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ORIGINAL ARTICLE

Backward bifurcations in fractional-order vaccination models



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Abstract We describe and analyze some fractional order models for disease transmission with vaccination. In particular, we give conditions for the existence of multiple endemic equilibria and backward bifurcations. Numerical solutions of the models are given. Numerical simulations have been used to verify the theoretical analysis.

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1. Introduction

Recently, mathematical models describing the dynamics of human infectious diseases have played an important role in the disease control in epidemiology. Researchers have proposed many epidemic models to understand the mechanism of disease transmission [1].

Usually, these classical epidemic models have only one endemic equilibrium when the basic reproduction number $R_0 > 1$, and the disease-free equilibrium is always stable when $R_0 < 1$ and unstable when $R_0 > 1$. So the bifurcation leading from a disease-free equilibrium to an endemic equilibrium is forward. For a forward bifurcation, the bifurcation curve is as shown in Fig. 1 [1–5].

But in recent years, the phenomenon of the backward bifurcations has caused interest in disease control (see [6–15]). In this case, the basic reproduction number cannot describe the necessary disease elimination effort any more. Thus, it is important to identify backward bifurcations and establish thresholds for the control of diseases.

When forward bifurcation occurs, the condition $R_0 < 1$ is usually a necessary and sufficient condition for disease eradication, whereas it is no longer sufficient when a backward bifurcation occurs. In fact, the backward bifurcation scenario involves the existence of a subcritical transcritical bifurcation at $R_0 = 1$ and of a saddle-node bifurcation at $R_0 = R_0^c < 1$. There may be multiple positive endemic equilibria for values of $R_0 < 1$ and a backward bifurcation at $R_0 = 1$. This means that the bifurcation curve has the form shown in Fig. 2 with a broken curve denoting an unstable endemic equilibrium that separates the domains of attraction of asymptotically stable equilibria.

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, economic, viscoelasticity, biology, physics and engineering. Recently, a large amount of

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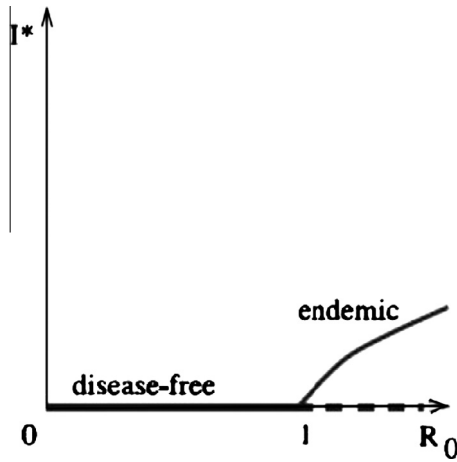


Figure 1 Forward bifurcation.

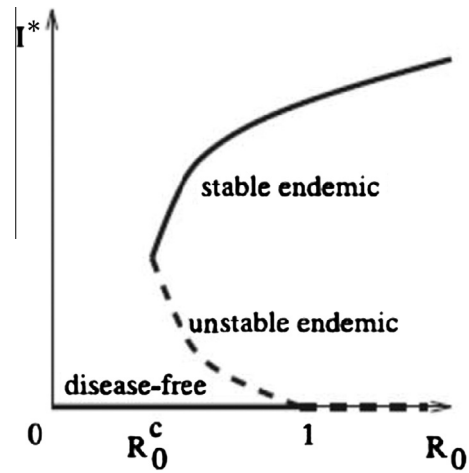


Figure 2 Backward bifurcation.

literature has been developed concerning the application of fractional differential equations in nonlinear dynamics [16].

In this paper we study some fractional order models for disease transmission with vaccination. The stability of equilibrium points is studied. Backward bifurcation in fractional order systems is studied. Numerical solutions of these models are given. Numerical simulations have been used to verify the theoretical analysis.

The reason for considering a fractional order system instead of its integer order counterpart is that fractional order differential equations are generalizations of integer order differential equations also fractional order calculus naturally includes memory effects which are important.

We like to argue that the fractional order models are at least as good as integer order ones in modeling biological, economic and social systems (generally complex adaptive systems) where memory effects are important.

In Section 2 the equilibrium points and their asymptotic stability of differential equations of fractional order are studied. In Sections 3–5 the models are presented and discussed. In Section 6 numerical solutions of the models are given.

