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ORIGINAL ARTICLE

A dynamic Cournot duopoly model with different strategies

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Abstract This paper analyzes the dynamics of a Cournot duopoly model with different strategies. We offer results on existence, stability and local bifurcations of the equilibrium points. The bifurcation diagrams and Lyapunov exponents of the model are presented to show that the model behaves chaotically with the variation in the parameters. The state variables feedback and parameter variation methods are used to control the chaos of the model.

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1. Introduction

A Cournot duopoly game is an oligopoly market with two players. Oligopoly was first introduced by Cournot [1]. In a typical Cournot oligopoly, there are two or more players, no other players can enter the market, and collusive behavior is prohibited. Each player in the oligopoly market aims to maximize its expected profit, and profit is maximized when marginal revenue equals marginal cost. Recently, it has also been shown that even oligopolistic markets may become chaotic under certain conditions [2–10]. In general, in order to adjust his output, a player can choose his expectation rule among many available strategies. Naive, adaptive, boundedly rational and local approximation expectations are only a few examples. In the literature on oligopoly games, most papers focus on games with homogeneous strategies, that is, players who adopt the same expectation rule. Another branch of the literature is made up of studies in which games with different strategies are taken into consideration. The assumption of players adopting heterogeneous rules to decide their production is, in our opinion, more realistic than the opposite case. This approach characterizes the works by Leonard and Nishimura [11], Den-Haan [12], Agiza et al. [13], Agiza and Elsadany [14,15]. Zhang et al. [16] used the technique of Agiza and Elsadany to analyze a duopoly game with heterogeneous players and nonlinear cost function. Angelini et al. [17] and Tramontana [18] studied a duopoly game with heterogeneous firms assuming a microfounded nonlinearity on the demand function.

The main purpose of this paper is to investigate the dynamic behavior and control of duopoly game with different strategies. We considered that each player forms a different strategy in order to compute its expected output. We assume that the first player represents a boundedly rational player and the second

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player has local approximation expectations. The main aim of this work is to investigate the dynamic behaviors of the two players game using different expectation rules. Moreover, from a mathematical point of view, it is shown that the loss of the market equilibrium stability may occur through a flip bifurcation and that a cascade of flip bifurcations may lead to periodic cycles and chaos.

The paper is organized as follows. In Section 2 we describe a nonlinear duopoly game model. In Section 3 we study the fixed points and the dynamics of the model, showing explicit parametric conditions of the existence and local stability of the market equilibrium. Section 4 the results of the previous section are numerically illustrated and showing the occurrence of complex behaviors. In Section 5, we exerted control on the duopoly game model. Section 6 concludes.

2. The model

We consider a duopoly Cournot game where $x_i^t, i = 1, 2$ represent the quantity supplied by ith player during period $t = 0, 1, 2 \ldots$. Assume that the inverse demand function [19,20] is

$$p(X) = a - b\sqrt{X} \tag{1}$$

where $X = x_1 + x_2$ is the total quantity in the market, a and b are positive constants. This function is convex as the isoelastic demand function but it does not tend to infinity as $p \to 0$. In fact, $(a/b)^2$ represents the maximum amount of output that can be brought to the market. Those properties are important/relevant from an economic point of view. Moreover this form has also used in others oligopoly models and in laboratory experiments economics dealing with learning and expectations formation (see e.g. [21–23]). The cost function of the players is as follows:

$$C_i(x_i) = c_i x_i, \qquad i = 1, 2. \tag{2}$$

where the positive parameters c_i are the marginal costs. The profit function of player i is

$$\Pi_i(x_1, x_2) = x_i\left(a - b\sqrt{X}\right) - c_i x_i, \quad i = 1, 2. \tag{3}$$

From the profit maximization by player i, the marginal profits are obtained as:

$$\Phi i = \frac{\partial \Pi_i}{\partial x_i} = a - c_i - b\sqrt{X} - \frac{b x_i}{2\sqrt{X}}, \quad i = 1, 2 \tag{4}$$

