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ORIGINAL ARTICLE

On asymptotically ideal equivalent sequences

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Abstract In this article we introduce the notion of asymptotically I-equivalent sequences. We prove the decomposition theorem for asymptotically I-equivalent sequences. Further, we will present four theorems that characterize asymptotically I-equivalent of multiple λ and the regularity of asymptotically I-convergence by using a sequence of infinite matrices.

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1. Introduction

Throughout $w, \ell_\infty, c, c_0, c^I, c_0^I, m^I$, and m_0^I denote *all, bounded, convergent, null, I-convergent, I-null, bounded I-convergent* and *bounded I-null* class of sequences, respectively. Also $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and set of real numbers, respectively. Further S_0^I denote the subset of the space m_0^I with non-zero terms.

The notion of statistical convergence is a very useful functional tool for studying the convergence problems of numerical sequences/matrices (double sequences) through the concept of density. It was first introduced by Fast [1] and Schoenberg [2], independently for the real sequences. Later on it was further investigated from sequence space point of view and linked with the summability theory by Fridy [3] and many others. The idea is based on the notion of natural density of subsets of $\mathbb{N}$, the set of positive integers, which is defined as follows. The natural density of a subset of $\mathbb{N}$ is denoted by $\delta(E)$ and is defined by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n : k \in E\}|,$$

where the vertical bar denotes the cardinality of the respective set.

The notion of I-convergence (I denotes an ideal of subsets of $\mathbb{N}$), which is a generalization of statistical convergence, was introduced by Kostyrko et al. [4]. Later on it was further investigated from sequence space point of view and linked with summability theory by Šalát et al. [5,6], Tripathy and Hazarika [7] and many others.

A non-empty family of sets $I \subseteq P(\mathbb{N})$ (power set of $\mathbb{N}$) is called an ideal of $\mathbb{N}$ if (i) for each $A, B \in I$, we have $A \cup B \in I$; (ii) for each $A \in I$ and $B \subseteq A$, we have $B \in I$. A family $F \subseteq P(\mathbb{N})$ (power set of $\mathbb{N}$) is called a filter of $\mathbb{N}$ if (i) $\phi \notin F$; (ii) for each $A, B \in F$, we have $A \cap B \in F$; and (iii) for each $A \in F$ and $B \supset A$, we have $B \in F$. An ideal I is called non-trivial if $I \neq \phi$ and $\mathbb{N} \notin I$. It is clear that $I \subseteq P(\mathbb{N})$ is a non-trivial ideal if and only if the class $F = F(I) = \{\mathbb{N} - A : A \in I\}$ is a filter on $\mathbb{N}$. The filter $F(I)$ is called the filter associated with the ideal I. A non-trivial ideal $I \subseteq P(\mathbb{N})$ is called an admissible

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ideal of $\mathbb{N}$ if it contains all singletons, i.e., if it contains $\{\{x\} : x \in \mathbb{N}\}$.

In [8], Marouf introduced the definition for asymptotically equivalent of two sequences. In [9], Pobyvancts introduced the concept of asymptotically regular matrices, which preserve the asymptotic equivalence of two nonnegative numbers sequences. The frequent occurrence of terms having zero value makes a term-by-term ratio inapplicable in many cases. In [3], Fridy introduced new ways of comparing rates of convergence. If x is in ℓ^1 , he used the remainder sum, whose nth term is $R_n(x) := \sum_{k=n}^{\infty}|x_k|$, and examined the ratio $\frac{R_n(x)}{R_n(y)}$ as $n \to \infty$. If x is a bounded sequence, he used the supremum of the remaining terms which is given by $\mu_n x := \sup_{k \geqslant n}|x_k|$. In [10], Patterson introduced the concept of asymptotically statistically equivalent sequences and natural regularity conditions for nonnegative summability matrices.

In present study we introduce the definition of asymptotically I-equivalent sequences and prove the decomposition theorem for asymptotically I-equivalent sequences and some interesting theorems related to this notion.