Now we give the definition of fractional-order integration and fractional-order differentiation:

Definition 1. The fractional integral of order $\beta \in \mathbb{R}^+$ of the function $f(t)$, $t > 0$ is defined by

$$I^\beta f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds \quad (1)$$

and the fractional derivative of order $\alpha \in (n-1, n]$ of $f(t)$, $t > 0$ is defined by

$$D^\alpha f(t) = I^{n-\alpha} D^n f(t), \quad D = \frac{d}{dt}. \quad (2)$$

For the main properties of the fractional-orders derivatives and integrals [17–22].

2. Equilibrium points and their asymptotic stability

Let $\alpha \in (0, 1]$ and consider the initial value problem [23,24]

$$D^\alpha x(t) = f(x(t)), t > 0 \text{ and } x(0) = x_0. \quad (3)$$

To evaluate the equilibrium points of (3) let $D^\alpha x(t) = 0$, then $f(x_{eq}) = 0$ from which we can get the equilibrium points.

To evaluate the asymptotic stability, let $x(t) = x_{eq} + \varepsilon(t)$, which implies that

$$D^\alpha \varepsilon(t) = f'(x_{eq})\varepsilon(t), t > 0 \text{ and } \varepsilon(0) = x_0 - x_{eq}. \quad (4)$$

Now let the solution $\varepsilon(t)$ of (4) be exists. So if $\varepsilon(t)$ is increasing, then the equilibrium point x_{eq} is unstable and if $\varepsilon(t)$ is decreasing, then the equilibrium point x_{eq} is locally asymptotically stable.

Let $\alpha \in (0, 1]$ and consider the system [25–29]

$$\begin{aligned} D^\alpha y_1(t) &= f_1(y_1, y_2), \\ D^\alpha y_2(t) &= f_2(y_1, y_2), \end{aligned} \quad (5)$$

with the initial values

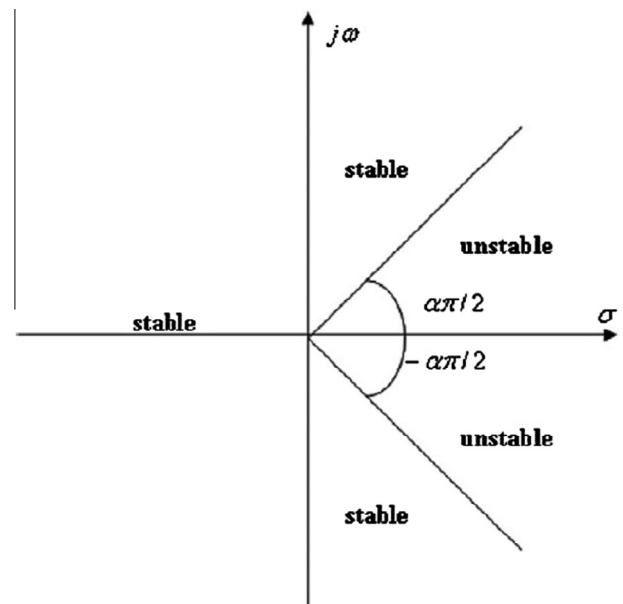


Figure 3 Stability region of the fractional-order system.

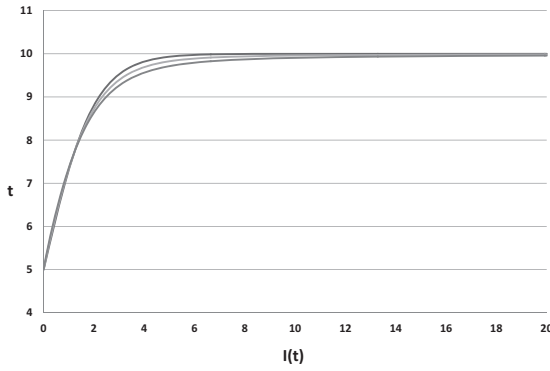


Figure 4

$$y_1(0) = y_{o1} \text{ and } y_2(0) = y_{o2}. \quad (6)$$

To evaluate the equilibrium points, let $D^\alpha y_i(t) = 0 \Rightarrow f_i(y_1^{eq}, y_2^{eq}) = 0, i = 1, 2$ from which we can get the equilibrium points y_1^{eq}, y_2^{eq} .

