Scientific Journal of Faculty of Science, Menoufia University

https://sjfsmu.journals.ekb.eg/

On correction of Sinoform distribution and studying its properties

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ARTICLE INFO. **ABSTRACT**

Received: 02/04/2024

Accepted: 05/05/2024

This paper provides a correction of the assumptions of the called Sinoform distribution and the corresponding properties of this distribution are studied. Moreover, applying the deformation technique for Sinoform distribution is discussed.

Keywords: deformation technique, Sine distribution, Sine transformation, Sinoform distribution, trigonometric distribution, uniform distribution.

1. Introduction

Recently, some new distributions using sine function has been presented in (Kumar and et al (2015); Hassan and Abdel-Salam (2010); Sinha (2012)). The called Sinoform distribution "Sinoform distribution" has been provided in (Sinha (2012)). This statistical distribution can be obtained if a sine term is incorporated in the uniform distribution. An insight has been given as to how a trigonometric function can change the density curve. The uniform distribution on $[a, b]$ "$U([a , b])$"is used to include the sine function. The sine term is incorporated in $U([a , b])$ and the obtained probability density function "pdf" has been given in the following form:

$$f_{\text{Sinoform}}(x; a, b, \delta, n) = \frac{1}{b-a}\left(1 + \delta\, sin\left(2\pi \frac{x-a}{n}\right)\right);$$
$$a \le x \le b, 0 \le \delta \le \frac{1}{b-a}, n \in \mathbb{N}, \qquad (1)$$

where $\mathbb{N}$ is the set of natural numbers. This form has been considered in (Sinha (2012))as Sinoform distribution with four parameters a, b, δ *and* n . The author referred that it is a statistical distribution, satisfying some conditions and it is considered a simplest example of distribution involving a trigonometric function with multi-modes.

From our point of view, the form (1) can't be considered as pdf of distribution under the given assumptions in (Sinha (2012)) because the conditions of the pdf can't be satisfied.

In this paper, one of the goals is concerned to suggest a correction of these assumptions which can be allowed to this given form (1) to be pdf. According to this correction, we study some properties and measures of Sinoform distribution.

The deformation technique has been applied for hyperbolic or trigonometric distributions (functions) in (Hassan and Abdel-Salam (2010); El-Shehawy (2012); El-Shehawy (2017)). The obtained deformed distribution is constructed by introducing two deformation parameters in]0, 1] as factors of the two exponential terms of the considered hyperbolic or trigonometric function. Since Sinoform distribution is one of the distributions, which includes the sine function, then the second goal in this paper is concerned to study effect of applying the deformation technique for this distribution. Moreover, we study applying this technique to transform Sinoform distribution to a corresponding deformed distribution.

This paper is organized as follows. Section 2 presents a correction of the assumptions of the parameters of Sinoform distribution and studies its properties in regard of this suggested correction. Section 3 discusses applying the deformation technique on Sinoform distribution. The main results are concluded in Section 4.

2. Correction of Sinoform-distribution and studying its properties

If a sine term is incorporated in $U([a, b])$, the form (1) can be obtained. In (Sinha (2012)), the form (1) has been provided as pdf of Sinoform distribution with four parameters a, b, δ *and* n, but it can't be considered under the given assumptions. In this section we suggest an appropriate assumption that makes the form (1) represent pdf.

Firstly, we give the following lemma which contains some results of integrations to be used directly in the proofs of the main results.

Lemma 1 *For* $n \in \mathbb{N}$ *and* $(b-a)n^{-1} \in \mathbb{N}$, *we find that:*

$$I_r = \int_a^b x^r \sin(2\pi \tfrac{x-a}{n})\, dx = \begin{cases} 0: r = 0 \\ -\frac{n}{2\pi}(b-a): r = 1 \\ -\frac{n}{2\pi}(b^2-a^2): r = 2 \\ -\frac{n}{2\pi}(b^3-a^3) + \frac{3n^3}{4\pi^3}(b-a): r = 3 \\ -\frac{n}{2\pi}(b^4-a^4) + \frac{3n^3}{2\pi^3}(b^2-a^2): r = 4 \end{cases} \quad (2)$$

$$I_5 = \int_a^b e^{tx} \sin(2\pi \tfrac{x-a}{n})\, dx = -\frac{2n\pi}{4\pi^2+n^2t^2}(e^{bt} - e^{at}), \quad (3)$$

$$I_6 = \int_a^b e^{itx} \sin(2\pi \tfrac{x-a}{n})\, dx = -\frac{2n\pi}{4\pi^2+i^2n^2t^2}(e^{ibt} - e^{iat}). \quad (4)$$