2. Definitions and notations

Definition 2.1. [1,3]. A sequence (x_k) is said to be textitstatistically convergent to x_0 if for each $\varepsilon > 0$, the set $E(\varepsilon) = \{k \in \mathbb{N} : |x_k - x_0| \geqslant \varepsilon\}$ has natural density zero.

Definition 2.2 [4]. A sequence (x_k) is said to be *I-convergent* if there exists a number x_0 such that for each $\varepsilon > 0$, the set

$$\{k \in \mathbb{N} : |x_k - x_0| \geqslant \varepsilon\} \in I.$$

Definition 2.3 [4]. Let (x_k) and (y_k) be two real sequences, then we say that $x_k = y_k$for almost all k related to I ($a.a.k.r.I$) if the set $\{k \in \mathbb{N} : x_k \neq y_k\}$ belongs to I.

Definition 2.4 [4]. An admissible ideal I is said to have the property (AP) if for any sequence $\{A_1, A_2, \ldots\}$ of mutually disjoint sets of I, there is sequence $\{B_1, B_2, \ldots\}$ of sets such that each symmetric difference $A_i \Delta B_i (i = 1, 2, 3, \ldots)$ is finite and $\bigcup_{i=1}^{\infty} B_i \in I$.

Example 2.1. If we take $I = I_f = \{A \subseteq \mathbb{N} : A \text{ is a finite subset}\}$. Then, I_f is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with usual convergence of sequences.

Example 2.2. If we take $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$, where $\delta(A)$ denote the asymptotic density of the set A. Then I_δ is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with statistical convergence of sequences.

Let $\ell^1 = \left\{x = (x_k) : \sum_{k=1}^{\infty}|x_k| < \infty\right\}$.

For a summability transformation A, we use $d(A)$ to denote the domain of A:

$$d(A) = \left\{x = (x_k) : \lim_n \sum_{k=1}^{\infty} a_{n,k} x_k \text{ exists}\right\}.$$

Also $S_\delta = \{x = (x_k) : x_k \geqslant \delta > 0 \text{ for all } k\}$ and

$S_0 = \{$the set of all nonnegative sequences which have at most a finite number of zero entries$\}$.

For a sequence $x = (x_k)$ in ℓ^1 or ℓ_∞, we also define

$$R_n(x) := \sum_{k=n}^{\infty}|x_k| \text{ and } \mu_n x := \sup_{k \geqslant n}|x_k| \text{ for } n \geqslant 0.$$

Definition 2.5 [8]. Two nonnegative sequences (x_k) and (y_k) are said to be *asymptotically equivalent*, written as $x \sim y$ if $\lim_k \dfrac{x_k}{y_k} = 1$.

Definition 2.6. If $A = (a_{n,k})$ is a sequence of infinite matrices, then a sequence $x = (x_k) \in \ell_\infty$ is said to be *A-summable* to the value x_0 if

$$\lim_n (Ax)_n = \lim_n \sum_{k=1}^{\infty} a_{n,k} x_k = x_0.$$

Definition 2.7. A summability matrix A is *asymptotically regular* provided that $Ax \sim Ay$ whenever $x \sim y, x \in S_0$ and $y \in S_\delta$ for some $\delta > 0$.

The following results will be used for establishing some results of this article.

Lemma 2.1 (*Pobyvancts [9]*). *A nonnegative matrix A is asymptotically regular if and only if for each fixed integer m,*

$$\lim_{n \to \infty} \frac{a_{n,m}}{\sum_{k=1}^{\infty} a_{n,k}} = 0.$$

Lemma 2.2. *A matrix A which maps c_0 to c_0 if and only if*

(a) $\lim_{n \to \infty} a_{n,k}$ *for* $k = 1, 2, 3, \ldots$
(b) *There exists a number $M > 0$ such that for each $n, \sum_{k=1}^{\infty}|a_{n,k}| < M$. Throughout the article I is an admissible ideal of subsets of* $\mathbb{N}$.

3. Asymptotically *I*-equivalent sequences

In this section we introduce the following definitions and prove the decomposition theorem and some interesting theorems.

