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RESULTS ON SOLUTIONS OF CERTAIN DIFFERENCE EQUATIONS

SHILPA N.

Abstract. In this article, we deal with the meromorphic solutions of certain q-difference equations and obtain results which generalize as well as improve the results of A. P. Singh and S. V. Dugane [2], Subhas S. Bhoosnurmath and K. S. L. N Prasad [3].

1. Introduction

For a meromorphic function f in the complex plane we assume that the reader is familiar with the standard notations of Nevanlinna theory such as, $T(r, f)$, $N(r, f)$ and $m(r, f)$ etc., as explained in [1].

Definition 1: If f is a meromorphic function of zero order, then we denote $\pi(f(qz))$ to be function which are polynomials in $f(qz)$ where $q \in \mathbb{C}$ with co-efficients $a(z)$ such that $T(r, a(z)) = o(T(r, f))$, on a set of logarithmic density 1, such functions will be called as "q-difference polynomials" in $f(qz)$.

$$\pi(f(qz)) = \sum_{j=1}^{s} a_j f^{n_{0j}} f(q_1 z)^{n_{1j}} f(q_2 z)^{n_{2j}} ... f(q_\nu z)^{n_{\nu j}},$$

where

$$\overline{d}(\pi) = max_{1 \leq j \leq s} \sum_{j=1}^{\nu} n_{ij}, \ \underline{d}(\pi) = min_{1 \leq j \leq s} \sum_{j=1}^{\nu} n_{ij}.$$

If $\overline{d}(\pi) = \underline{d}(\pi) = n(say)$ then the q-difference polynomial is called Homogeneous otherwise Nonhomogeneous.

In [2] A. P. Singh and S. V. Dukane proved the following result.

Theorem A. *No transcendental meromorphic function f with $N(r, f) = S(r, f)$ will satisfy an equation of the form*

$$a_1(z)[f(z)]^n \pi_k(f) + a_2(z)\pi_k(f) + a_3(z) = 0,$$

where $n \geq 1$, $a_1(z)(\neq 0)$ and $\pi_k(f)$ is a non-zero homogeneous differential polynomial in f of degree k having p terms where p and k satisfy the relation $(p-1)k < n$.

Later in [3]. Subhas S. Bhoosnurmath and K. S. L. N Prasad improved Theorem

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A and obtained the following result.

Theorem B. *No transcendental meromorphic function f with $N(r,f) = S(r,f)$ will satisfy an equation of the form*

$$a_1(z)[f(z)]^n \pi(f) + a_2(z)\pi(f) + a_3(z) = 0$$

where $n \geq 1$, $a_1(z)(\neq 0)$ and $\pi(f) = M_i(f) + \sum_{j=1}^{i-1} a_j(z) M_j(z)$ is a differential polynomial in f of degree n and each $M_i(f)$ is a monomial in f.

In this section we prove that in Theorem B, f^n can be replaced by $P(f)$, where $P(f)$ is a linear combination of powers of f and we also improve the above theorem by considering any q-difference polynomial in $f(qz)$.

Theorem 1.1. *No non-constant zero-order meormorphic function f with $N(r,f) = S(r,f)$ will satisfy an equation of the form*

$$a_1(z)P(f(qz))\pi(f(qz)) + a_2(z)\pi(f(qz)) + a_3(z) = 0, \tag{1.1}$$

where $a_1(z)(\neq 0)$, $a_2(z)$ and $a_3(z)$ are small functions of f, $P(f) = b_n f^n + b_{n-1} f^{n-1} + \cdots + b_1 f + b_0$, where n is a positive integer, $b_n(\neq 0), b_{n-1}, ..., b_0$ are small functions of f and $\pi(f) = M_i(f(qz)) + \sum_{j=1}^{i-1} a_j(z) M_j(f(qz))$ is a q-difference polynomial in $f(qz)$ of degree n and each $M_i(f(qz))$ is a monomial in $f(qz)$.

2. Lemmas

In order to prove our main result, we need to prove the following Lemmas.

Lemma 2.1. *Suppose that f is a non-constant zero-order meromorphic function in the plane and that $f^n P(qz) = Q(qz)$, where $P(qz)$ and $Q(qz)$ are q-difference polynomials in $f(qz)$ and degree of $Q(qz)$ is atmost n, then $m(r, P(qz)) = S(r,f)$ as $r \to \infty$.*

Proof. We have

$$\begin{aligned} 2\pi m(r, P(qz)) &= \int_0^{2\pi} log^+ |P(re^{i\theta})| d\theta \\ &\leq \int_{E_1} log^+ |P(re^{i\theta})| d\theta + \int_{E_2} log^+ |P(re^{i\theta})| d\theta, \end{aligned}$$

where E_1 is the set of θ in $0 \leq \theta \leq 2\pi$ for which $|f(re^{i\theta})| < 1$ and E_2 is the complementary set.