Proof: Since $(b-a)n^{-1} \in \mathbb{N}$, then $\cos(2\pi\frac{b-a}{n}) = 1$ and $\sin(2\pi\frac{b-a}{n}) = 0$. Using the integration by parts we can obtain on $I_0, I_1, I_2, I_3, I_4, I_5$ and I_6 as follows:

$$I_0 = \int_a^b \sin(2\pi\tfrac{x-a}{n})\, dx = -\frac{n}{2\pi}(\cos(2\pi\tfrac{b-a}{n}) - \cos 0) = -\frac{n}{2\pi}(1-1) = 0,$$

$$I_1 = \int_a^b x\sin(2\pi\tfrac{x-a}{n})\, dx = -\frac{n}{2\pi}[x\cos(2\pi\tfrac{x-a}{n})]_a^b + \frac{n}{2\pi}\int_a^b \cos(2\pi\tfrac{x-a}{n})\, dx$$

$$= -\frac{n}{2\pi}(b\cos(2\pi\tfrac{b-a}{n}) - a) + \frac{n^2}{4\pi^2}(\sin(2\pi\tfrac{b-a}{n}) - \sin 0) = -\frac{n}{2\pi}(b-a),$$

$$I_2 = \int_a^b x^2\sin(2\pi\tfrac{x-a}{n})\, dx = -\frac{n}{2\pi}[x^2\cos(2\pi\tfrac{x-a}{n})]_a^b + \frac{n}{\pi}\int_a^b x\cos(2\pi\tfrac{x-a}{n})\, dx$$

$$= -\frac{n}{2\pi}(b^2\cos(2\pi\tfrac{b-a}{n}) - a^2) + \frac{n^2}{2\pi^2}\{0 - I_0\} = -\frac{n}{2\pi}(b^2-a^2),$$

$$I_3 = \int_a^b x^3\sin(2\pi\tfrac{x-a}{n})\, dx = -\frac{n}{2\pi}[x^3\cos(2\pi\tfrac{x-a}{n})]_a^b + \frac{3n}{2\pi}\int_a^b x^2\cos(2\pi\tfrac{x-a}{n})\, dx$$

$$= -\frac{n}{2\pi}(b^3\cos(2\pi\tfrac{b-a}{n}) - a^3) + \frac{3n^2}{4\pi^2}\{0 - 2I_1\} = -\frac{n}{2\pi}(b^3-a^3) + \frac{3n^3}{4\pi^3}(b-a),$$

$$I_4 = \int_a^b x^4\sin(2\pi\tfrac{x-a}{n})\, dx = -\frac{n}{2\pi}[x^4\cos(2\pi\tfrac{x-a}{n})]_a^b + \frac{2n}{\pi}\int_a^b x^3\cos(2\pi\tfrac{x-a}{n})\, dx$$

$$= -\frac{n}{2\pi}(b^4\cos(2\pi\tfrac{b-a}{n}) - a^4) + \frac{n^2}{\pi^2}\{0 - 3I_2\} = -\frac{n}{2\pi}(b^4-a^4) + \frac{3n^3}{2\pi^3}(b^2-a^2),$$

$$I_5 = \int_a^b e^{tx}\sin(2\pi\tfrac{x-a}{n})\, dx = -\frac{n}{2\pi}[e^{tx}\cos(2\pi\tfrac{x-a}{n})]_a^b + \frac{nt}{2\pi}\int_a^b e^{tx}\cos(2\pi\tfrac{x-a}{n})\, dx$$

$$= -\frac{n}{2\pi}[e^{bt} - e^{at}] + \frac{n^2t}{4\pi^2}\{[e^{tx}\sin(2\pi\tfrac{x-a}{n})]_a^b - tI_5\} = -\frac{n}{2\pi}[e^{bt} - e^{at}] - \frac{n^2t^2}{4\pi^2}I_5,$$

i.e. $I_5 = -\frac{2n\pi}{4\pi^2+n^2t^2}(e^{bt} - e^{at})$,

$$I_6 = \int_a^b e^{itx}\sin(2\pi\tfrac{x-a}{n})\, dx = -\frac{n}{2\pi}[e^{itx}\cos(2\pi\tfrac{x-a}{n})]_a^b + \frac{int}{2\pi}\int_a^b e^{itx}\cos(2\pi\tfrac{x-a}{n})\, dx$$

$= -\frac{n}{2\pi}[e^{ibt} - e^{iat}] + \frac{in^2t}{4\pi^2}\{[e^{itx}\sin(2\pi\frac{x-a}{n})]_a^b - itI_6\} = -\frac{n}{2\pi}[e^{ibt} - e^{iat}] - \frac{i^2n^2t^2}{4\pi^2}I_6,$

i.e. $I_6 = -\frac{2n\pi}{4\pi^2+i^2n^2t^2}(e^{ibt} - e^{iat})$. The proof is completed.

For the considered form (1), we conclude its properties in the following theorem.

Theorem 1 *The constructed form (1) under the assumption that* $(b-a)/n \in \mathrm{N}$ *satisfies following properties:*

(i) This form can be considered as pdf of Sinoform distribution with 4 parameters a, b, δ *and* n*, where* $n \in \mathbb{N}$.