Definition 3.1. Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be *asymptotically I-equivalent* of multiple $\lambda \in \mathbb{R}$, written as $x \overset{I_\lambda}{\sim} y$, provided for every $\varepsilon > 0$, and $y_k \neq 0$, the set

$$\left\{k \in \mathbb{N} : \left|\frac{x_k}{y_k} - \lambda\right| \geqslant \varepsilon\right\}$$

belongs to I and in this case we write $I - \lim_k \frac{x_k}{y_k} = \lambda$, simply asymptotically I-equivalent if $\lambda = 1$. It is easy to observe that $x \overset{I_\lambda}{\sim} y$ is equivalent to $\frac{x_k}{\lambda} \overset{I}{\sim} y_k$. From this observation it follow, that we obtain the same notion if we use all real λ's, some $\lambda \neq 0$, or just $\lambda = 1$.

Example 3.1. Let us consider the sequences $x = (x_k)$ and $y = (y_k)$ as follows:

$$x_k = \begin{cases} 3k^{-1}, & \text{if } k \text{ is even;} \\ (k+2)^{-1}, & \text{if } k \text{ is odd} \end{cases}$$

and

$$y_k = \begin{cases} k^{-1}, & \text{if } k \text{ is even;} \\ 3(k+2)^{-1}, & \text{if } k \text{ is odd} \end{cases}$$

Therefore we have

$$\frac{x_k}{y_k} = \begin{cases} 3, & \text{if } k \text{ is even;} \\ 3, & \text{if } k \text{ is odd} \end{cases}$$

Thus

$$\left\{k \in \mathbb{N} : \left|\frac{x_k}{y_k} - 3\right| \geqslant \varepsilon\right\} \in I$$

Hence $x = (x_k)$ and $y = (y_k)$ are asymptotically I-equivalent of multiple 3.

Definition 3.2. A summability matrix A is said to be *asymptotically I-regular* provided that $Ax \overset{I_\lambda}{\sim} Ay$ whenever $x \overset{I_\lambda}{\sim} y, x \in S_0^I$ and $y \in S_\delta$ for some $\delta > 0$.

Example 3.2. Let us consider the sequences $x = (x_k)$ and $y = (y_k)$ as follows:

$x_k = 3 = y_k$ for all $k \in \mathbb{N}$.

Let A be defined as follows:

$$\begin{pmatrix} 3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & \dots \\ cdots & cdots & cdots & cdots & cdots & cdots & cdots & cdots & cdots & \dots \end{pmatrix}.$$

We have

$Ax = (18, 18, \dots) = Ay$

Then we have

$$I - \lim_k \frac{x_k}{y_k} = 1 \text{ and } I - \lim_n \frac{(Ax)_n}{(Ay)_n} = 1.$$

i.e. $x \overset{I_1}{\sim} y$ implies $Ax \overset{I_1}{\sim} Ay$.

Theorem 3.1. *Let $x = (x_k)$ and $y = (y_k)$ be two elements in S_0^I be such that $x \overset{I}{\sim} y$. Then there exists a sequence $z = (z_k)$ in S_0^I such that $x \overset{I}{\sim} y \overset{I}{\sim} z$.*

Proof. The proof of the theorem is trivial, thus omitted. □

Theorem 3.2. *Let I has the property (AP). Let $x = (x_k), y = (y_k) \in S_0^I$, then the followings are equivalent:*

(i) $x \overset{I}{\sim} y$.

(ii) *There exist $x' = (x'_k), y' = (y'_k) \in S_0$ such that $x_k = x'_k$ for a.a.k.r.I; $y_k = y'_k$ for a.a.k.r.I and $x' \sim y'$.*

(iii) *There exists a subset $K = \{k_i : i \in \mathbb{N}\}$ of $\mathbb{N}$ such that $K \in F$ and $(x_{k_i}) \sim (y_{k_i})$.*

Proof. (i) ⇒ (ii) Let $x = (x_k) \in S_0^I$, then there exists a subset A_1 of $\mathbb{N}$ with $A_1 \in F$ such that

$$\lim_k x_k = 0, \text{ over } A_1.$$

Again if $y = (y_k) \in S_0^I$, then there exists a subset A_2 of $\mathbb{N}$ with $A_2 \in F$ such that

$$\lim_k y_k = 0, \text{ over } A_2.$$

Let $x \overset{I}{\sim} y$, then there exists a subset A_3 of $\mathbb{N}$ with $A_3 \in F$ such that

$$\lim_k \frac{x_k}{y_k} = 0, \text{ over } A_3.$$

Let $A = A_1 \cap A_2 \cap A_3$, then $A \in F$.