We assume different expectations: i.e., player 1 is boundedly rational and player 2 is local approximation. The boundedly rational player 1 has no complete knowledge of the market; hence they try to use local information based on the marginal profit $\frac{\partial \Pi_1}{\partial x_1}$. It decides to increase (decrease) its quantity if it has a positive (negative) marginal profit. This adjustment mechanism has been called myopic by Dixit [2]. Thus, the dynamic adjustment mechanism can be modeled as follows:

$$x_1^{t+1} = x_1^t + \alpha_1 x_1^t \frac{\partial \pi_1\left(x_1^t, x_2^t\right)}{\partial x_1^t}, \tag{5}$$

where α_1 is a positive parameter which represents the speed of adjustment of first player.

The second duopolist is a local approximation player (see [19,23]), i.e.

$$x_2^{t+1} = \frac{x_2^{t+1}}{2} + \frac{c_1 - f(X^t)}{2f'(X^t)} \tag{6}$$

Therefore, given these types of strategies formation, the two-dimensional system that characterizes the dynamics of a Cournot duopoly game is the following:

$$\begin{cases} x_1^{t+1} = x_1^t + \alpha_1 x_1^t\left(a - c_1 - b\sqrt{x_1^t + x_2^t} - \frac{b x_1^t}{2\sqrt{x_1^t + x_2^t}}\right) \\ x_2^{t+1} = \frac{2(a - c_2)\sqrt{x_1^t + x_2^t} - 2b x_1^t - b x_2^t}{2b} \end{cases} \tag{7}$$

We are interested only in positive trajectories. Note also that the game is not defined in the origin $(0, 0)$. In the next section, we study the dynamical behaviors of the map (7).

3. Equilibrium points and local stability

In this section, we determine the equilibrium points of the map (7) by solving the following nonlinear algebraic system:

$$\begin{cases} x_1^*\left(a - c_1 - b\sqrt{x_1^* + x_2^*} - \frac{b x_1^*}{2\sqrt{x_1^* + x_2^*}}\right) = 0 \\ 2(a - c_2)\sqrt{x_1^* + x_2^*} - 2b x_1^* - 3b x_2^* = 0 \end{cases} \tag{8}$$

This map has two equilibrium points:

$$E_1 = \left(0, \frac{4(a - c_2)^2}{9b^2}\right) \tag{9}$$

$E_2 = \left(x_1^*, x_2^*\right)$

where

$$x_1^* = \frac{4(2a - c_1 - c_2)(a + 2c_2 - 3c_1)}{25b^2}, \quad x_2^* = \frac{4(2a - c_1 - c_2)(a + 2c_1 - 3c_2)}{25b^2} \tag{10}$$

E_2 is called Nash equilibrium point and has positive coordinates provided that

$$\begin{cases} 2a > c_1 + c_2 \\ a > 3c_1 - 2c_2 \\ a > 3c_2 - 2c_1 \end{cases} \tag{11}$$

In order to investigate the local stability of the equilibrium points, we must consider the Jacobian matrix of the map (7) is the following:

$$J = \begin{bmatrix} 1 + \alpha_1\left[a - c_1 - b\sqrt{X} - \frac{b x_1}{2\sqrt{X}} - \frac{3b x_1^2 + 4b x_1 x_2}{4\sqrt{X^3}}\right) & -\frac{\alpha_1\left(b x_1^2 + 2b x_1 x_2\right)}{4\sqrt{X^3}} \\ \frac{a - c_2}{2b\sqrt{x_1 + x_2}} - 1 & \frac{a - c_2}{2b\sqrt{x_1 + x_2}} - \frac{1}{2} \end{bmatrix} \tag{12}$$

