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# D-HYPERCYCLIC AND D-CHAOTIC PROPERTIES OF ABSTRACT DIFFERENTIAL EQUATIONS OF FIRST ORDER

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Abstract. The main aim of this paper is to contribute to the existing theory of disjoint hypercyclic and disjoint topologically transitive abstract non-degenerate differential equations of first order as well as to initiate the study of disjoint chaoticity for strongly continuous semigroups and $C$-distribution semigroups in Banach and Fréchet function spaces. We also investigate disjoint topologically mixing property for $C$-distribution semigroups, and prove a disjoint analogue of the Desch-Schappacher-Webb criterion in this context. Some new results on disjoint transitivity and disjoint chaoticity of strongly continuous families of composition operators and strongly continuous semigroups induced by semiflows are shown, as well.

## 1. Introduction and Preliminaries

Let $E$ be a Fréchet space. A linear operator $T$ on $E$ is said to be hypercyclic iff there exists an element $x \in D_\infty(T)$, whose orbit $\{T^n x \ : \ n \in \mathbb{N}_0\}$ is dense in $E$. A periodic point for $T$ is an element $x \in D_\infty(T)$ for which there exists $n \in \mathbb{N}$ with $T^n x = x$. We say that $T$ is chaotic iff $T$ is hypercyclic and the set of periodic points of $T$ is dense in $E$.

The first examples of hypercyclic operators were given on the space of entire functions $H(\mathbb{C})$ equipped with topology of uniform convergence on compact subsets of $\mathbb{C}$. In 1929, G. D. Birkhoff proved that the translation operator is hypercyclic in $H(\mathbb{C})$ and, in 1952, G. R. MacLane proved that the derivative operator is hypercyclic in $H(\mathbb{C})$. The first example of a hypercyclic operator on a Banach space was given by S. Rolewicz [38] in 1969 (see the monographs [4] by F. Bayart, E. Matheron and [23] by K.-G. Grosse-Erdmann, A. Peris for a comprehensive survey of results on topological dynamical properties of linear operators). The first systematic study of hypercyclic and chaotic strongly continuous semigroups in Banach spaces was conducted by W. Desch, W. Schappacher and G. F. Webb [20] in 1997.

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Also, hypercyclic behaviour of operators in a hypercyclic $C_0$-semigroup was studied deeply in [16].

On the other hand, L. Bernal-González [6] and J. Bés, A. Peris [7] have introduced various notions of disjoint hypercyclicity for continuous linear operators in Fréchet spaces (cf. [8]-[13], [27], [33]-[34], [37], [39] and [42] for more details on the subject). The notion of disjoint hypercyclicity for strongly continuous semigroups in Banach spaces and the notion of disjoint hypercyclicity for $C$-distribution cosine functions (global fractionally integrated $C$-cosine functions) in Banach spaces have been introduced by the second named author in [32] and [29], respectively. One of the main aims of this paper is to fill the gap in the existing theory of disjoint hypercyclic abstract PDEs by enquiring into the basic disjoint hypercyclic and disjoint chaotic properties of $C$-distribution semigroups and global fractionally integrated $C$-semigroups in Fréchet spaces (for further information about $C$-distribution semigroups and fractionally integrated $C$-semigroups in locally convex spaces, as well as about hypercyclic and chaotic properties of various classes of abstract (degenerate) Volterra integro-differential equations in locally convex spaces, the reader may consult the monographs [26]-[28] and references cited therein; in this paper, we will focus our attention entirely on the abstract non-degenerate differential equations of first order). We provide the first examples of abstract Cauchy problems of first order whose solutions are not governed by strongly continuous semigroups and which possess a certain disjoint hypercyclic behaviour.