We define the subsequences $x' = (x'_k)$, $y' = (y'_k)$ as follows:

$$x'_k = \begin{cases} x_k, & \text{if } k \in A; \\ k^{-3}, & \text{otherwise} \end{cases}$$

and

$$y'_k = \begin{cases} y_k, & \text{if } k \in A; \\ k^{-3}, & \text{otherwise} \end{cases}$$

Clearly $x' = (x'_k), y' = (y'_k) \in S_0$ and $x_k = x'_k$ for *a.a.k.r.I*; $y_k = y'_k$ for *a.a.k.r.I*. Also we have $x' \sim y'$.

(ii) ⇒ (iii) Let $x' = (x'_k), y' = (y'_k) \in S_0$ be such that $x_k = x'_k$ for *a.a.k.r.I*; $y_k = y'_k$ for *a.a.k.r.I* and $x' \sim y'$.

Let $B_1 = \{k \in \mathbb{N} : x_k = x'_k\}$ and $B_2 = \{k \in \mathbb{N} : y_k = y'_k\}$. Then $B_1, B_2 \in F$.

Put $K = B_1 \cap B_2$. Then $K \in F$.

Since $K \subset \mathbb{N}$, we can enumerate K as $K = \{k_i : i \in \mathbb{N}\}$. Then $(x_{k_i}) = (x'_{k_i}) \in S_0$ and $(y_{k_i}) = (y'_{k_i}) \in S_0$. Also we have

$$\frac{x_{k_i}}{y_{k_i}} = \frac{x'_{k_i}}{y'_{k_i}} \to 1 \text{ as } l \to \infty.$$

Hence $(x_{k_i}) \sim (y_{k_i})$.

(iii) ⇒ (i) Let $K = \{k_i : i \in \mathbb{N}\}$ be a subset of $\mathbb{N}$ with $K \in F$ and $(x_{k_i}) \sim (y_{k_i})$. Then, we have

$$\lim_i \frac{x_{k_i}}{y_{k_i}} = 1.$$

Therefore we have

$$\left\{k \in \mathbb{N} : \left|\frac{x_k}{y_k} - 1\right| \geqslant \varepsilon\right\} \in I$$

Hence $x \overset{I}{\sim} y$. □

Theorem 3.3. *A necessary and sufficient condition for a sequence of summability matrices A to be asymptotically I-regular is that for each fixed positive integer k_0:*

(i) $\sum_{p=1}^{k_0} a_{n,p}$ is bounded for each n;
(ii) For $\varepsilon > 0$ and for each k_0 such that

$$\left\{p \in \mathbb{N} : \left|\frac{\sum_{p=1}^{k_0} a_{n,p}}{\sum_{p=1}^{\infty} a_{n,p}}\right| \geqslant \varepsilon\right\} \in I.$$

Proof. The necessary part of this theorem is easy, so omitted. To establish the sufficient part, let $\varepsilon > 0$ be given and $x \overset{I_\lambda}{\sim} y, x \in S_0^I$ and $y \in S_\delta$ for some $\delta > 0$, then we have for some $t = 1, 2, 3, \ldots,$

$$(\lambda - \varepsilon)y_{k+t} \leqslant x_{k+t} \leqslant (\lambda + \varepsilon), \text{ for } a.a.k.r.I. \tag{3.1}$$

Let us consider the following:

$$\frac{(Ax)_n}{(Ay)_n} = \frac{\sum_{p=1}^{t} a_{n,p}x_p + \sum_{p=1+t}^{\infty} a_{n,p}x_p}{\sum_{p=1}^{t} a_{n,p}y_p + \sum_{p=1+t}^{\infty} a_{n,p}y_p} = \frac{\frac{\sum_{p=1}^{t} a_{n,p}x_p}{\sum_{p=1+t}^{\infty} a_{n,p}y_p} + \frac{\sum_{p=1+t}^{\infty} a_{n,p}x_p}{\sum_{p=1+t}^{\infty} a_{n,p}y_p}}{\frac{\sum_{p=1}^{t} a_{n,p}y_p}{\sum_{p=1+t}^{\infty} a_{n,p}y_p} + 1}. \tag{3.2}$$