To evaluate the asymptotic stability, let $y_i(t) = y_i^{eq} + \varepsilon_i(t)$, so the equilibrium point (y_1^{eq}, y_2^{eq}) is locally asymptotically stable if both the eigenvalues of the Jacobian matrix

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix}$$

evaluated at the equilibrium point satisfies [25–29].

$$(|\arg(\lambda_1)| > \alpha\pi/2, |\arg(\lambda_2)| > \alpha\pi/2). \quad (7)$$

The stability region of the fractional-order system with order α is illustrated in Fig. 3 (in which σ, ω refer to the real and imaginary parts of the eigenvalues, respectively, and $j = \sqrt{-1}$). From Fig. 3, it is easy to show that the stability region of the fractional-order case is greater than the stability region of the integer-order case.

The eigenvalues equation of the equilibrium point (y_1^{eq}, y_2^{eq}) is given by the following polynomial [25]:

$$p(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0. \quad (8)$$

The conditions for (7) are either Routh–Hurwitz conditions or:

$$a_1 < 0, 4a_2 > (a_1)^2, \quad \left| \tan^{-1} \left(\sqrt{4a_2 - (a_1)^2} / a_1 \right) \right| > \alpha\pi/2. \quad (9)$$

3. Fractional-order SIS model

Let $S(t)$ and $I(t)$ denote the numbers of susceptible, infective individuals at time t respectively. The fractional-order SIS model is given by [3]

$$\begin{aligned} D^\alpha S(t) &= A - \beta SI - \mu S + \gamma I, \\ D^\alpha I(t) &= \beta SI - (\mu + \gamma)I, \end{aligned} \quad (10)$$

with the incorporation of a constant birth rate A in the susceptible class and a proportional natural death rate μ in each class and no disease deaths, β is the infectious contact rate, γ is the recovery rate for the disease and $0 < \alpha \leq 1$.

In (10) the total population size $N = S + I$ and $D^\alpha N = A - \mu N$.

Also we can reduce the dimension of the system (10) by using $S = N - I = K - I$ where $K = A/\mu$ is the population

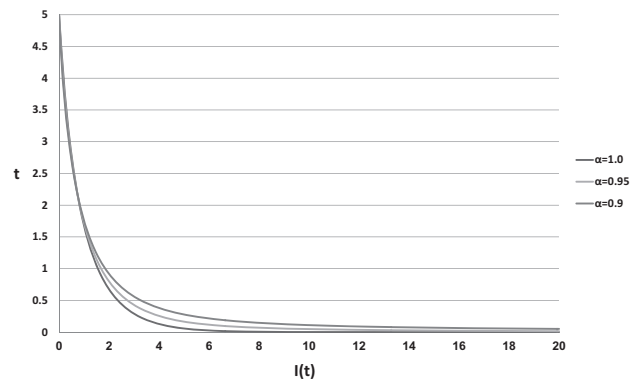


Figure 5

carrying capacity to give the single fractional order differential equation

$$D^\alpha I(t) = \beta(K - I)I - (\mu + \gamma)I. \quad (11)$$

To evaluate the equilibrium points, let $D^\alpha I(t) = 0$, then $I_{eq} = 0, K\left(1 - \frac{1}{R_0}\right), R_0 = \frac{\beta K}{(\mu + \gamma)}$ are the equilibrium points.

Now, to study the stability of the equilibrium points [3,24], we have

$$D^\alpha I(t) = f(I(t)), \quad f(I(t)) = \beta(K - I)I - (\mu + \gamma)I,$$

$$f'(0) = \beta K - (\mu + \gamma) < 0 \text{ if } R_0 < 1.$$

Hence a disease-free equilibrium $I = 0$ is locally asymptotically stable if $R_0 < 1$.

Also for the equilibrium point $I = K\left(1 - \frac{1}{R_0}\right)$ we have $f'\left(K\left(1 - \frac{1}{R_0}\right)\right) = -\beta K\left(1 - \frac{1}{R_0}\right) < 0$ if $R_0 > 1$. Hence if $R_0 > 1$ the disease-free equilibrium $I = 0$ is unstable but there is an endemic equilibrium $K\left(1 - \frac{1}{R_0}\right) > 0$ which is locally asymptotically stable.