(ii) Sinoform distribution has the following cumulative distribution function "cdf" :

$$F_{Sinoform}(x; a, b, \delta, n) = \frac{x-a}{b-a} + \frac{n\delta}{2\pi}(1 - \frac{1}{b-a}\cos(2\pi\frac{x-a}{n})), \forall x \in [a, b]. \tag{5}$$

(iii) The 1st and 2nd non-central moments of Sinoform distribution are given respectively as follows:

$$\mu_{Sinoform} = \mu'_{1,Sinoform} = \frac{b+a}{2} - \frac{n\delta}{2\pi} \quad \text{and}$$
$$\mu'_{2,Sinoform} = \frac{b^2+ab+a^2}{3} - \frac{n\delta}{2\pi}(b+a). \tag{6}$$

(iv) The variance (the 2nd central moment) of Sinoform distribution is a real valued function of the parameters a, b, δ *and* n *in the following form:*

$$\sigma^2_{Sinoform} = \frac{(b-a)^2}{12} - (\frac{n\delta}{2\pi})^2. \tag{7}$$

(v) The 3rd and 4th non-central moments of Sinoform distribution are given respectively as follows:

$$\mu'_{3,Sinoform} = \frac{b^4-a^4}{4(b-a)} - \frac{n\delta}{2\pi}((b^2+ab+a^2) - \frac{3n^2}{2\pi^2}) \tag{8}$$

and

$$\mu'_{4,Sinoform} = \frac{b^5-a^5}{5(b-a)} + \frac{n\delta(b+a)}{2\pi}(\frac{3n^2}{\pi^2} - (b^2+a^2)). \tag{9}$$

(vi) The 3rd and 4th central moments of Sinoform distribution are given respectively as follows:

$$\mu_{3,Sinoform} = -\frac{(b+a)^3}{4} + \frac{3n\delta}{4\pi}(\frac{n^2}{\pi^2} - \frac{n\delta}{\pi}(b+a) + (b+a)^2) + 2(\frac{b+a}{2} - \frac{n\delta}{2\pi})^3 \tag{10}$$

And

$$\mu_{4,Sinoform} = \frac{b^5-a^5}{5(b-a)} + \frac{n\delta}{2\pi}(12\frac{n^2}{4\pi^2} - (b^2+a^2))(b+a) - 4(\frac{b+a}{2} - \frac{n\delta}{2\pi})(\frac{(b+a)(b^2+a^2)}{4} - \frac{n\delta}{2\pi}(b^2+ab+a^2 - \frac{3n^2}{2\pi^2}))$$
$$-6(\frac{b+a}{2} - \frac{n\delta}{2\pi})^2(\frac{b^2+ab+a^2}{3} - \frac{n\delta}{\pi}\frac{b+a}{2}) - 3(\frac{b+a}{2} - \frac{n\delta}{2\pi})^4. \tag{11}$$

(vii) The moment generating function and the characteristic function of Sinoform distribution are given respectively as follows:

$$M_{Sinoform}(t) = \frac{e^{bt}-e^{at}}{(b-a)t}[1 - \frac{2n\pi\delta t}{4\pi^2+n^2t^2}], t \neq 0 \tag{12}$$

and

$$\Psi_{Sinoform}(t) = \frac{e^{ibt}-e^{iat}}{i(b-a)t}[1 - \frac{2in\pi\delta t}{4\pi^2+i^2n^2t^2}], |t| \neq 0, \frac{2\pi}{n}. \tag{13}$$

Proof:

(i) Using (2) for r=0 in Lemma 1, we find that:

$$\int_{-\infty}^{\infty} f_{Sinoform}(x; a, b, \delta, n)\,dx = \int_a^b \frac{1}{b-a}(1 + \delta\sin(2\pi\frac{x-a}{n}))\,dx = 1 + \frac{\delta}{b-a}I_0 = 1$$

Moreover, it is clear that $f_{\text{Sinoform}}(x; a, b, \delta, n) \geq 0$, since

$$0 \leq 2\pi\frac{x-a}{n} \leq 2\pi\frac{b-a}{n}.$$

This implies that the constructed form (1) is pdf of Sinoform distribution.

