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HERMITE POLYNOMIALS AND HAHN'S THEOREM WITH RESPECT TO THE RAISING OPERATOR

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Abstract. Let $\{H_n\}_{n\geq 0}$ be the monic Hermite polynomial sequence, It is well known that $\mathscr{H} H_n(x) = H_{n+1}(x),\ n \geq 0$, where $\mathscr{H}$ is the raising operator associated to the monic Hermite polynomial and given by $\mathscr{H} := x\mathbb{I} - (1/2)D$, with $\mathbb{I}$ represents the identity operator. In this paper, we introduce the notion of $\mathscr{H}_\epsilon$-classical orthogonal polynomials, where $\mathscr{H}_\epsilon := x\mathbb{I} + \epsilon D$ $(\epsilon \in \mathbb{C}^*)$. Then we show that the scaled Hermite polynomial sequence $\{a^{-n} H_n(ax)\}_{n\geq 0}$, where $a^2 = -(2\epsilon)^{-1}$, is the only $\mathscr{H}_\epsilon$-classical orthogonal sequence. As an illustration, we give some properties related to this operator.

1. Introduction and main results

Let $\mathbb{P}$ be the linear space of polynomials in one variable with complex coefficients. Let $\mathbb{P}'$ be the algebraic linear dual of $\mathbb{P}$. We write $\langle u, p\rangle := u(p)$ $(u \in \mathbb{P}',\ p \in \mathbb{P})$. A linear functional $u \in \mathbb{P}'$ is said to be regular [10, 14] if it is quasi-definite, i.e., $\det\langle u, x^{i+j}\rangle_{i,j=1,\dots,n} \neq 0$ for $n \geq 0$. This is equivalent to the existence of a unique sequence of monic polynomials $\{p_n\}_{n\geq 0}$ of degree n such that $\langle u, p_n p_m\rangle = r_n\delta_{n,m},\ n,\ m \geq 0$, with $r_n \neq 0$ $(n \geq 0)$. Then the sequence $\{p_n\}_{n\geq 0}$ is said to be the sequence of monic orthogonal polynomials (SMOP) with respect to u.

Proposition 1.1. (Favard's Theorem[10]). *Let $\{P_n\}_{n\geq 0}$ be a monic polynomial sequence. Then $\{P_n\}_{n\geq 0}$ is orthogonal if and only if there exist two sequences of complex number $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$, such that $\gamma_n \neq 0,\ n \geq 1$ and satisfies the three-term recurrence relation*

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ n \geq 0. \end{cases} \tag{1}$$

When $\{P_n\}_{n\geq 0}$ is a SMOP, then $\{\tilde{P}_n\}_{n\geq 0}$, where $\tilde{P}_n(x) = a^{-n}P_n(ax+b),\ (a,b) \in \mathbb{C}^* \times \mathbb{C}$, is also a SMOP and satisfies [12, 13]

$$\begin{cases} \tilde{P}_0(x) = 1, \quad \tilde{P}_1(x) = x - \tilde{\beta}_0, \\ \tilde{P}_{n+2}(x) = (x - \tilde{\beta}_{n+1})\tilde{P}_{n+1}(x) - \tilde{\gamma}_{n+1}\tilde{P}_n(x),\ n \geq 0, \end{cases}$$

where $\tilde{\beta}_n = a^{-1}(\beta_n - b)$ and $\tilde{\gamma}_{n+1} = a^{-2}\gamma_{n+1}$.

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An orthogonal polynomial sequence $\{P_n\}_{n\geq 0}$ is called classical, if $\{P'_n\}_{n\geq 0}$ is also orthogonal (Hermite, Laguerre, Bessel or Jacobi), (Hahn-property[7, 8]).