4. Existence of uniformly stable solution

Let $x_1(t) = S(t), x_2(t) = I(t), f_1(x_1(t), x_2(t)) = A - \beta x_1(t)x_2(t) - \mu x_1(t) + \gamma x_2(t)$, and $f_2(x_1(t), x_2(t)) = \beta x_1(t)x_2(t) - (\mu + \gamma)x_2(t)$. Let $D = \{x_1, x_2 \in \mathbb{R} : |x_i(t)| \leq a, t \in [0, T], i = 1, 2\}$, then on D we have $|\frac{\partial}{\partial x_1} f_1(x_1, x_2)| \leq k_1, |\frac{\partial}{\partial x_2} f_1(x_1, x_2)| \leq k_2, |\frac{\partial}{\partial x_1} f_2(x_1, x_2)| \leq k_3$ and $|\frac{\partial}{\partial x_2} f_2(x_1, x_2)| \leq k_4$, where k_1, k_2, k_3 and k_4 are positive constants.

This implies that each of the two functions f_1, f_2 satisfies the Lipschitz condition with respect to the two arguments x_1 and x_2 , then each of the two functions f_1, f_2 is absolutely continuous with respect to the two arguments x_1 and x_2 .

Consider the following initial value problem which represents the fractional-order SIS model (12) and (13)

$$D^\alpha x_1(t) = f_1(x_1(t), x_2(t)), \quad t > 0 \quad \text{and} \quad x_1(0) = x_{o1}, \quad (12)$$

$$D^\alpha x_2(t) = f_2(x_1(t), x_2(t)), \quad t > 0 \quad \text{and} \quad x_2(0) = x_{o2}. \quad (13)$$

Definition 2. By a solution of the fractional-order SIS model (12) and (13) we mean a column vector $(x_1(t) \ x_2(t))^T, x_1$ and $x_2 \in C[0, T], T < \infty$ where $C[0, T]$ is the class of continuous functions defined on the interval $[0, T]$ and τ denote the transpose of the matrix.

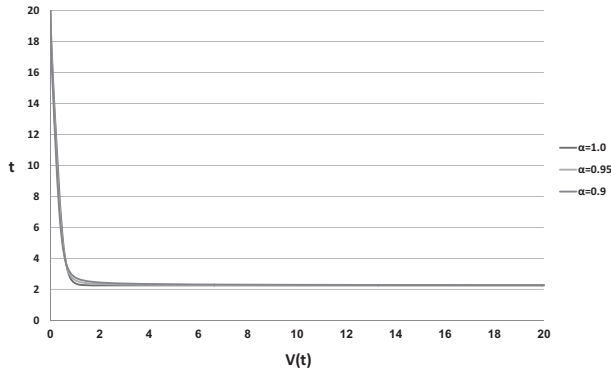


Figure 6

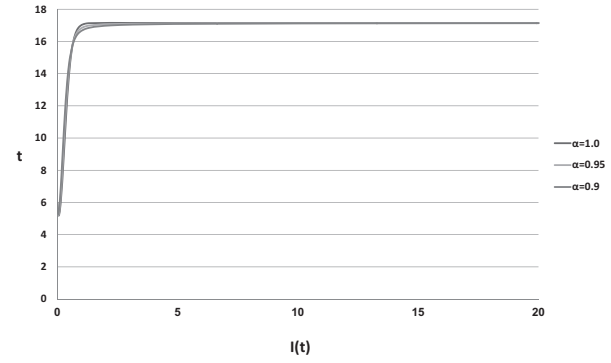


Figure 7

Now we have the following theorem:

Theorem 1. *The fractional-order SIS model (12) and (13) has a unique uniformly Lyapunov stable solution.*

Proof. Write the model (12) and (13) in the matrix form

$$D^\alpha X(t) = F(X(t)), \quad t > 0 \quad \text{and} \quad X(0) = X_0 \quad (14)$$

where $X(t) = (x_1(t) \ x_2(t))^T$, and $F(X(t)) = (f_1(x_1(t), x_2(t)) \ f_2(x_1(t), x_2(t)))^T$. Now applying Theorem 2.1 [30] we deduce that the fractional-order SIS model (12) and (13) has a unique solution. Also by Theorem 3.2 [30] this solution is uniformly Lyapunov stable. $\square$

5. Fractional-order vaccination model

According to the theory of asymptotically autonomous systems, this result extends to the system [3]

$$\begin{aligned} D^\alpha S(t) &= \Lambda(N) - \beta(N)SI - \mu S + \gamma I, \\ D^\alpha I(t) &= \beta SI - (\mu + \gamma)I, \end{aligned} \quad (15)$$

where the population carrying capacity K is now defined by $\Lambda(K) = \mu K$, $\Lambda'(K) < \mu$ and the contact rate $\beta(N)$ is now a function of total population size with $N\beta(N)$ nondecreasing and $\beta(N)$ non-increasing.