(ii) The cdf of Sinoform distribution can be deduced as follows:

$$F_{Sinoform}(x; a, b, \delta, n) = \int_{-\infty}^{x} f_{Sinoform}(u; a, b, \delta, n))\,du = \int_a^x \frac{1}{b-a}(1 + \delta\sin(2\pi\frac{u-a}{n}))\,du$$

$$= \frac{x-a}{b-a} + \frac{n\delta}{2\pi}(1 - \frac{1}{b-a}\cos(2\pi\frac{x-a}{n})), \forall x \in [a, b].$$

(iii) Using (2) for r=1 in Lemma 1, the 1st non-central moments of Sinoform distribution can be obtained as follows:

$$\mu_{Sinoform} = \int_{-\infty}^{\infty} x f_{Sinoform}(x; a, b, \delta, n)\, dx = \frac{b+a}{2} + \frac{\delta}{b-a} I_1 = \frac{b+a}{2} - \frac{n\delta}{2\pi}$$

Similarly, using (2) for r=2 in Lemma 1 the 2nd non-central moment can be derived as follows;

$$\mu'_{2,Sinoform} = \int_{-\infty}^{\infty} x^2 f_{Sinoform}(x; a, b, \delta, n)\, dx = \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right) + \frac{\delta}{b-a} I_2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{n\delta}{2\pi}(b+a).$$

(iv) The variance (the 2nd central moment) of Sinoform distribution can be derived directly from the relation between the central and non-central moments as follows:

$$\sigma^2_{Sinoform} = \mu'_{2,Sinoform} - (\mu'_{1,Sinoform})^2 = \frac{b^2+ab+a^2}{3} - \frac{n\delta}{2\pi}(b+a) - \left(\frac{b+a}{2} - \frac{n\delta}{2\pi}\right)^2 = \frac{(b-a)^2}{12} - \left(\frac{n\delta}{2\pi}\right)^2.$$

(v) Using (2) for r=3 and r=4 in Lemma 1, the 3rd and 4th non-central moments of Sinoform distribution can be derived respectively as follows:

$$\mu'_{3,Sinoform} = \int_{-\infty}^{\infty} x^3 f_{Sinoform}(x; a, b, \delta, n)\, dx$$

$$= \frac{1}{b-a}\left(\frac{b^4-a^4}{4}\right) + \frac{\delta}{b-a} I_3 = \frac{b^4-a^4}{4(b-a)} - \frac{n\delta}{2\pi}\left(b^2 + ab + a^2 - \frac{3n^2}{2\pi^2}\right)$$

and

$$\mu'_{4,Sinoform} = \int_{-\infty}^{\infty} x^4 f_{Sinoform}(x; a, b, \delta, n)\, dx$$

$$= \frac{1}{b-a}\left(\frac{b^5-a^5}{5}\right) + \frac{\delta}{b-a} I_4 = \frac{b^5-a^5}{5(b-a)} + \frac{n\delta(b+a)}{2\pi}\left(\frac{3n^2}{\pi^2} - (b^2+a^2)\right).$$

(vi) From the relation between the central and non-central moments, we find that

$$\mu_{3,Sinoform} = \mu'_{3,Sinoform} - 3\mu'_{1,Sinoform}\mu'_{2,Sinoform} + 2(\mu'_{1,Sinoform})^3$$

$$= -\frac{(b+a)^3}{4} + \frac{3n\delta}{4\pi}\left(\frac{n^2}{\pi^2} - \frac{n\delta}{\pi}(b+a) + (b+a)^2\right) + 2\left(\frac{b+a}{2} - \frac{n\delta}{2\pi}\right)^3$$

And

$$\mu_{4,Sinoform} = \mu'_{4,Sinoform} - 4\mu'_{1,Sinoform}\mu'_{3,Sinoform} + 6(\mu'_{1,Sinoform})^2\mu'_{2,Sinoform} - 3(\mu'_{1,Sinoform})^4$$

$$= \frac{b^5-a^5}{5(b-a)} + \frac{n\delta}{2\pi}\left(12\frac{n^2}{4\pi^2} - (b^2+a^2)\right)(b+a) - 4\left(\frac{b+a}{2} - \frac{n\delta}{2\pi}\right)\left(\frac{(b+a)(b^2+a^2)}{4} - \frac{n\delta}{2\pi}\left(b^2+ab+a^2 - \frac{3n^2}{2\pi^2}\right)\right) - 6\left(\frac{b+a}{2} - \frac{n\delta}{2\pi}\right)^2\left(\frac{b^2+ab+a^2}{3} - \frac{n\delta}{\pi}\frac{b+a}{2}\right) - 3\left(\frac{b+a}{2} - \frac{n\delta}{2\pi}\right)^4.$$

(vii) Using (3) the moment generating function can be obtained as follows:

$$M_{Sinoform}(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f_{Sinoform}(x; a, b, \delta, n)\, dx = \frac{1}{b-a}\int_a^b e^{tx}\left(1 + \delta \sin\left(2\pi\frac{x-a}{n}\right)\right) dx$$

$$= \frac{e^{bt}-e^{at}}{(b-a)t} + \frac{\delta}{b-a} I_5 = \frac{e^{bt}-e^{at}}{(b-a)t}\left[1 - \frac{2n\pi\delta t}{4\pi^2+n^2t^2}\right], t \neq 0.$$

Similarly, using (4) in Lemma 1, the characteristic function of Sinoform distribution can be obtained in the following form

$$\Psi_{Sinoform}(t) = E[e^{itx}] = \frac{e^{ibt}-e^{iat}}{i(b-a)t}\left[1 - \frac{2in\pi\delta t}{4\pi^2+i^2n^2t^2}\right], |t| \neq 0, \frac{2\pi}{n}$$

The proof is completed.