The inequality (3.1) implies that

$$\lim_n \frac{\sum_{p=1+t}^{\infty} a_{n,p}x_p}{\sum_{p=1+t}^{\infty} a_{n,p}y_p} = \lambda \text{ for } a.a.n.r.I.$$

Since $x \in S_0^I$ and $y \in S_\delta$ for some $\delta > 0$ and condition (ii) holds, we obtain the following:

$$\lim_n \frac{\sum_{p=1}^{t} a_{n,p}x_p}{\sum_{p=1+t}^{\infty} a_{n,p}y_p} = 0, \text{ for } a.a.n.r.I.$$

and

$$\lim_n \frac{\sum_{p=1}^{t} a_{n,p}y_p}{\sum_{p=1+t}^{\infty} a_{n,p}y_p} = 0, \text{ for } a.a.n.r.I.$$

Thus from the relation (3.2), we have

$$\lim_n \frac{(Ax)_n}{(Ay)_n} = \lambda \text{ for } a.a.n.r.I$$

i.e.

$$\left\{n \in \mathbb{N} : \left|\frac{(Ax)_n}{(Ay)_n} - \lambda\right| \geqslant \varepsilon\right\} \in I.$$

This implies that $Ax \overset{I_\lambda}{\sim} Ay$, whenever $x \overset{I}{\sim} y, x \in S_0^I$ and $y \in S_\delta$, for some $\delta > 0$.

This completes the proof. □

Theorem 3.4. *Let $A = (a_{n,k})$ be an infinite nonnegative matrix. Suppose $x \overset{I}{\sim} y$ and $x \in S_0^I$ and $y \in S_\delta$, for some $\delta > 0$. Then $(\mu Ax) \overset{I}{\sim} (\mu Ay)$, if and only if for each $i = 1, 2, 3, \ldots$ and for $\varepsilon > 0$ such that*

$$\left\{n \in \mathbb{N} : \left|\frac{a_{n,i}}{\sum_{j=1}^{\infty} a_{n,j}}\right| \geqslant \varepsilon\right\} \in I.$$

Proof. Suppose for for $\varepsilon > 0$ and for each $i = 1, 2, 3, \ldots$ such that

$$\left\{n \in \mathbb{N} : \left|\frac{a_{n,i}}{\sum_{j=1}^{\infty} a_{n,j}}\right| \geqslant \varepsilon\right\} \in I.$$

We want to show that $(\mu Ax) \overset{I}{\sim} (\mu Ay)$.

Since $x \overset{I}{\sim} y$, then there exists a bounded sequence $z = (z_k)$ with I-limit zero such that $x_k = y_k(1 + z_k), k = 1, 2, 3, \ldots$. Then for each n, we have the following:

$$\begin{aligned}\frac{(\mu Ax)_n}{(\mu Ay)_n} &= \frac{\sup_{k \geqslant n}(Ax)_k}{\sup_{k \geqslant n}(Ay)_k} \\ &= \frac{\sup_{k \geqslant n}\sum_{i=1}^{\infty} a_{k,i}x_i}{\sup_{k \geqslant n}\sum_{i=1}^{\infty} a_{k,i}y_i} \\ &= \frac{\sup_{k \geqslant n}\sum_{i=1}^{\infty} a_{k,i}(y_i + y_i z_i)}{\sup_{k \geqslant n}\sum_{i=1}^{\infty} a_{k,i}y_i} \\ &\leqslant \frac{\sup_{k \geqslant n}\left|\sum_{i=1}^{\infty} a_{k,i}(y_i + y_i z_i)\right|}{\sup_{k \geqslant n}\sum_{i=1}^{\infty} a_{k,i}y_i} \\ &\leqslant 1 + \frac{\sup_{k \geqslant n}\sum_{i=1}^{t_0} a_{k,i}y_i|z_i|}{\sup_{k \geqslant n}\sum_{i=1}^{\infty} a_{k,i}y_i} \\ &\quad + \frac{\sup_{k \geqslant n}\sum_{i=t_0+1}^{\infty} a_{k,i}y_i|z_i|}{\sup_{k \geqslant n}\sum_{i=1}^{\infty} a_{k,i}y_i},\end{aligned}$$

where t_0 is a positive integer.