Next we collect some properties of the monic Hermite polynomials that we will need in the sequel [4, 10].
The monic Hermite polynomial sequence $\{H_n\}_{n\geq 0}$ can be expressed by the Rodrigues formula (see [11, 15])

$$H_n(x) = \frac{(-1)^n}{2^n}\ e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}), \quad n \geq 0. \tag{2}$$

The monic sequence of Hermite polynomials $\{H_n\}_{n\geq 0}$ is an Appell sequence [4], i.e.,

$$H'_{n+1}(x) = (n+1)H_n(x), \quad n \geq 0. \tag{3}$$

So $\{H_n\}_{n\geq 0}$ also satisfies the three-term recurrence relation (1), where

$$\beta_n = 0,\ n \geq 0; \quad \gamma_{n+1} = \frac{n+1}{2},\ n \geq 0. \tag{4}$$

By starting from (2), with n replaced by $n+1$, we obtain

$$H_{n+1}(x) = \frac{(-1)^{n+1}}{2^{n+1}}\ e^{x^2}\frac{d^n}{dx^n}\big(-2xe^{-x^2}\big), \quad n \geq 0.$$

But according to the Leibniz rule

$$\frac{d^n}{dx^n}\big(f(x)g(x)\big) = \sum_{k=0}^{n}\binom{n}{k}f^{(k)}(x)g^{(n-k)}(x),$$

we have $H_{n+1}(x) = xH_n(x) - \frac{1}{2}H'_n(x),\ n \geq 0$, or equivalently

$$H_{n+1}(x) = \mathscr{H} H_n(x),\ n \geq 0, \tag{5}$$

where $\mathscr{H} := x\mathbb{I} - (1/2)D$ is called the raising operator associated to the monic Hermite polynomials (for more details see [16]).

In view of (5), we can say that $\{H_n\}$ is an $\mathscr{H}$-classical polynomial sequence, since it satisfies the Hahn-property with respect to the operators $\mathscr{H}$ i.e., it is an orthogonal polynomial sequence, whose sequence of $\mathscr{H}$ is also orthogonal. See further examples in [1, 2, 5, 7, 8, 9]

In this paper, we introduce the raising operator $\mathscr{H}_\epsilon := x\mathbb{I} + \epsilon D,\ \epsilon \neq 0$, and we show that the scaled Hermite polynomial sequence $\{a^{-n}H_n(ax)\}_{n\geq 0}$ where $a^2 = -(2\epsilon)^{-1}$, is actually the only monic orthogonal polynomial sequence which is $\mathscr{H}_\epsilon$-classical. As an illustration, we give some properties related to the above operator. Finally, we represent certain sequences by a triple integrals in terms of Hermite polynomials.

2. Raising operator associated to the Hermite polynomials

Recall the operator

$$\begin{aligned} \mathscr{H}_\epsilon : \mathbb{P} &\longrightarrow \mathbb{P} \\ f &\longmapsto xf + \epsilon f',\ \epsilon \neq 0. \end{aligned}$$

Clearly, the operator $\mathscr{H}_\epsilon$ raises the degree of any polynomial. Such operator is called raising operator.

Definition 2.1. *We call a sequence $\{P_n\}_{n\geq 0}$ of orthogonal polynomials $\mathscr{H}_\epsilon$-classical if there exist a sequence $\{Q_n\}_{n\geq 0}$ of orthogonal polynomials such that $\mathscr{H}_\epsilon P_n = Q_{n+1},\ n \geq 0$.*

The aim of this paper is to find the sequences of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ such that the monic sequence $\{Q_n\}_{n\geq 0}$, where

$$Q_{n+1}(x) := xP_n(x) + \epsilon P_n'(x),\ n \geq 0, \quad (Q_0(x) = 1), \tag{6}$$

is also orthogonal.

Assume that $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are SMOP satisfying

$$\begin{cases} P_0(x) = 1,\ P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ \gamma_{n+1} \neq 0,\ n \geq 0, \end{cases} \tag{7}$$

$$\begin{cases} Q_0(x) = 1,\ Q_1(x) = x - \rho_0, \\ Q_{n+2}(x) = (x - \rho_{n+1})Q_{n+1}(x) - \varrho_{n+1}P_n(x),\ \varrho_{n+1} \neq 0,\ n \geq 0. \end{cases} \tag{8}$$

We have the following fundamental result.