The organization and main ideas of paper are briefly described as follows. In the preliminary part, we remind ourselves of the basic properties of generalized function spaces used, $C$-distribution semigroups and global fractionally integrated $C$-semigroups in Fréchet spaces, and recall the assertion of $d$-Blow-up/Collapse Criterion for single-valued linear operators (cf. [12, Proposition 3.7] for a slight generalization); in Proposition 1.2, we transfer the assertion of [30, Lemma 6(i)] to $C$-distribution semigroups in Fréchet spaces. The main purpose of Definition 2.1 is to introduce various topological dynamical properties of $C$-distribution semigroups. After that, we explain how these notions can be extended to arbitrary families of linear operators (Remark 2.2) and reformulate them for global fractionally integrated $C$-semigroups (Definition 2.3). In [19], R. deLaubenfels, H. Emamirad and K.-G. Grosse-Erdmann have initiated the study of hypercyclic and chaotic properties of distribution semigroups and $C$-regularized semigroups. The main objective in Theorem 2.4 and Theorem 2.5 is to prove $d$-Hypercyclicity Criterion for linear, not necessarily continuous, operators (this is, actually, a disjoint analogue of [19, Theorem 2.3]; see also [7, Definition 2.5, Proposition 2.6, Theorem 2.7] for continuous case) and $d$-Hypercyclicity Criterion for $C$-distribution semigroups, respectively; $d$-Blow-up/Collapse Criterion for $C$-distribution semigroups is stated in Proposition 2.6. In Theorem 2.7, we reconsider the Desch-Schappacher-Webb criterion for $C$-distribution semigroups and prove its disjoint version following the analysis of L. Bernal-González [6, Theorem 4.3]; to the best knowledge of the authors, this statement is new even for strongly continuous semigroups of operators in Banach spaces. After giving a few noteworthy observations in Remark 2.9-Remark 2.11, we present some illustrative applications of our abstract results in Example 2.12 and Example 2.13. To motivate our investigations in Section 3, let us recall that T. Kalmes [25] has scrutinized the hypercyclicity and chaoticity of strongly continuous semigroups induced by semiflows. In his analysis, the pivot function space is chosen to be $C_{0,\rho}(X,\mathbb{K})$, resp. $L^p(X,\mu,\mathbb{K})$, where $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, $X$ is a locally compact, Hausdorff space and $\rho : X\to(0,\infty)$ is an upper semicontinuous function, resp. $X$ is a locally compact, $\sigma$-compact Hausdorff space, $p\in[1,\infty)$ and $\mu$ is a locally finite Borel measure on $X$ (for the sake of simplicity and better exposition, we shall work henceforth only with Fréchet spaces over the field of complex numbers). Disjoint hypercyclicity of strongly continuous semigroups induced by semiflows has been investigated in [32] (cf. also [27, Subsection 3.1.1]). In Section 3, our intention is to continue the research studies raised in [25] and [32] by exploring disjoint transitivity and disjoint chaoticity of strongly continuous families constituted of composition operators (cf. Theorem 3.1, Theorem 3.2 for weighted $L^p$-spaces, and Theorem 3.7, Theorem 3.8 for weighted $C_0$-spaces) and strongly continuous semigroups induced by semiflows (cf. Corollary 3.3, Corollary 3.4, Theorem 3.5, Theorem 3.6 for weighted $L^p$-spaces, and Corollary 3.9, Corollary 3.10, Theorem 3.11, Theorem 3.12 for weighted $C_0$-spaces). We investigate disjoint chaoticity of strongly continuous semigroups on Fréchet space $C(\Omega)$ in Theorem 3.13, and present several examples of disjoint chaotic strongly continuous semigroups in Example 3.14.

We use the standard terminology throughout the paper. By $E$ we denote a non-trivial, separable, Fréchet space over the field of complex numbers. If $X$ is also a non-trivial, separable, Fréchet space over the same field of scalars as $E$, then we denote by $L(E,X)$ the space consisting of all continuous linear mappings from $E$ into $X$; $L(E)\equiv L(E,E)$. By $\circledast_E$ ($\circledast$, if there is no risk for confusion), we denote the fundamental system of seminorms which defines the topology of $E$; the dual space of $E$ is denoted by $E^*$. Let us recall that a subset of $E$ is called total iff its linear span is dense in $E$.