The following figure 1and figure 2 exhibit pdf of Sinoform distribution in two cases.

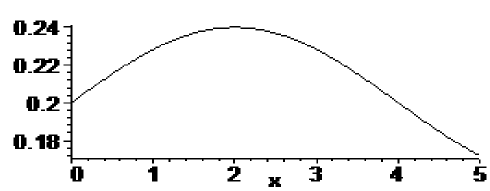

Figure 1 the pdf of Sinoform distribution in the form (1) on $[0, 5]$ with n = 8 and δ = 0.02.

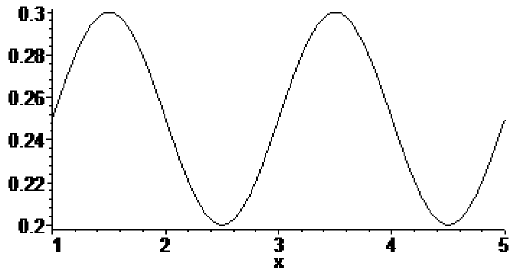

Figure 2 the pdf of Sinoform distribution in the form (1) on [1, 5] with n = 2 and δ = 0.02

Here figure 1 shows pdf for Sinoform distribution with only one mode when there are no restrictions on $(b-a)n^{-1}$, while Fig. 2 illustrates the same function with multi-modes when $(b-a)n^{-1} \in \mathbb{N}$.

3. A study of applying the *pq*-deformation on Sinoform distribution

In this section, some main results about applying the *pq*-deformation technique by introducing two real values of the parameters p and q in]0, 1] on Sinoform distribution is provided.

The *pq*-Deformed form of the pdf of Sinoform distribution can be given in the following form:

$$f_{pq-DSinoform}(x;a,b,\delta,n,p,q) = \frac{1}{b-a}\left(1+\delta\, sin_{pq}\left(2\pi\frac{x-a}{n}\right)\right);\ a \le x \le b, \quad (14)$$

$0 \le \delta \le \frac{1}{b-a}, n \in \mathbb{N}, (b-a)n^{-1} \in \mathbb{N}, p, q \in]0,1],$

where $\sin_{pq} x = \dfrac{pe^{ix} + qe^{-ix}}{2}$.

For more details of the deformed trigonometric functions $\sin_{pq} x$ or $\cos_{pq} x$, see refs. (Hassan and Abdel-Salam (2010); El-Shehawy (2012); El-Shehawy (2017)).

The following theorem gives one the main results on applying of the *pq*-deformation technique, where $p \neq q$ and $p, q \in]0,1]$, for Sinoform distribution with pdf in the form (1).

Theorem 2 *For unequal real valued deformation parameters* p *and* q *in the interval]0, 1], Sinoform distribution with pdf (1) can't be transformed to a pq-deformed distribution on the considered corresponding interval.*

Proof: The proof can be explained as follow.

Firstly, consider the *pq*-deformed function $f_{pq-DSinoform}(x;a,b,\delta,n,p,q)$ in (14) for Sinoform distribution with pdf $f_{Sinoform}(x;a,b,\delta,n)$ in the form (1) on $[a,b]$, where p and q are two unequal deformation parameters in]0, 1].

To test the two conditions that the *pq*-deformed function $f_{pq-DSinoform}(x;a,b,\delta,n,p,q)$ on $[a,b]$ is a pdf for $p \neq q$, $p, q \in]0,1]$:

Using the facts $e^{\pm 2i\pi\frac{b-a}{n}} = 1$ and $cos_{pq}(2\pi\frac{b-a}{n}) = \frac{pe^{2i\pi\frac{b-a}{n}} + qe^{-2i\pi\frac{b-a}{n}}}{2} = \frac{p+q}{2}$, the integration of the function $f_{pq-DSinoform}(x;a,b,\delta,n,p,q)$ on R equals 1, i.e.

$$\int_{-\infty}^{\infty} f_{pq-DSinoform}(x;a,b,\delta,n,p,q)\,dx = \frac{1}{b-a}\int_a^b \left(1+\delta sin_{pq}\left(2\pi\frac{x-a}{n}\right)\right)dx$$
$$= 1 + \frac{\delta}{2i(b-a)}\int_a^b \left[pe^{2i\pi\frac{x-a}{n}} - qe^{-2i\pi\frac{x-a}{n}}\right]dx$$
$$= 1 + \frac{n\delta}{2i(b-a)(2i\pi)}\left(2cos_{pq}\left(2\pi\frac{b-a}{n}\right) - [p+q]\right) = 1.$$