Theorem 2.1. *the scaled Hermite polynomial sequence $\{a^{-n}H_n(ax)\}_{n\geq 0}$ where $a^2 = -(2\epsilon)^{-1}$, is actually the only monic orthogonal polynomial sequence which is $\mathscr{H}_\epsilon$-classical. More precisely, $Q_n(x) = P_n(x) = a^{-n}H_n(ax)$ where $a^2 = -(2\epsilon)^{-1}$.*

Proof. By differentiating (7), we obtain

$$P_{n+2}'(x) = (x - \beta_{n+1})P_{n+1}'(x) - \gamma_{n+1}P_n'(x) + P_{n+1}(x),\ n \geq 0.$$

Multiplying the last equation by ϵ and the relation (7) by x, and we summarize, we get

$$Q_{n+3}(x) = (x - \beta_{n+1})P_{n+2}(x) - \gamma_{n+1}Q_{n+1}(x) + \epsilon Q_{n+1}(x),\ n \geq 0.$$

By using (8), we finally get

$$(\beta_{n+1} - \rho_{n+2})Q_{n+2}(x) + (\gamma_{n+1} - \varrho_{n+2})Q_{n+1}(x) = \epsilon P_{n+1}(x),\ n \geq 0, \tag{9}$$

and

$$(\beta_0 - \rho_1)Q_1(x) - \varrho_1 Q_0(x) = \epsilon P_0(x). \tag{10}$$

By comparing the degrees in(9) and (10), we obtain

$$\rho_{n+1} = \beta_n,\ n \geq 0, \tag{11}$$

$$\varrho_{n+2} = \gamma_{n+1} - \epsilon,\ n \geq 0. \tag{12}$$

$$\epsilon = -\varrho_1. \tag{13}$$

Then, (9) gives $Q_n(x) = P_n(x),\ n \geq 0$, since $Q_0(x) = P_0(x)$. Hence, (11) gives $\beta_{n+1} = \beta_n = \beta_0 = \rho_0 = 0,\ n \geq 0$ by using (6) for $n = 0$. On the other hand, (12) gives, by induction, $\gamma_{n+1} = -(n+1)\epsilon,\ n \geq 0$. This implies that $Q_n(x) = P_n(x) = a^{-n}H_n(ax)$ where $a^2 = -(2\epsilon)^{-1}$, with $\{a^{-n}H_n(ax)\}_{n\geq 0}$ is the scaled Hermite polynomial sequence. ■

3. Some properties of the obtined polynomials

In this section, we firstly deduce some consequences of the operator $\mathscr{H}$ and Hermite polynomials. Secondly, we represent some integer (or real) sequences by a triple integral representations in terms of Hermite polynomials.

3.1. **Higher order $\mathscr{H}$-differential relations.** From (6) and as a consequence of our problem, we have

$$\mathscr{H} H_n(x) = H_{n+1}(x),\ n \geq 0. \tag{14}$$

In the other hand, the relation (3) of *Appell property* can be written as follow

$$DH_{n+1}(x) = (n+1)H_n(x),\ n \geq 0. \tag{15}$$

Then, we obtain $\mathscr{H} \circ DH_{n+1}(x) = (n+1)H_{n+1}(x),\ n \geq 0$ and $D \circ \mathscr{H} H_n(x) = (n+1)H_n(x),\ n \geq 0$, or equivalently the Böchner's charactrisation [6] of Hermite polynomials

$$H''_{n+1}(x) - 2xH'_{n+1}(x) + 2(n+1)H_{n+1}(x) = 0,\ n \geq 0.$$

By using (15), we have

$$D^m H_{n+1}(x) = (n+1)n \cdots (n+2-m)H_{n+1-m}(x),\ m \leq n+1,\ n \geq 0.$$

In particular, $D^n H_n(x) = n!H_0(x)$.