Since z is a bounded null sequence, therefore $\sup_j|z_j| < \infty$ and for any $\varepsilon > 0$, there exists a positive integer t_0 such that $|z_j| < \varepsilon$ for $j \geqslant t_0$. Therefore we have

$$\begin{aligned}\frac{(\mu Ax)_n}{(\mu Ay)_n} &\leqslant 1 + \sup_j|z_j|\sum_{i=1}^{t_0}\frac{\sup_{k \geqslant n} a_{k,i}y_i}{\sup_{k \geqslant n}\sum_{i=1}^{\infty} a_{k,i}y_i} \\ &\quad + \frac{\varepsilon\sup_{k \geqslant t_0}\sum_{i=t_0+1}^{\infty} a_{k,i}y_i}{\sup_{k \geqslant n}\sum_{i=1}^{\infty} a_{k,i}y_i}\end{aligned}$$

$$\leqslant 1 + \sup_j|z_j|\sum_{i=1}^{t_0}\frac{\sup_{k \geqslant n} a_{k,i}y_i}{\sup_{k \geqslant n}\sum_{i=1}^{\infty} a_{k,i}y_i} + \varepsilon.$$

According to the hypothesis, we obtain the following:

$$\frac{a_{k,i}}{\sum_{i=1}^{\infty} a_{k,i}} < \frac{\varepsilon}{t_0}\sup_j|z_j| \sup_{0 < i \leqslant t_0} y_i, \text{ for } a.a.k.r.I.$$

For $n \geqslant t_0$ we have

$$\frac{(\mu Ax)_n}{(\mu Ay)_n} \leqslant 1 + \varepsilon + \varepsilon, \text{ for } a.a.n.r.I.$$

This implies that

$$\lim_n \frac{(\mu Ax)_n}{(\mu Ay)_n} \leqslant 1, \text{ for } a.a.n.r.I.$$

In a similar manner, we can prove that

$$\lim_n \frac{(\mu Ax)_n}{(\mu Ay)_n} \geqslant 1, \text{ for } a.a.n.r.I.$$

Thus we have

$$\lim_n \frac{(\mu Ax)_n}{(\mu Ay)_n} = 1, \text{ for } a.a.n.r.I.$$

i.e.

$$\left\{n \in \mathbb{N} : \left|\frac{(\mu Ax)_n}{(\mu Ay)_n} - 1\right| \geqslant \varepsilon\right\} \in I.$$

Hence $(\mu Ax) \overset{I}{\sim} (\mu Ay)$.

Next, suppose that $(\mu Ax) \overset{I}{\sim} (\mu Ay)$, for $x \overset{I}{\sim} y$ and $x \in S_0^I$ and $y \in S_\delta$, for some $\delta > 0$. If we consider the sequences x and y defined by

$x_k = 1 = y_k$ for all $k \in \mathbb{N}$.

Then $(\mu Ax) \overset{I}{\sim} (\mu Ay)$. i.e.

$$\lim_n \frac{\sup_{k \geqslant n} \sum_{i=1}^{\infty} a_{k,i}}{\sup_{k \geqslant n} \sum_{i=1}^{\infty} a_{k,i}} = 1, \text{ for } a.a.n.r.I.$$

Therefore, there exists $K > 0$ such that $\left\{\sum_{i=1}^{\infty} a_{k,i}\right\}_{k=1}^{\infty}$ is bounded by K.