The curve of the function $f_{pq-DSinoform}(x;a,b,\delta,n,p,q)$ on $[a,b]$ can't be plotted for $p \neq q$ and moreover the value of $f_{pq-DSinoform}(x;a,b,\delta,n,p,q)$ is not real on $[a, b]$ for $p \neq q$, e.g. at $x = 2; a = 1, b = 5, \delta = 1/5, n = 2$, we find that

$$f_{pq-DSinoform}\left(2;1,5,\frac{1}{5},2,p,q\right) = \frac{1}{4}\left(1+\frac{1}{5}sin_{pq}\pi\right) = \frac{1}{4}\left(1-\frac{1}{5}\left(\frac{p-q}{2i}\right)\right) = \frac{1}{4} + \frac{i}{40}(p-q) \notin R$$

for $p \neq q$, where $e^{\pm i\pi} = -1$ and $\sin_{pq}\pi = -\frac{p-q}{2i}$.

Then, one of the conditions of pdf is not satisfied for the *pq*-deformed form (14). This means that, for two unequal deformation parameters $p, q \in]0,1]$ the *pq* deformation technique can't transform Sinoform distribution with pdf in the form (1) on $[a,b]$ to a *pq*-deformed distribution. The proof is completed.

In the rest of this section, we explain the possibility of the *pp*-deformation for Sinoform distribution where $p \in]0,1]$ and study some properties of this case.

The following theorem concludes some results on applying the *pq*-deformation technique for Sinoform distribution with pdf in the form (1) in the case of two equal deformation parameters $p, q \in]0,1]$.

Theorem 3 *If the deformation technique has been applied by introducing two equal parameters (i.e. $p = q$) in]0,1] for Sinoform distribution with pdf (1), then:*

(i) The constructed pp-deformed form

$$f_{pp-DSinoform}(x;a,b,\delta,n,p)=\frac{1}{b-a}(1+\delta\, sin_{pp}(2\pi\frac{x-a}{n}));\, a\le x\le b, \tag{15}$$

$0\le\delta\le\frac{1}{b-a}, n\in\mathbb{N}, (b-a)n^{-1}\in\mathbb{N}, 0<p<1$, $sin_{pp}(2\pi\frac{x-a}{n})=p\,sin(2\pi\frac{x-a}{n})$,

can be considered as pdf of the pp-deformed Sinoform distribution "pp-DSinoform distribution".

(ii) pp-DSinoform distribution has the following cdf :

$$F_{pp-DSinoform}(x;a,b,\delta,n,p)=\frac{x-a}{b-a}+\frac{np\delta}{2\pi}(1-\frac{1}{b-a}cos(2\pi\frac{x-a}{n})), \forall x\in[a,b]. \tag{16}$$

(iii) The 1st and 2nd non-central moments of pp-DSinoform distribution are given respectively as follows:

$$\mu'_{1,pp-DSinoform}=\frac{b+a}{2}-\frac{np\delta}{2\pi} \tag{17}$$

and

$$\mu'_{2,pp-DSinoform}=\frac{b^2+ab+a^2}{3}-\frac{np\delta}{2\pi}(b+a). \tag{18}$$

(iv) The variance (the 2nd central moment) of pp-DSinoform distribution is

$$\sigma^2_{pp-DSinoform}=\frac{(b-a)^2}{12}-(\frac{np\delta}{2\pi})^2. \tag{19}$$

(v) The 3rd and 4th non-central moments of pp-DSinoform distribution are given respectively as follows:

$$\mu'_{3,pp-DSinoform}=\frac{b^4-a^4}{4(b-a)}-\frac{np\delta}{2\pi}\left((b^2+ab+a^2)-\frac{3n^2}{2\pi^2}\right), \tag{20}$$

and

$$\mu'_{4,pp-DSinoform}=\frac{b^5-a^5}{5(b-a)}+\frac{np\delta(b+a)}{2\pi}\left(\frac{3n^2}{\pi^2}-(b^2+a^2)\right). \tag{21}$$

(vi) The 3rd and 4th central moments are given respectively as follows:

$$\mu_{3,pp-DSinoform}=-\frac{(b+a)^3}{4}+\frac{3np\delta}{4\pi}(\frac{n^2}{\pi^2}-\frac{np\delta(b+a)}{\pi}+(b+a)^2)+2(\frac{b+a}{2}-\frac{np\delta}{2\pi})^3, \tag{22}$$

And

$$\mu_{4,pp-DSinoform}=\frac{b^5-a^5}{5(b-a)}+\frac{np\delta}{2\pi}(12\frac{n^2}{4\pi^2}-(b^2+a^2))(b+a)-4(\frac{b+a}{2}-\frac{np\delta}{2\pi})(\frac{(b+a)(b^2+a^2)}{4}-\frac{np\delta}{2\pi}(b^2+ab+a^2-\frac{3n^2}{2\pi^2}))-6(\frac{b+a}{2}-\frac{np\delta}{2\pi})^2(\frac{b^2+ab+a^2}{3}-\frac{np\delta}{\pi}\frac{b+a}{2})-3(\frac{b+a}{2}-\frac{np\delta}{2\pi})^4. \tag{23}$$