The equilibrium points will be stable if the eigenvalues λ_i, $i = 1, 2$ of the above Jacobian matrix satisfy inequalities $|\lambda_i| < 1, \quad i = 1, 2$. By applying the stability condition to the equilibrium E_1 we have the following result:

Proposition 1. *If the Nash equilibrium E_2 is strictly positive, then the equilibrium point E_1 is a saddle point.*

Proof. In fact at E_1, the Jacobian matrix becomes a triangular matrix:

$$J(E_1) = \begin{bmatrix} 1 + \alpha_1 \frac{(a + 2c_2 - 3c_1)}{3} & 0 \\ \frac{-1}{4} & \frac{1}{4} \end{bmatrix}$$

whose eigenvalues are given by the diagonal entries. They are $\lambda_1 = 1 + \alpha_1 \frac{(a + 2c_2 - 3c_1)}{3}$ and $\lambda_2 = \frac{1}{4}$. If the Nash equilibrium point has positive coordinates, then $a > 3c_1 - 2c_2$, hence $|\lambda_1| > 1$ and $|\lambda_2| < 1$. Then E_1 is a saddle point. □

3.1. Stability of the Nash equilibrium point E_2

In order to investigate the local stability properties of the Nash equilibrium point (10) of the two-dimensional system (7), the Jacobian matrix evaluated at E_2, that is:

$$J(E_2) = \begin{bmatrix} 1 - \frac{\alpha_1\left(3bx_1^{*2} + 4bx_1^* x_2^*\right)}{4\sqrt{X^{*3}}} & -\frac{\alpha_1\left(bx_1^{*2} + 2bx_1^* x_2^*\right)}{4\sqrt{X^{*3}}} \\ \frac{a-c_2}{2b\sqrt{x_1^*+x_2^*}} - 1 & \frac{a-c_2}{2b\sqrt{x_1^*+x_2^*}} - \frac{1}{2} \end{bmatrix} \tag{13}$$

where $X^* = x_1^* + x_2^*$.

Whose trace and determinant are given by:

$$T := Tr(J(E_2)) = \frac{1}{2} - \alpha_1\theta + \phi$$

$$D := Det(J(E_2)) = \phi - \frac{1}{2} - \alpha_1\psi\phi + \alpha_1\sigma$$

where

$$\theta = \frac{\left(3bx_1^{*2} + 4bx_1^* x_2^*\right)}{4\sqrt{X^{*3}}}, \phi = \frac{a-c_2}{2b\sqrt{X^*}}, \psi = \frac{\left(bx_1^{*2} + bx_1^* x_2^*\right)}{2\sqrt{X^{*3}}} \tag{14}$$

and $\sigma = \dfrac{bx_1^{*2}}{8\sqrt{X^{*3}}}$

so the characteristic polynomial of (13) is:

$$P(\lambda) = \lambda^2 - T\lambda + D, \tag{15}$$

whose discriminant is $\Delta = T^2 - 4D$.

From the stability theory we know that the fixed point E_2 is locally asymptotically stable as long as the eigenvalues of Jacobian matrix $J(E_2)$are inside the unit circle of the complex plane. This is true if and only if the following Jury's [24] stability criteria are hold:

$$\begin{cases} (1): \ F := 1 + T + D > 0, \\ (2): \ TC := 1 - T + D > 0, \\ (3): \ H := 1 - D > 0. \end{cases} \tag{16}$$

The above inequalities (16) define a region in which the Nash equilibrium point E_2 is local stable. The violation of any single inequality in (16), with other two being simultaneously fulfilled leads to: (1) a flip bifurcation (real eigenvalue that passes through -1) when $F = 0$; (2) a fold or transcritical bifurcation (a real eigenvalue that passes through $+1$) when $TC = 0$; (3) a Neimark-Sacker bifurcation (i.e., the modulus of a complex eigenvalue pair that passes through 1) when $H = 0$ and $|T| < 2$ (see [25,26]).