Suppose

$$\left\{ n \in \mathbb{N} : \left| \frac{a_{n,i}}{\sum_{j=1}^{\infty} a_{n,j}} \right| < \varepsilon \right\} \in F \text{ for some } i \text{ and } \varepsilon > 0.$$

Then there exists $\gamma > 0$ and a sequence $n_1 < n_2 < \dots$ such that

$$\frac{a_{u,i}}{\sum_{j=1}^{\infty} a_{u,j}} \geqslant \gamma \text{ for } u = 1, 2, 3 \dots.$$

For $s > 0$ and define the sequences x and y by

$$x_k = \begin{cases} 1 + s, & \text{if } k = i; \\ 1, & \text{otherwise} \end{cases}$$

and

$y_k = 1$ for all $k \in \mathbb{N}$.

Clearly $x \overset{I}{\sim} y$ and $x, y \in S_1$. Consider the following limit:

$$\begin{aligned} \lim_{u \to \infty} \frac{\sup_{k \geqslant u} \sum_{j=1}^{\infty} a_{n_k,j} x_j}{\sup_{k \geqslant u} \sum_{j=1}^{\infty} a_{n_k,j} y_j} &= \lim_{u \to \infty} \frac{\sup_{k \geqslant u} \sum_{j=1}^{\infty} \left(a_{n_k,j} + s a_{n_k,j}\right)}{\sup_{k \geqslant u} \sum_{j=1}^{\infty} a_{n_k,j}} \\ &\geqslant \lim_{u \to \infty} \frac{\sup_{k \geqslant u} \left(\sum_{j=1}^{\infty} a_{n_k,j} + s\gamma \sum_{j=1}^{\infty} a_{n_k,j} \right)}{\sup_{k \geqslant u} \sum_{j=1}^{\infty} a_{n_k,j}} \\ &= 1 + s\gamma. \end{aligned}$$

We choose $s = \frac{1}{\gamma}$, then we have

$$\lim_{u \to \infty} \frac{(\mu Ax)_{n_u}}{(\mu Ay)_{n_u}} \geqslant 2.$$

This contradicts that $(\mu Ax) \overset{I}{\sim} (\mu Ay)$. □

Example 3.3. We consider the sequences x and y defined by

$x_k = 3 = y_k$ for all $k \in \mathbb{N}$.

Let A be defined as follows:

$$\begin{pmatrix} 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & \dots \\ . & . & . & . & . & . & . & . & . & \dots \end{pmatrix}.$$

We have

$(\mu Ax) = 12 = (\mu Ay)$

Then we have

$x \overset{I}{\sim} y$ and $(\mu Ax) \overset{I}{\sim} (\mu Ay)$.

Also for $i = 1, 2, 3, \dots$ we have

$$\left\{ n \in \mathbb{N} : \left| \frac{a_{n,i}}{\sum_{j=1}^{\infty} a_{n,j}} \right| \geqslant \varepsilon \right\} \in I.$$

Example 3.4. Consider the sequences x and y defined by

$x = (4, 4, 4, \dots)$ and $y = (4, 2, 4, 2, \dots)$.

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & . & \dots \\ 0 & \frac{1}{4} & 0 & 0 & . & \dots \\ 0 & 0 & \frac{1}{8} & 0 & . & \dots \\ 0 & 0 & 0 & \frac{1}{16} & . & \dots \\ . & . & . & . & . & \dots \end{pmatrix}.$$

We have

$$Ax = \left(4, 1, \frac{1}{2}, \frac{1}{4}, \dots\right)$$

$$Ay = \left(4, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \dots\right)$$

Also we have

$$\frac{\mu_n x}{\mu_n y} = 1, n = 1, 2, 3, \dots$$

i.e. $\mu_n x \overset{I}{\sim} \mu_n y$.

Again

$$\frac{(\mu Ax)_n}{(\mu Ay)_n} = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n \text{ is even} \end{cases}$$

But

$\frac{(\mu Ax)_n}{(\mu Ay)_n}$ has no limits as $n \to \infty$.

and

$\frac{x}{y}$ has no limits as $k \to \infty$.

i.e

$x \overset{I}{\nsim} y$ and $(\mu Ax) \overset{I}{\nsim} (\mu Ay)$.

4. Applications

We can used this concept to the derivation of the continuous time Gaussian channel capacity.

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