By hypothesis $P(qz)$ is the sum of finite number of terms of the type

$$F(qz) = a(z) f^{n_{0j}} f(q_1 z)^{n_{1j}} f(q_2 z)^{n_{2j}} ... f(q_\nu z)^{n_{\nu j}}, \tag{2.1}$$

where $n_{0j}, n_{1j}, n_{2j}, ..., n_{\nu j}$ are non-negative integers.
Hence in E_1

$$\begin{aligned} \int_{E_1} log^+ |F(re^{i\theta})| &\leq m(r,a) + o\left\{ \sum_{t=0}^{\nu} m\left(r, \frac{f(q_t z)}{f(qz)}\right) \right\} \\ &= S(r, f(qz)). \end{aligned}$$

Therefore $T(r, a(z)) = S(r, f(qz))$ as $r \to \infty$.

Thus by addition

$$\int_E log^+|P(re^{i\theta})| \leq \sum_F \int_{E_1} log^+|F(re^{i\theta})|d\theta + O(1) = S(r, f(qz)).$$

Next let E_2,

$$\begin{aligned} |P(qz)| &= \left| \frac{1}{(f(qz))^n} \sum_{t=0}^{\nu} a(z) f^{n_{0j}} f(q_1 z)^{n_{1j}} f(q_2 z)^{n_{2j}} ... f(q_\nu z)^{n_{tj}} \right| \\ &\leq \sum |a(z)| \left| \frac{f(q_1 z)}{f(qz)} \right|^{n_{1j}} ... \left| \frac{f(q_\nu z)}{f(qz)} \right|^{n_{\nu j}}. \end{aligned}$$

Thus again

$$\int_{E_2} log^+|P(re^{i\theta})|d\theta \leq O\left[\sum_{t=0}^{\nu} m\left(r, \frac{f(t)}{f(qz)}\right) + m(r, a(z)) \right] = S(r, f(qz)).$$

This proves the lemma.

Lemma 2.2. *Suppose that f is a non-constant zero order meromorphic function in the plane and $g(qz) = [f(qz)]^n + P_{n-1}(f(qz))$ where $P_{n-1}(f(qz))$ is a q-difference polynomial of degree atmost $n-1$ in $f(qz)$ and that $N(r, f(qz)) + N\left(r, \frac{1}{g(qz)}\right) = S(r, f(qz))$, then $g(qz) = [h(qz)]^n$, $h(qz)$, $f(qz) + \frac{1}{n}a(z)$ and $[h(qz)]^{n-1}a(z)$ is obtained by substituting $h(qz)$ for $f(qz)$, $h'(qz)$ for $f'(qz)$ etc., in the terms of degree $n-1$ in $P_{n-1}(f(qz))$.*

Proof. We have $g(qz)$ of the form $[f(qz) + \frac{a}{n}]^n$, where a is determined by the terms of degree $n-1$ in $P_{n-1}(f(qz))$ and by $g(qz)$. We note the following special cases.

If $P_{n-1}(f(qz)) = a_0(z)(f(qz))^{n-1}$+terms of degree $n-2$ atmost, then $h^{n-1}a(z) = a_0(z)h^{n-1}$ so that $a(z) = a_0(z)$ and $g(qz) = [f(qz) + \frac{a_0(z)}{n}]^n$.

In this case $h^{n-1}a(z) = a_0(z)h'h^{n-2}$ or $a(z) = a_0(z)\frac{h'}{h} = \frac{a_0(z)}{n}\frac{g'(qz)}{g(qz)}$,

$$g(qz) = \left[f(qz) + \frac{a_0(z)}{n^2}\frac{g'(qz)}{g(qz)} \right]^n.$$

Lemma 2.3. *Let $f(z)$ be a non-constant zero order meromorphic function and $\pi(f(qz))$ be a q-difference polynomial in $f(qz)$ of degree $n \geq 1$ with coefficients $a(z)$ and degree $\overline{d}(\pi)$ and lower degree $\underline{d}(\pi)$ then,*

$$m\left(r, \frac{\pi(f(qz))}{f^{\overline{d}(\pi)}}\right) \leq [\overline{d}(\pi) - \underline{d}(\pi)]m\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.2}$$

Proof. Let $F(qz)$ be defined as in (2.1) then,

$$\frac{F(qz)}{f^{\overline{d}(\pi)}} = a(z)\left(\frac{f(qz)}{f(qz)}\right)^{n_{0j}} \left(\frac{f(q_1 z)}{f(qz)}\right)^{n_{1j}} ... \left(\frac{f(q_k z)}{f(qz)}\right)^{n_{\nu j}}$$