To the model (15) we add the assumption that in unit time a fraction ϕ of the susceptible class is vaccinated. The vaccination may reduce but not completely eliminate susceptibility to infection. We model this by including a factor σ , $0 \leq \sigma \leq 1$, in the infection rate of vaccinated members with $\sigma = 0$ meaning that the vaccine is perfectly effective and $\sigma = 1$ meaning that the vaccine has no effect. We assume also that the vaccination loses effect at a proportional rate θ . We describe the new fractional order model by including a vaccinated class V , with [3]

$$\begin{aligned} D^\alpha S(t) &= \Lambda(N) - \beta(N)SI - (\mu + \phi)S + \gamma I + \theta V, \\ D^\alpha I(t) &= \beta(N)SI + \sigma\beta(N)VI - (\mu + \gamma)I, \\ D^\alpha V(t) &= \phi S - \sigma\beta(N)VI - (\mu + \theta)V, \end{aligned} \quad (16)$$

where $0 < \alpha \leq 1$

In (16) the total population size $N = S + I + V$ and $D^\alpha N = \Lambda(N) - \mu N$.

We may replace N by K and S by $K - I - V$ to give the qualitatively equivalent fractional order system

$$\begin{aligned} D^\alpha I(t) &= \beta[K - I - (1 - \sigma)V]I - (\mu + \gamma)I, \\ D^\alpha V(t) &= \phi[K - I] - \sigma\beta VI - (\mu + \theta + \phi)V, \end{aligned} \quad (17)$$

with $\beta = \beta(K)$. The system (17) is the basic vaccination fractional order model which we will analyze. We remark that if the vaccine is completely ineffective, $\sigma = 1$, then (17) is equivalent to the fractional order SIS model (11). If there is no loss of effectiveness of vaccine, $\theta = 0$, and if all susceptibles are vaccinated immediately (formally, $\phi \rightarrow \infty$), the model (17) is equivalent to $D^\alpha I(t) = \sigma\beta I(K - I) - (\mu + \gamma)I$, which is the same as (11) with β replaced by $\sigma\beta$ and has basic reproductive number $R_0^* = \frac{\sigma\beta K}{(\mu + \gamma)} \leq \sigma R_0 \leq R_0$.

For the fractional order model (17) to evaluate the equilibrium points, let

$$D^\alpha I = 0, \quad D^\alpha V = 0,$$

then there is a disease-free equilibrium $(I_{eq}, V_{eq}) = (0, \frac{\phi}{(\mu + \theta + \phi)}K)$.

For $(I_{eq}, V_{eq}) = (0, \phi \frac{K}{(\mu + \theta + \phi)})$ we find that

$$A = \begin{bmatrix} -(1 - \sigma)\beta V - (\mu + \gamma) + \beta K & 0 \\ -(\phi + \sigma\beta V) & -(\mu + \theta + \phi) \end{bmatrix},$$

and its eigenvalues are

$$\lambda_1 = -(\mu + \theta + \phi) < 0,$$

$$\lambda_2 = -(1 - \sigma)\beta \frac{\phi}{(\mu + \theta + \phi)}K - (\mu + \gamma) + \beta K < 0 \quad \text{if}$$

$$R(\phi) = \frac{(\mu + \theta + \sigma\phi)}{(\mu + \theta + \phi)} R_0 < 1.$$

Hence the disease-free equilibrium $(I_{eq}, V_{eq}) = (0, \frac{\phi}{(\mu + \theta + \phi)}K)$ is locally asymptotically stable if $R(\phi) < 1$.

For $\phi = 0$ is that of no vaccination with $R(0) = R_0$, and if $\phi > 0$ then $R(\phi) < R_0$, that if $R_0 < 1$ the disease-free equilibrium is locally asymptotically stable. We note that $\lim_{\phi \rightarrow \infty} R(\phi) = \sigma R_0 < R_0$ [3].