The discriminant of (13) is

$$\Delta = T^2 - 4D = \left(\frac{3}{2} - \alpha_1\theta - \phi\right)^2 + 2\alpha_1(2\psi\phi - 2\theta\phi + \theta - 2\sigma),$$

From Eqs. (10), (11), (14) and since a, b, c_1, c_2 and α_1 are positive parameters. Then $\psi > \theta$ and $\theta > 2\sigma$. Hence $\Delta > 0$, then the eigenvalues of Nash equilibrium are real.

For the special case of the Jacobian matrix (13), the stability conditions in (16) can be written as follows:

$$\begin{cases} (1): \ F := 1 - \alpha_1\theta + 2\phi - \alpha_1\psi\phi + \alpha_1\sigma > 0, \\ (2): \ TC := \alpha_1\theta + \alpha_1\sigma - \alpha_1\psi\phi > 0, \\ (3): \ H := \frac{3}{2} - \phi + \alpha_1\psi\phi - \alpha_1\sigma > 0. \end{cases} \tag{17}$$

Since the discriminant is positive, the existence of complex eigenvalues of $J(E_2)$ is prevented. Then the condition (3) is always fulfilled. While from Eqs. (10), (11) and (14) it is clear that the condition (2) is satisfied. Therefore, the Nash equilibrium point E_2 can loose stability only through a flip bifurcation. So, we have the following proposition about local stability of Nash Equilibrium point E_2.

Proposition 2. *The Nash equilibrium point E_2 is asymptotically stable if $\alpha < \frac{1+2\phi}{\psi+\phi\theta-\sigma}$. The system (7) undergoes a flip bifurcation at E_2 when $\alpha^* = \frac{1+2\phi}{\psi+\phi\theta-\sigma}$. Moreover, period-2 points bifurcate from E_2 when $\alpha > \frac{1+2\phi}{\psi+\phi\theta-\sigma}$.*

Proof. From above results, the conditions (2) and (3) are always satisfied. However, condition (1) can be violated, since the flip bifurcation occurs when $F = 0$. Then

$$1 + T + D = 0;$$
$$1 - \alpha_1\theta + 2\phi - \alpha_1\psi\phi + \alpha_1\sigma = 0$$

Then from stability situation, the Nash equilibrium point can loose stability when $\alpha^* = \frac{1+2\phi}{\psi+\phi\theta-\sigma}$. Hence the Nash equilibrium point is asymptotically stable if $\alpha < \alpha^*$. Moreover, period-2 points bifurcate from E_2 when $\alpha > \alpha^*$. □

Next we present the results of the numerical simulations for the system (7).

4. Analysis and numerical simulation

The main purpose of this section is to show that the qualitative behavior of the solutions of the nonlinear duopoly game with heterogeneous players (described by the dynamic system (7)). To provide some numerical evidence for the existence of chaotic motions, we use several standard tools, bifurcations diagrams, basin of attraction, lyapunov exponents, strange attractors, sensitive dependence on initial conditions and so on.

Fig. 1 presents a bifurcation diagram of system (7) in $(\alpha_1 - x_1x_2)$ plane when $a = 10, b = 1, c_1 = 1$ and $c_2 = 3$. From Fig. 1, we can see that the orbit with initial values $(0.25, 0.2)$ approaches to the stable fixed point for $\alpha_1 < 0.415$. With α_1 increasing, a flip bifurcation for system (7) takes place at $\alpha_1 = 0.415$ and period-2 points bifurcate as $\alpha_1 = .415$, which is verifies Proposition 2. Furthermore, the period-2 points are attracting when $\alpha_1$varies in the interval $(0.415, 0.565)$. As long