Case(i). When $|f(qz)| \leq 1$

$$\left|\frac{\pi(f(qz))}{f^{\overline{d}(\pi)}}\right| = \sum_{j=1}^{s} |a_j| \left|\frac{M_j(f)}{f^{\gamma_{M_j}}}\right| \left|\frac{1}{f}\right|^{\overline{d}(\pi) - \gamma_{M_j}}$$

where γ_{M_j} is the degree of the monomial $M_j(f)$.
As $|f| \leq 1$, $\left|\frac{1}{f}\right| \geq 1$ and $\left|\frac{1}{f}\right|^{\overline{d}(\pi)-\gamma_{M_j}} \geq 1$ and we have

$$\left|\frac{1}{f}\right|^{\overline{d}(\pi)-\gamma_{M_j}} \leq \left|\frac{1}{f}\right|^{\overline{d}(\pi)-min_{1\leq j\leq s}\gamma_{M_j}} = \left|\frac{1}{f}\right|^{\overline{d}(\pi)-\underline{d}(\pi)}$$

Hence, we get

$$\left|\frac{\pi(f)}{f^{\overline{d}(\pi)}}\right| \leq \left|\frac{1}{f}\right|^{\overline{d}(\pi)-\underline{d}(\pi)} \left[\sum_{j=1}^{s} |a_j| \left|\frac{f(q_1 z)}{f}\right|^{n_{1j}} ... \left|\frac{f(q_k z)}{f}\right|^{n_{\nu j}}\right]$$

Using logarithmic derivative lemma we get (2.2).
Case(ii). When $|f(z)| \geq 1$

$$\left|\frac{\pi(f)}{f^{\overline{d}(\pi)}}\right| = \sum_{j=1}^{s} |a_j| \left|\frac{M_j(f)}{f^{\gamma_{M_j}}}\right| \left|\frac{1}{f}\right|^{\overline{d}(\pi)-\gamma_{M_j}},$$

but as $|f| \geq 1$, $\left|\frac{1}{f}\right| \leq 1$ and $\left|\frac{1}{f}\right|^{\overline{d}(\pi)-\gamma_{M_j}} \leq 1$. So $log^+ \left|\frac{1}{f}\right|^{\overline{d}(\pi)-\gamma_{M_j}} = 0$ and

$$log^+ \left|\frac{\pi(f(qz))}{f^{\overline{d}(\pi)}}\right| \leq \sum_{j=1}^{s} log^+ \left|\frac{M_j(f)}{f^{\gamma_{M_j}}}\right| + c$$

i.e.,

$$m\left(r, \frac{\pi(f(qz))}{f^{\overline{d}(\pi)}}\right) \leq S(r,f).$$

Hence we get (2.2).

3. Proofs of the Theorem.

In this section we present the proof of our main result.
Proof of Theorem 1.1
We first consider the case when $n \geq 2$.
Suppose there exists a transcendental meromorphic function f with $N(r,f) = S(r,f)$ satisfying (1.1) then

$$a_1\left[b_n f^n + b_{n-1}f^{n-1} + \cdots + b_0\right]\pi(f(qz)) + a_2\pi(f(qz)) + a_3 = 0$$

or

$$a_1 b_n f^n \pi(f(qz)) + P_1(f(qz))\pi(f(qz)) + a_3 = 0,$$

where $P_1(f(qz)) = a_1 b_{n-1} f(qz)^{n-1} + \cdots + a_1 b_0 + a_2$.
Since from our assumption we have $N(r,f) = S(r,f)$, then by applying Lemma 2.1 to $\pi(f(qz))$, we get

$$N(r, \pi(f(qz))) = S(r,f). \tag{3.1}$$

Now let,

$$H(z) = [f(z)]^n + \frac{P_1(f(qz))}{a_1 b_n} = -\frac{a_3}{a_1 b_n \pi(f(qz))} \tag{3.2}$$

from (3.1) and (3.2) we have

$$N\left(r, \frac{1}{H}\right) = N\left(r, -\frac{a_1 b_n \pi(f(qz))}{a_3}\right) = S(r,f).$$