If $\sigma = 1$, meaning that the vaccine has no effect, we have seen that (17) is equivalent to the fractional order SIS model (11) and if $R_0 > 1$ there is a unique endemic equilibrium which is locally asymptotically stable. If $0 \leq \sigma < 1$ there is an endemic equilibrium (I_{eq}, V_{eq}) .

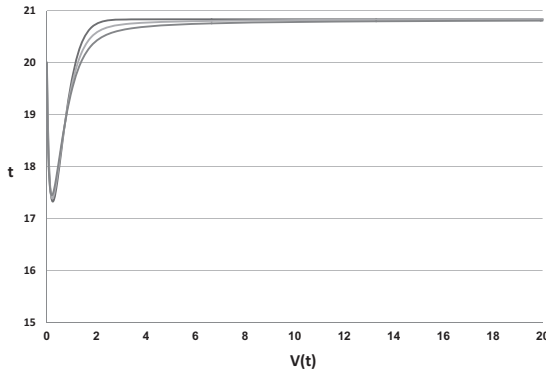


Figure 8

An endemic equilibria are solutions of the pair of equations

$$\beta[K - I - (1 - \sigma)V] = \mu + \gamma,$$

$$\phi[K - I] = \sigma\beta VI + (\mu + \theta + \phi)V,$$

which give an equation of the form

$$AI^2 + BI + C = 0 \quad (18)$$

with

$$A = \sigma\beta, B = (\mu + \theta + \sigma\phi) + \sigma(\mu + \gamma) - \sigma\beta K,$$

$$C = \frac{(\mu + \gamma)(\mu + \theta + \phi)}{\beta} - (\mu + \theta + \sigma\phi)K,$$

and its eigenvalues are given by an equation

$$\lambda^2 + a_1\lambda + a_2 = 0,$$

where

$$a_1 = [\mu + \theta + \phi + (\sigma + 1)\beta I] > 0,$$

$$a_2 = \beta I[2AI + B] > 0 \text{ if } (2AI + B) > 0.$$

Hence if $(2AI + B) > 0$ an endemic equilibrium (I_{eq}, V_{eq}) is locally asymptotically stable [25].

If $\sigma = 0$ Eq. (17) give $I = K[1 - \frac{1}{R(\phi)}] > 0$ if $R(\phi) > 1$. Thus for $\sigma = 0$ there is a unique endemic equilibrium if $R(\phi) > 1$ which approaches zero as $R(\phi) \rightarrow 1$ and there cannot be an endemic equilibrium if $R(\phi) < 1$. In this case it is not possible to have a backward bifurcation at $R(\phi) = 1$.

If $\sigma > 0$ so that (18) is quadratic and if $R(\phi) > 1$ ($C < 0$), then there is a unique positive root of (18)

$$I = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

and thus there is a unique endemic equilibrium.

If $R(\phi) = 1$ ($C = 0$), we obtain

$$AI^2 + BI = 0$$

$$I = 0, -B/A > 0, \text{ if } B < 0.$$

Then if $B < 0$ when $C = 0$ ($R(\phi) = 1$) there is a unique positive endemic equilibrium.

If $B = -2\sqrt{AC} < 0$, Then there is a unique positive endemic equilibrium $I = -B/2A$.

If $R(\phi) < 1$ ($C > 0$), we obtain

$$I = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} > 0 \text{ if}$$

$$B < 0, B^2 > 4AC, \text{ or } B < -2\sqrt{AC}.$$

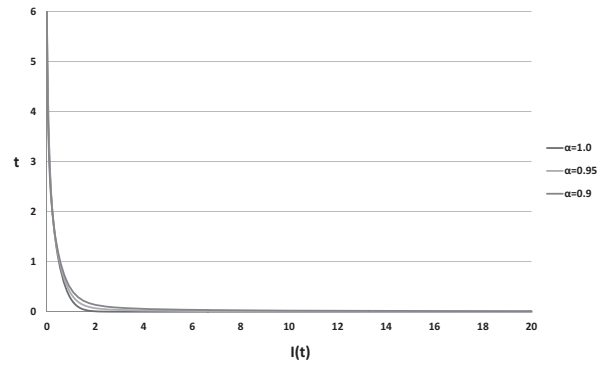


Figure 9

Then if $B < 0, B^2 > 4AC$, or $B < -2\sqrt{AC}$ when $C > 0$ ($R(\phi) < 1$) there are two positive endemic equilibrium and if $C > 0$ and either $B \geq 0, B^2 < 4AC$, there are no endemic equilibria.

