\|g\|_{\mathcal{B}^\beta}.$$

Proof.

(i) Given $M > 0$, let $\sigma = \frac{\beta-1}{M}$. Since ϕ is an entire function of order less than one or of order one and type zero, the same holds for $\phi^{(n)}$. Then there exists M_1 such that

$$|\phi^{(n)}(w)|, |\phi(w)| \leqslant M_1 e^{\sigma|w|} \quad \text{for} \quad w \in \mathbb{C}.$$

Thus,

$$|\phi(f(0))| \leqslant M_1 e^{\sigma|f(z)|} \leqslant M_1 e^{\sigma M} = M_1 e^{\beta-1}$$

$$\begin{aligned}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|(S^{(n)}_{g,\phi}f)(z)| &\leqslant M_1 \sup_{z\in\mathbb{D}}(1-|z_n|^2)^\beta|g(z_n)|e^{\sigma|f(z)|}\\ &\leqslant M_1 \sup_{z\in\mathbb{D}}(1-|z_n|^2)^\beta|g(z_n)|\left(\frac{e}{1-|z|}\right)^{\beta-1}\\ &\leqslant 2^{\beta-1}M_1e^{\beta-1}\|g\|_{\mathcal{B}^\beta}.\end{aligned}$$

Hence,

$$\|S^{(n)}_{g,\phi}f\|_{\mathcal{B}^\beta} \leqslant C\|g\|_{\mathcal{B}^\beta}. \tag{6}$$

(ii) Now, suppose that $\phi^{(n)}$ is of order bigger than 1, or of order one and positive type. Thus, there exists an $\eta > 0$ and a sequence $w_n \to \infty$ such that

$$|\phi^{(n)}(w_n)| \geqslant e^{\eta|w_n|} \quad \text{for} \quad n \in N. \tag{7}$$

Let $\delta > 0$ be given so that $\frac{\eta\delta}{\sigma} > \beta$. As in the proof of Theorem 2.1, we may assume that w_n satisfies the condition in Lemma 1.2 with the given δ. Thus, there exists a domain Ω and a conformal mapping $f(f(0) = 0)$ with all properties in Lemma 1.1. Let $l = f^{-1}(L)$, $z_n = f^{-1}(w_n)$, and denote the part of l from 0 to z_n by l_n for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We have $z_n \to \partial\mathbb{D}$ since $w_n \to \infty$. For $w' \in L$, let $z' = f^{-1}(w')$ and $\psi_1 = \frac{z'-z}{1-\bar{z}z}$ for $z \in \mathbb{D}$. Since $(1-|z'|^2)|f'(z')| \geqslant \delta$. Thus, for $n \in \mathbb{N}$, by (5) and (7)

$$\begin{aligned}3|w_n| &\geqslant \sum_{k=1}^{n}|w_k - w_{k-1}| = \int_{l_n}|f'(z)||dz|\\ &= \int_{l_n}(1-|z|^2)|f'(z)|\frac{|dz|}{(1-|z|^2)} \geqslant \delta\int_0^{|z_n|}\frac{dt}{(1-t^2)} = \delta C,\end{aligned}$$

and

$$\begin{aligned}(1-|z_n|^2)^\beta|(S^{(n)}_{g,\phi}f)(z_n)| &= (1-|z_n|^2)^\beta|\phi^{(n)}(f(w_n))||g(z_n)|\\ &\geqslant (1-|z_n|^2)^\beta e^{\eta|f(w_n)|}|g(z_n)| \geqslant C\|g\|_{\mathcal{B}^\beta}.\end{aligned}$$

This completes the proof. □

3. Conclusion

we introduced a new integration operator $S^{(n)}_{g,\phi}$, We characterized all entire functions that transform a Bloch-type space into another by this new integration operator. We proved that all generalized superposition operators induced by such entire functions are bounded. Also, we studied the generalized superpositions on Bloch-type spaces in terms of the order and type of ϕ and the degree of polynomials.

4. Future work

It is still an open problem to extend the obtained results in this paper by using the generalized superposition operators in new hyperbolic classes of functions which introduced in [10].

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