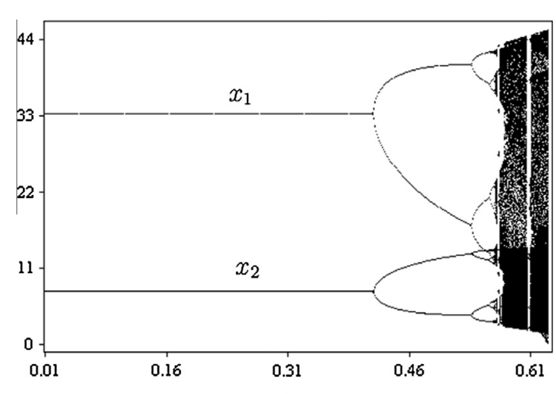

Figure 1 Bifurcation diagram of the model (7) with respect to α_1.

as the parameter $\alpha_1$increases, the Nash equilibrium point E_2 becomes unstable and the bifurcation scenario occurs and ultimately leads to unpredictable (chaotic) motions that are observed. This means that immediately after the NS bifurcation the system displays complicated dynamics, differently from what happens after the period doubling bifurcation in which aperiodic attractor appears. While bifurcation and chaos occur, the output of players acutely fluctuate making it difficult for players to forecast their output and to make decisions in the future.

In order to classify the dynamic behavior of system (7) for many different parameters set, we compute the largest lyapunov exponent. Fig. 2 shows the largest lyapunov exponent corresponding to the bifurcation diagram Fig. 1. As it is positive, there is evidence of chaos and beyond that it is even possible to differentiate between cycles of very high order and aperiodic (chaotic) behavior of the system. Moreover, by comparing the standard bifurcation diagram in Fig. 1 with the diagram of largest lyapunov exponent, one can obtain a better understanding of the particular properties of the dynamic behavior of the system.

Fig. 3 shows the bifurcation diagram with respect to the parameter a with $b = 1$, $c1 = 1$, $c2 = 3$ and $\alpha_1 = 0.6$. The Nash equilibrium E_2 is locally stable for small values of a. As a increases, the Nash equilibrium point becomes unstable and complex dynamic behavior occurs, including higher-order cycles and chaos. The bifurcation diagram with respect to c_1

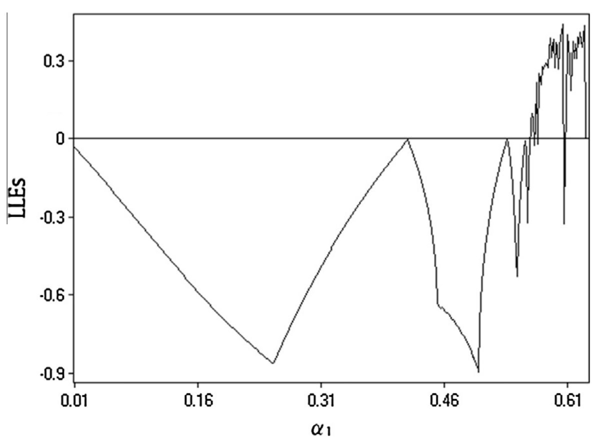

Figure 2 Largest Lyapunov exponents corresponding to Fig.1.

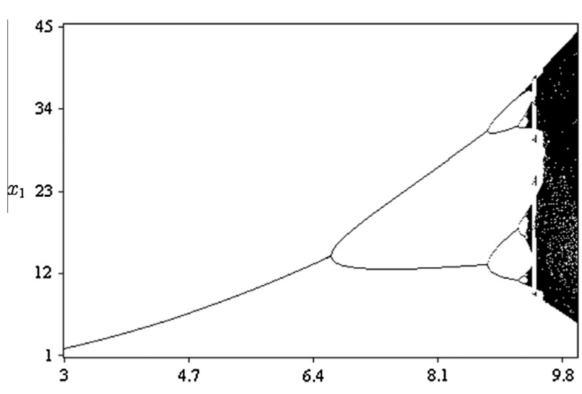

Figure 3 Bifurcation diagram of the model (7) with respect to a.