According to (14) we can obtain a similar result for the *raising operator* $\mathscr{H}$

$$\mathscr{H}^m H_n(x) = H_{n+m}(x), \quad n,\ m \geq 0. \tag{16}$$

In particular, $\mathscr{H}^n(H_0(x)) = H_n(x),\ n \geq 0$, and then

$$\mathscr{H}^n \circ D^n\big(H_n(x)\big) = n!H_n(x),\ n \geq 0.$$

In the following theorem, we prove that the SMP $\{\mathscr{H}^n H_m\}_{n,\ m \geq 0}$ can be expressed by the so-called Rodrigues formula.

Theorem 3.1. *For every integer $m \geq 0$, the following relation holds*

$$\mathscr{H}^n H_m(x) = \frac{(-1)^n}{2^n}\ e^{x^2} \frac{d^n}{dx^n}\big(H_m(x)e^{-x^2}\big), \quad n \geq 0. \tag{17}$$

Proof. By induction, taking into account $\mathscr{H}^{n+1}H_m(x) = \mathscr{H}\big(\mathscr{H}^n H_m(x)\big)$, it follows that

$$\begin{aligned} \mathscr{H}^{n+1}H_m(x) &= \mathscr{H}\Big(\frac{(-1)^n}{2^n}\ e^{x^2} \frac{d^n}{dx^n}\big(H_m(x)e^{-x^2}\big)\Big) \\ &= (x\mathbb{I} - \frac{1}{2}D)\Big(\frac{(-1)^n}{2^n}\ e^{x^2} \frac{d^n}{dx^n}\big(H_m(x)e^{-x^2}\big)\Big) \\ &= \frac{(-1)^{n+1}}{2^{n+1}}\ e^{x^2} \frac{d^{n+1}}{dx^{n+1}}\big(H_m(x)e^{-x^2}\big), \quad n \geq 0. \end{aligned}$$

Hence the desired result. ∎

Corollary 3.2. *By using* (16)*, we have the following formula*

$$H_{n+m}(x) = \frac{(-1)^n}{2^n}\ e^{x^2} \frac{d^n}{dx^n}\big(H_m(x)e^{-x^2}\big), \quad n,\ m \geq 0.$$

3.2. **Representations in terms of Hermite polynomials.** Let us recall the integral relation between Laguerre and Hermite polynomials: *Uspensky's formula* [17]

$$L_n^{(\alpha)}(x) = \frac{n!\Gamma(n+\alpha+1)}{\sqrt{\pi}(2n)!\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{1} (1-y^2)^{\alpha-\frac{1}{2}} H_{2n}\big(y\sqrt{x}\big)\ \mathrm{d}y,\ \alpha > -\frac{1}{2},\ n \geq 0,$$

which gives, with x replaced by tx and $\alpha = 1$

$$L_n^{(1)}(tx) = \frac{n!(n+1)!}{(2n)!} \frac{2}{\pi} \int_{-1}^{1} (1-y^2)^{\frac{1}{2}} H_{2n}\big(y\sqrt{tx}\big)\ \mathrm{d}y,\ n \geq 0. \tag{18}$$

In the other hand, we have the following results.

Lemma 3.1. [3] *The following representations in terms of Laguerre polynomials, (with parameter $\alpha = 1$), hold*

$$n!(n+1)! = \int_0^{+\infty}\int_0^{+\infty} te^{-(x+t)}L_n^{(1)}\big(t(x+1)\big)\ \mathrm{d}x\mathrm{d}t. \tag{19}$$

$$\frac{(2n)!(n+1)\sqrt{\pi}}{4^n} = \int_0^{+\infty}\int_0^{+\infty} \frac{t}{\sqrt{x}}\ e^{-(x+t)}L_n^{(1)}\big(t(x+1)\big)\ \mathrm{d}x\mathrm{d}t. \tag{20}$$

$$n! = \int_0^{+\infty}\int_0^{1} te^{-t}L_n^{(1)}\big(t(x+1)\big)\ \mathrm{d}x\mathrm{d}t. \tag{21}$$

$$(-1)^n n! = \int_0^{+\infty}\int_0^{1} te^{-t}L_n^{(1)}(tx)\ \mathrm{d}x\mathrm{d}t. \tag{22}$$

$$(n+1)!(-1)^n\Big(\ln 2 + \sum_{k=1}^{n}\frac{(-1)^k}{k}\Big) = \int_0^{+\infty}\int_0^{1} \frac{te^{-t}}{1+x}L_n^{(1)}\big(t(x+1)\big)\ \mathrm{d}x\mathrm{d}t. \tag{23}$$

$$n!(n+1)!(-1)^n\Big(e - \sum_{k=0}^{n}\frac{1}{k!}\Big) = \int_0^{+\infty}\int_0^{1} te^{x-t}L_n^{(1)}(tx)\ \mathrm{d}x\mathrm{d}t. \tag{24}$$