Now we have the following theorem:

Theorem 2. The fractional order system (17) has a backward bifurcation at $R(\phi) = 1$ if and only if $B < 0$ when β is chosen to make $C = 0$ [3].

We can give an explicit criterion in terms of the parameters $\mu, \gamma, \theta, \phi, \sigma$ for the existence of a backward bifurcation at $R(\phi) = 1$ [3]. When $R(\phi) = 1$, ($C = 0$) so that

$$(\mu + \theta + \sigma\phi)\beta K = (\mu + \gamma)(\mu + \theta + \phi), \quad (19)$$

which reduces to

$$\sigma(1 - \sigma)(\mu + \gamma)\phi > (\mu + \theta + \sigma\phi)^2. \quad (20)$$

A backward bifurcation occurs at $R(\phi) = 1$, with βK given by (19) if and only if (20) is satisfied.

If (20) is satisfied, so that there is a backward bifurcation at $R(\phi) = 1$, there are two endemic equilibria for an interval of values of β [3] from

$$\beta K = \frac{(\mu + \gamma)(\mu + \theta + \phi)}{(\mu + \theta + \sigma\phi)}$$

corresponding to $R(\phi) = 1$ to a value β_c defined by $B = -2\sqrt{AC}$ so that

$$\beta_c K = \frac{\sigma(\mu + \gamma) + 2\sqrt{\sigma(1 - \sigma)(\mu + \gamma)\phi} - (\mu + \theta + \sigma\phi)}{\sigma}, \quad (21)$$

and the critical basic reproductive number R_c is given by [3]

$$R_c = \frac{(\mu + \theta + \sigma\phi)}{(\mu + \theta + \phi)} \left[\frac{\sigma(\mu + \gamma) + 2\sqrt{\sigma(1 - \sigma)(\mu + \gamma)\phi} - (\mu + \theta + \sigma\phi)}{\sigma(\mu + \gamma)} \right] < 1. \quad (22)$$

6. Numerical methods and results

For the single fractional order SIS Eq. (11), the approximate solutions are displayed in Figs. 4 and 5 for $I(0) = 5.0$ and different $0 < \alpha \leq 1$.

In Fig. 4 we take $\beta = 0.1, \mu = 0.5, K = 20, \gamma = 0.5$, and found that the equilibrium point $(I_{eq} = K(1 - \frac{1}{R_0}) = 10.0)$ is locally asymptotically stable where $R_0 = 2.0 > 1$.

In Fig. 5 we take $\beta = 0.1, \mu = 0.5, K = 2.0, \gamma = 0.5$, and found that the equilibrium point ($I_{eq} = 0$) is locally asymptotically stable where $R_0 = 0.2 < 1$.

For the fractional-order vaccination model (17), the approximate solutions are displayed in Figs. 6–9 for $I(0) = 6.0, V(0) = 20.0$ and different $0 < \alpha \leq 1$.

In Figs. 6 and 7 we take $\beta = 0.2, K = 25.0, \mu = 0.1, \gamma = 12.0, \phi = 3.0, \theta = 0.5, \sigma = 0.2$, and found that the equilibrium point (17.1484, 2.252) is locally asymptotically stable where $R(\phi) = 1.37741 > 1$.

In Figs. 8 and 9 we take $\beta = 1.0, K = 25.0, \mu = 0.1, \gamma = 12.0, \phi = 3.0, \theta = 0.5, \sigma = 0.2$, and found that the equilibrium point (0, 20.8333) is locally asymptotically stable where $R(\phi) = 0.688705 < 1$.

7. Conclusions

In this paper we study some fractional order models for disease transmission with vaccination. The stability of equilibrium points is studied. Backward bifurcation in fractional order systems is studied. Numerical solutions of these models are given. Numerical simulations have been used to verify the theoretical analysis.

The reason for considering a fractional order system instead of its integer order counterpart is that fractional order differential equations are generalizations of integer order differential equations. Also bifurcation in fractional order systems may differ from that of integer order.

We like to argue that fractional order equations are more suitable than integer order ones in modeling biological, economic and social systems (generally complex adaptive systems) where memory effects are important.

To the best of my knowledge this is the first paper on backward bifurcation in fractional order system.

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