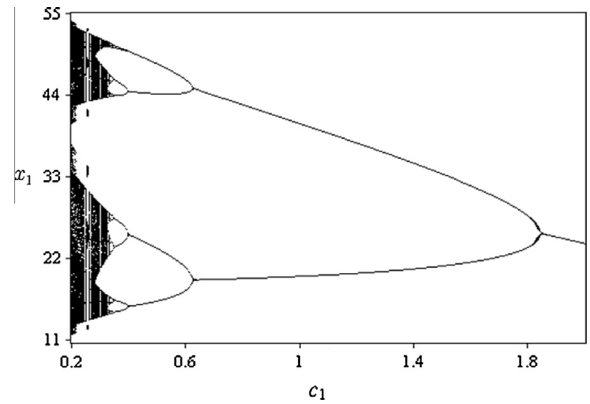

Figure 4 Bifurcation diagram of the model (7) with respect to c_1.

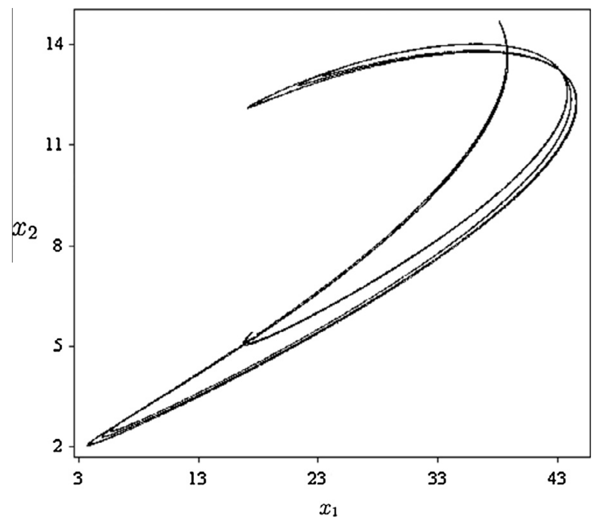

Figure 5 Strange attractor for the model (7).

when the other parameters take the values $(a, b, c_2, \alpha_1) = (10, 1, 3, 0.6)$ is given Fig. 4. We can see that the system dynamics is chaotic if the return rate c_1 is small. As c_1 increases, there exist period-halving bifurcations. The game experiences chaos and period-halving bifurcation.

A strange attractor is another characteristic of chaos of the system, and it reflects the inherent regularity of the complex phenomena in a chaotic state. Thus, players can forecast the market output in a short term according to inherent regularity while the system is in a chaotic state. Fig. 5 shows the graph of strange attractors of system (7) when $a = 10$, $b = 1$, $c_1 = 1$, $c_2 = 3$ and $\alpha_1 = 0.6$. Strange attractors show the complexity of players dynamic output competition in chaos.

5. Chaos control

The appearance of chaos in the economic system is not expected and even is harmful. Thus, people hope to find some methods to control the chaos of economic system. A wide variety of methods have been proposed for controlling chaos in oligopoly models, for example, chaos control with OGY method in the Kopel duopoly game model was applied in [27], chaos control with modified straight-line stabilization method in an output duopoly competing evolution model have been studied by Du et al. [28], Holyst and Urbanowicz [29] had studied chaos control with time-delayed feedback method in an economical model. The dynamics and adaptive control of a duopoly advertising model based on heterogeneous expectations is presented in Ding et al. [30], and so on. Elabbasy et al. [31] have considered such a feedback control in their triopoly with heterogeneous players. Also Ding et al. [32] have applied feedback control on multi-team Bertrand model.