Then, by inserting (18) in (19)−(24), we can easily obtain the following result.

Theorem 3.3. *For $n \in \mathbb{N}$, we have the following representations in terms of Hermite polynomials*

$$(2n)!\frac{\pi}{2} = \int_0^{+\infty}\int_0^{+\infty}\int_{-1}^{1} te^{-(x+t)}(1-y^2)^{\frac{1}{2}}H_{2n}\big(y\sqrt{t(x+1)}\big)\ \mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

$$\frac{[(2n)!]^2}{(n!)^2 2^{2n+1}}\pi^{\frac{3}{2}} = \int_0^{+\infty}\int_0^{+\infty}\int_{-1}^{1} \frac{t}{\sqrt{x}}\ e^{-(x+t)}(1-y^2)^{\frac{1}{2}}H_{2n}\big(y\sqrt{t(x+1)}\big)\ \mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

$$\frac{(2n)!}{(n+1)!}\frac{\pi}{2} = \int_0^{+\infty}\int_0^{1}\int_{-1}^{1} te^{-t}(1-y^2)^{\frac{1}{2}}H_{2n}\big(y\sqrt{t(x+1)}\big)\ \mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

$$(-1)^n\frac{(2n)!}{(n+1)!}\frac{\pi}{2} = \int_0^{+\infty}\int_0^{1}\int_{-1}^{1} te^{-t}(1-y^2)^{\frac{1}{2}}H_{2n}\big(y\sqrt{tx}\big)\ \mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

$$\frac{(-1)^n(2n)!}{n!}\Big(\ln 2+\sum_{k=1}^{n}\frac{(-1)^k}{k}\Big)\frac{\pi}{2} = \int_0^{+\infty}\int_0^{1}\int_{-1}^{1} \frac{te^{-t}}{1+x}(1-y^2)^{\frac{1}{2}}H_{2n}\big(y\sqrt{t(x+1)}\big)\ \mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

$$(-1)^n(2n)!\Big(e-\sum_{k=0}^{n}\frac{1}{k!}\Big)\frac{\pi}{2} = \int_0^{+\infty}\int_0^{1}\int_{-1}^{1} te^{x-t}(1-y^2)^{\frac{1}{2}}H_{2n}\big(y\sqrt{tx}\big)\ \mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

Corollary 3.4. *For $n = 0$, we have the special cases*

$$\frac{\pi}{2} = \int_0^{+\infty}\int_0^{+\infty}\int_{-1}^{1} te^{-(x+t)}(1-y^2)^{\frac{1}{2}}\ \mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

$$\sqrt{\pi}\ \frac{\pi}{2} = \int_0^{+\infty}\int_0^{+\infty}\int_{-1}^{1} \frac{t}{\sqrt{x}}\ e^{-(x+t)}(1-y^2)^{\frac{1}{2}}\ \mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

$$\frac{\pi}{2} = \int_0^{+\infty}\int_0^{1}\int_{-1}^{1} te^{-t}(1-y^2)^{\frac{1}{2}}\ \mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

$$\ln 2\ \frac{\pi}{2} = \int_0^{+\infty}\int_0^1\int_{-1}^1 \frac{te^{-t}}{1+x}(1-y^2)^{\frac{1}{2}}\ \mathrm{d}y\mathrm{d}x\mathrm{d}t.$$
$$(e-1)\frac{\pi}{2} = \int_0^{+\infty}\int_0^1\int_{-1}^1 te^{x-t}(1-y^2)^{\frac{1}{2}}\ \mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

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