Also $\frac{P_1(f(qz))}{a_1 b_n}$ is a q-difference polynomial in f of degree $n-1$. Hence by Lemma 2.2 $H(z) = (h(z))^n$, where $h(z) = f(z) + \frac{a(z)}{n}$ and $(h(z))^{n-1}a(z)$ is obtained by substituting $h(z)$ for $f(z)$, $h'(z)$ for $f'(z)$ etc., in the terms of degree $n-1$ in $\frac{P_1(f)}{a_1 b_n}$. Since $n \geq 2$ and the term in $\frac{P_1(f)}{a_1 b_n}$ with degree $n-1$ is $\frac{b_{n-1}}{b_n} f^{n-1}$.
Thus
$(h(z))^{n-1}a(z) = \frac{b_{n-1}}{b_n}(h(z))^{n-1}$ or $a(z) = \frac{b_{n-1}}{b_n}$.
Therefore

$$H(z) = \left(f(z) + \frac{b_{n-1}}{nb_n}\right)^n. \tag{3.3}$$

From (3.2) and (3.3), we have

$$\left(f(z) + \frac{b_{n-1}}{nb_n}\right)^n \pi(f(qz)) = -\frac{a_3}{a_1 b_n}.$$

Thus

$$T\left(r, \left(f(z) + \frac{b_{n-1}}{nb_n}\right)^n \pi(f(qz))\right) = S(r,f). \tag{3.4}$$

From the first fundamental theorem of Nevanlinna, (3.1), (3.4) and Lemma 2.3, we get

$$\begin{aligned}
T\left(r, f^{\overline{d}[\pi(f(qz))]}\left[f(z) + \frac{b_{n-1}}{nb_n}\right]^n\right) &= T\left(r, \frac{1}{f^{\overline{d}[\pi(f(qz))]}\left[f(z) + \frac{b_{n-1}}{nb_n}\right]^n}\right) + O(1), \\
&\leq T\left(r, \frac{\pi(f(qz))}{f^{\overline{d}[\pi(f(qz))]}}\right) + T\left(r, \frac{1}{\pi(f(qz))\left[f(z) + \frac{b_{n-1}}{nb_n}\right]^n}\right) + O(1), \\
&\leq N\left(r, \frac{\pi(f(qz))}{f^{\overline{d}[\pi(f(qz))]}}\right) + m\left(r, \frac{\pi(f(qz))}{f^{\overline{d}[\pi(f(qz))]}}\right) + O(1), \\
&\leq N(r, \pi(f(qz))) + \overline{d}[\pi(f(qz))]N\left(r, \frac{1}{f}\right) \\
&+ [\overline{d}[\pi(f(qz))] - \underline{d}[\pi(f(qz))]]m\left(r, \frac{1}{f}\right) + S(r,f), \\
&\leq \overline{d}[\pi(f(qz))]N\left(r, \frac{1}{f}\right) + \overline{d}[\pi(f(qz))]m\left(r, \frac{1}{f}\right) + S(r,f), \\
&\leq \overline{d}[\pi(f(qz))]T(r,f) + S(r,f).
\end{aligned} \tag{3.5}$$

But

$$\begin{aligned}
T\left(r, f^{\overline{d}[\pi(f(qz))]}\left[f(z) + \frac{b_{n-1}}{nb_n}\right]^n\right) &= T\left(r, f^{\overline{d}[\pi(f(qz))]}\right) + T\left(r, \left[f(z) + \frac{b_{n-1}}{nb_n}\right]^n\right) \\
&= \overline{d}[\pi(f(qz))]T(r,f) + nT\left(r, f(z) + \frac{b_{n-1}}{nb_n}\right) + S(r,f), \\
&= [\overline{d}[\pi(f(qz))] + n]T(r,f) + S(r,f).
\end{aligned} \tag{3.6}$$

Thus from (3.5) and (3.6), we get

$$[\overline{d}[\pi(f(qz))] + n]T(r,f) = \overline{d}[\pi(f(qz))]T(r,f) + S(r,f).$$

Which is a contradiction to our assumption.

We shall now consider the case when $n = 1$.

If $n = 1$ then equation (3.1) becomes $a_1[b_1 f + b_0]\pi(f(qz)) + a_2\pi(f(qz)) + a_3 = 0$, that is

$$\left(f + \frac{(a_1 b_0 + a_2)}{a_1 b_1}\right)\pi(f(qz)) = -\frac{a_3}{a_1 b_1}.$$

Hence from lemma 2.2 and the equation (3.1), we have

$$T(r, \pi(f(qz))) = S(r, f).$$

Also

$$T\left(r, \left(f + \frac{(a_1 b_0 + a_2)}{a_1 b_1}\right)\right) = T\left(r, -\frac{a_3}{\pi(f(qz)) a_1 b_1}\right) + S(r, f).$$

Thus

$$T(r, f) = S(r, f).$$

Which is again a contradiction to our assumption. Hence the theorem.

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Shilpa N.

Department of Mathematics, SOE, Presidency University, Near Rajanukunte, Bangalore, Karnataka, India.

E-mail address: shilpajaikumar@gmail.com