The above section results show that the duopoly market will become unstable and fall into chaos when output adjustment speed parameter out of the stable region. Therefore, it is necessary to take a control strategy to delay or eliminate the occurrence of bifurcation and chaos. Feedback and parameter variation are two methods for the chaos control. Recently, Luo et al. [33] proposed a new control method which is called as control strategy of the state variables feedback and parameter variation. Pu and Ma [34] have considered the state variables feedback and parameter variation method in their four oligopolist model. In this article, the same method will be used to control the chaos of system (7). We change two-dimensional discrete dynamic system (7) into the following format:

$$\begin{cases} x_1^{t+1} = (1-\mu)\left\{x_1^t + \alpha_1 x_1^t\left(a - c_1 - b\sqrt{x_1^t + x_2^t} - \frac{bx_1^t}{2\sqrt{x_1^t + x_2^t}}\right)\right\} + \mu x_1^t \\ x_2^{t+1} = (1-\mu)\left\{\frac{2(a-c_2)\sqrt{x_1^t + x_2^t} - 2bx_1^t - bx_2^t}{2b}\right\} + \mu x_2^t \end{cases} \tag{18}$$

System (7) will fall into instability region and chaos with the change of output modification speed for player 1. The chaotic state of system (7) with the change of speed adjustment α_1 for player 1 is controlled. Fig. 6 is the bifurcation diagram of controlled system (18) with the change of control parameter μ after adding control to the chaotic state ($a = 10, b = 1, c_1 = 1, c_2 = 3$ and $\alpha_1 = 0.6$). It can be seen from Fig. 6 that the system (18) is in a chaotic state when $\mu < 0.0465$; the first players output and the second players output are controlled in the four-fold period bifurcation state when $\mu = 0.0475$. The first player output and the second player output are controlled in the period-doubling bifurcation when $\mu = 0.0875$. Also, the period-doubling bifurcation disappears, and the system stabilizes at the Nash equilibrium point when $\mu > 0.27$. Fig. 7 shows that the chaotic system is controlled at fixed point when $\mu = 0.3$. Thus chaos control is successful. The method presented here can be applied to many chaotic dynamical systems.

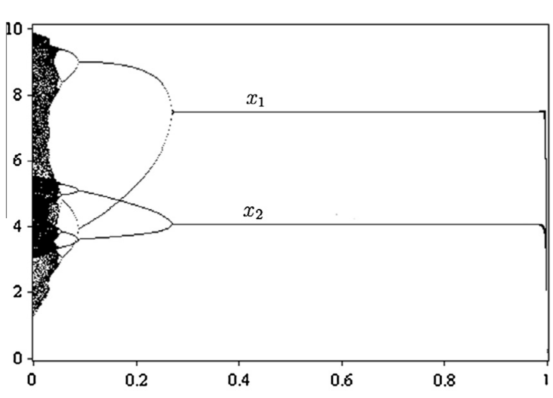

Figure 6 Bifurcation diagram of system (18) with respect to control parameter μ.

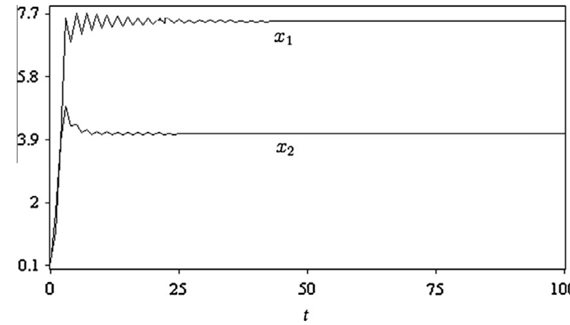

Figure 7 Time series of system (18) when control parameter $\mu = 0.3$.

6. Conclusion

A Cournot duopoly game model with different strategies is analyzed. The stability of the equilibrium points has been analyzed. From bifurcation diagrams and phase portraits, the basic properties of the game are presents. The results show that the quantity adjustment speed of boundedly rational player has an obvious impact on the stability of the players' dynamic quantity competition model. When it continues to increase, a series of chaotic phenomena occur: period doubling bifurcation, a positive Lyapunov exponent and strange attractors have been obtained. Finally, the model is quickly arrived at the Nash equilibrium point when a suitable controlling factor is chosen.

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