(vii) The moment generating function and the characteristic function of pp-DSinoform distribution are given respectively as follows:

$$M_{\ pp-DSinoform}(t)=\frac{e^{bt}-e^{at}}{(b-a)t}[1-\frac{2n\pi p\delta t}{4\pi^2+n^2t^2}], t\neq 0 \tag{24}$$

and

$$\Psi_{pp-DSinoform}(t)=\frac{e^{ibt}-e^{iat}}{i(b-a)t}[1-\frac{2in\pi p\delta t}{4\pi^2+i^2n^2t^2}], |t|\neq 0,\frac{2\pi}{n} \tag{25}$$

Proof:

(i) For two equal deformation parameters (i.e. p=q) in]0, 1], we find that the pp-Deformed form with pdf "$f_{pp-DSinoform}(x;a,b,\delta,n,p)$" of Sinoform distribution in the given form (15). For $(b-a)n^{-1}\in\mathbb{N}$, we find that $e^{2i\pi\frac{b-a}{n}}+e^{-2i\pi\frac{b-a}{n}}=2$. This implies that

$$\int_{-\infty}^{\infty}f_{pp-DSinoform}(x;a,b,\delta,n,p)\,dx=\frac{1}{b-a}\int_a^b(1+\delta sin_{pp}(2\pi\frac{x-a}{n}))dx$$

$$=1+\frac{p\delta}{2i(b-a)}\int_a^b[e^{2i\pi\frac{x-a}{n}}-e^{-2i\pi\frac{x-a}{n}}]dx$$

$$=1+\frac{np\delta}{2i(b-a)(2i\pi)}([e^{2i\pi\frac{b-a}{n}}+e^{-2i\pi\frac{b-a}{n}}]-2)=1.$$

Since $-1 \le sin(2\pi\frac{x-a}{n}) \le 1,$ then $1-\delta p \le 1+\delta p\, sin(2\pi\frac{x-a}{n}) \le 1+\delta p.$

But $0 < \delta p \le 1$ and so $1-\delta p > 0$. This implies that the following inequality is satisfied: $1+\delta p\, sin(2\pi\frac{x-a}{n}) \ge 0.$

Hence $\frac{1}{b-a}(1+p\delta\, sin(2\pi\frac{x-a}{n})) \ge 0.$

i.e. $f_{pp-DSinoform}(x;a,b,\delta,n,p)$ is a nonnegative real valued function on [*a*, *b*].

Then this function is pdf of *pp*-DSinoform distribution.

(ii) The cdf of this distribution can be derived as follows:

$$F_{pp-DSinoform}(x;a,b,\delta,n,p) = \int_{-\infty}^{x} f_{pp-DSinoform}(u;a,b,\delta,n,p)\,du$$

$$= \frac{x-a}{b-a} + \frac{np\delta}{2\pi}(1-\frac{1}{b-a}cos(2\pi\frac{x-a}{n})), \forall x \in [a,b].$$

(iii) Using (2) for *r*=1 in Lemma 1 the expectation of *pp*-DSinoform-distribution can be obtained as follows:

$$\mu_{pp-DSinoform} = \int_{-\infty}^{\infty} x f_{pp-DSinoform}(x;a,b,\delta,n,p)\,dx = \frac{b+a}{2} + \frac{p\delta}{b-a}I_1 = \frac{b+a}{2} - \frac{np\delta}{2\pi}.$$

Similarly, using (2) for *r*=2 in Lemma 1 the 2nd non-central moments of *pp*-DSinoform distribution can be also derived as follows:

$$\mu'_{2,pp-DSinoform} = \int_{-\infty}^{\infty} x^2 f_{pp-DSinoform}(x;a,b,\delta,n,p)\,dx = \frac{1}{b-a}(\frac{b^3-a^3}{3}) + \frac{p\delta}{b-a}I_2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{np\delta}{2\pi}(b+a).$$

(iv) The variance (the 2nd central moment) of *pp*-DSinoform distribution can be obtained from the relation between the central and non-central moments as follows:

$$\sigma^2_{pp-DSinoform} = \mu'_{2,pp-DSinoform} - (\mu'_{1,pp-DSinoform})^2 = \frac{b^2+ab+a^2}{3} - \frac{np\delta}{2\pi}(b+a) - (\frac{b+a}{2} - \frac{np\delta}{2\pi})^2$$

This implies that:

$$\sigma^2_{pp-DSinoform} = \frac{(b-a)^2}{12} - (\frac{n\,p\,\delta}{2\pi})^2.$$

(v) Using (2) for *r*=3 and *r*=4 in Lemma 1 the 3rd and 4th non-central moments of *pp*-DSinoform distribution can be respectively obtained as follows:

$$\mu'_{3,pp-DSinoform} = \int_{-\infty}^{\infty} x^3 f_{pp-DSinoform}(x;a,b,\delta,n,p)\,dx = \frac{1}{b-a}(\frac{b^4-a^4}{4}) + \frac{p\delta}{b-a}I_3$$

$$= \frac{b^4-a^4}{4(b-a)} - \frac{np\delta}{2\pi}((b^2+ab+a^2) - \frac{3n^2}{2\pi^2})$$

and

$$\mu'_{4,pp-DSinoform} = \int_{-\infty}^{\infty} x^4 f_{pp-DSinoform}(x;a,b,\delta,n,p)\,dx$$

$$= \frac{1}{b-a}(\frac{b^5-a^5}{5}) + \frac{p\delta}{b-a}I_4 = \frac{b^5-a^5}{5(b-a)} + \frac{np\delta(b+a)}{2\pi}(\frac{3n^2}{\pi^2} - (b^2+a^2)).$$

(vi) From the relation between the central and non-central moments, we find that

$$\mu_{3,pp-DSinoform} = \mu'_{3,pp-DSinoform} - 3\mu'_{1,pp-DSinoform}\mu'_{2,pp-DSinoform} + 2(\mu'_{1,pp-DSinoform})^3$$

$$= -\frac{(b+a)^3}{4} + \frac{3np\delta}{4\pi}(\frac{n^2}{\pi^2} - \frac{np\delta}{\pi}(b+a) + (b+a)^2) + 2(\frac{b+a}{2} - \frac{np\delta}{2\pi})^3$$

And

$$\mu_{4,pp-DSinoform} = \mu'_{4,pp-DSinoform} - 4\mu'_{1,pp-DSinoform}\mu'_{3,pp-DSinoform} + 6(\mu'_{1,pp-DSinoform})^2\mu'_{2,pp-DSinoform} - 3(\mu'_{1,pp-DSinoform})^4$$

$$= \frac{b^5-a^5}{5(b-a)} + \frac{np\delta}{2\pi}(12\frac{n^2}{4\pi^2} - (b^2+a^2))(b+a) - 4(\frac{b+a}{2} - \frac{np\delta}{2\pi})(\frac{(b+a)(b^2+a^2)}{4} - \frac{np\delta}{2\pi}(b^2+ab+a^2 - \frac{3n^2}{2\pi^2}))$$

$-6(\frac{b+a}{2}-\frac{np\delta}{2\pi})^2(\frac{b^2+ab+a^2}{3}-\frac{np\delta}{\pi}\frac{b+a}{2})-3(\frac{b+a}{2}-\frac{np\delta}{2\pi})^4.$

(vii) Using (3) in Lemma 1 the moment generating and characteristic functions of *pp*-DSinoform distribution can be deduced respectively as follows:

$$M_{pp-DSinoform}(t)=E[e^{tx}]=\int_{-\infty}^{\infty}e^{tx}f_{pp-DSinoform}(x;a,b,\delta,n,p)\,dx$$

$$=\frac{1}{b-a}\int_a^b e^{tx}(1+p\delta\,sin(2\pi\frac{x-a}{n}))\,dx=\frac{e^{bt}-e^{at}}{(b-a)t}+\frac{p\delta}{b-a}I_5$$

$$=\frac{e^{bt}-e^{at}}{(b-a)t}[1-\frac{2n\pi p\delta t}{4\pi^2+n^2t^2}],t\neq 0.$$

Similarly, using (4) the characteristic function can be obtained in the following form:

$$\Psi_{pp-DSinoform}(t)=E[e^{itx}]=\frac{e^{ibt}-e^{iat}}{i(b-a)t}[1-\frac{2in\pi p\delta t}{4\pi^2+i^2n^2t^2}],|t|\neq 0,\frac{2\pi}{n}$$

Then, the theorem is proved.

Corollary 1 *For $\delta=0$, each of Sinoform and pp-DSinoform distributions on [a, b] is reduced to U([a, b]) with the corresponding properties.*

Proof: The proof is obtained directly by putting $\delta=0$ in the forms of pdf of Sinoform and *pp*-DSinoform distributions. Moreover, the corresponding properties in this case can be directly obtained by putting $\delta=0$ in the results of Theorem 2.

4. Conclusions

In this paper, we corrected the assumptions of the parameters of the obtained form when a sine term is incorporated in *U*([*a*, *b*]) by considering $(b-a)n^{-1}\in\mathbb{N}$ to construct Sinoform distribution on [*a*, *b*]. Some properties of this constructed distribution are discussed.

Effect of applying the *pq*-deformation technique on Sinoform distribution has been discussed. In case of unequal values of the deformation parameters p and q in]0, 1], Sinoform distribution on [*a*, *b*] can't be transformed to a *pq*-deformed distribution on the same interval, but when p and q are equal the transformation of Sinoform distribution to a corresponding *pp*-DSinoform distribution is applicable. Moreover, some measures of *pp*-DSinoform distribution have been deduced.

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