Let $0<\tau\leq\infty$. In our framework, any strongly continuous operator family $(W(t))_{t\in[0,\tau)}\subseteq L(E,X)$ is locally equicontinuous, i.e., for every $T\in(0,\tau)$ and for every $p\in\circledast_X$, there exist $q_p\in\circledast_E$ and $c_p>0$ such that $p(W(t)x)\leq c_p q_p(x)$, $x\in E$, $t\in[0,T]$; the notions of equicontinuity of $(W(t))_{t\in[0,\tau)}$ and the exponential equicontinuity of $(W(t))_{t\geq0}$ are defined similarly.

By $\mathcal{B}$ we denote the family consisting of all bounded subsets of $E$. Define $p_B(T):=\sup_{x\in B}p(Tx)$, $p\in\circledast_E$, $B\in\mathcal{B}$, $T\in L(E,X)$. Then $p_B(\cdot)$ is a seminorm on $L(E,X)$ and the system $(p_B)_{(p,B)\in\circledast_X\times\mathcal{B}}$ induces the Hausdorff locally convex topology on $L(E,X)$. Suppose that $A$ is a closed linear operator acting on $E$. Then we denote the domain, kernel space, range and point spectrum of $A$ by $D(A)$, $N(A)$, $R(A)$ and $\sigma_p(A)$, respectively. Since no confusion seems likely, we will identify $A$ with its graph. Set $D_\infty(A):=\bigcap_{n\in\mathbb{N}}D(A^n)$. We will always assume henceforth that $C\in L(E)$ and $C$ is injective. Put $p_C(x):=p(C^{-1}x)$, $p\in\circledast$, $x\in R(C)$. Then $p_C(\cdot)$ is a seminorm on $R(C)$ and the calibration $(p_C)_{p\in\circledast}$ induces a Fréchet topology on $R(C)$; we denote this space by $[R(C)]_\circledast$. Set $\mathbb{C}_+:=\{\lambda\in\mathbb{C}:\Re\lambda>0\}$, $\mathbb{C}_-:=\{\lambda\in\mathbb{C}:\Re\lambda<0\}$, and, by common consent, $0^0:=1$. By $\Gamma(\cdot)$ we denote the Gamma function. Set $\mathbb{N}_n:=\{1,2,\cdot\cdot\cdot,n\}$, $\mathbb{N}_n^0:=\mathbb{N}_n\cup\{0\}$ $(n\in\mathbb{N})$, $g_\zeta(t):=t^{\zeta-1}/\Gamma(\zeta)$ $(\zeta>0,\ t>0)$ and $g_0(t):=$ the Dirac $\delta$-distribution. Given $s\in\mathbb{R}$ in advance, set $\lceil s\rceil:=\inf\{l\in\mathbb{Z}:s\leq l\}$.

Suppose that $V$ is a general topological vector space. As it is well-known, a function $f:\Omega\to V$, where $\Omega$ is an open non-empty subset of $\mathbb{C}$, is said to be analytic iff it is locally expressible in a neighborhood of any point $z\in\Omega$ by a uniformly convergent power series with coefficients in $V$. The reader may consult

[1] and [27, Section 1.1] and references cited there for the basic information about vector-valued analytic functions. In our framework, the analyticity of a mapping $f : \Omega \to E$ is equivalent with its weak analyticity.

In what follows, we will remind ourselves of the basic facts concerning vector-valued distribution spaces used henceforth. The Schwartz spaces of test functions $\mathcal{D} = C_0^\infty(\mathbb{R})$ and $\mathcal{E} = C^\infty(\mathbb{R})$ are equipped with the usual inductive limit topologies; the topology of space of rapidly decreasing functions $\mathcal{S}$ defines the following system of seminorms $p_{m,n}(\psi) := \sup_{x\in\mathbb{R}} |x^m \psi^{(n)}(x)|$, $\psi \in \mathcal{S}$, $m$, $n \in \mathbb{N}_0$. If $\emptyset \neq \Omega \subseteq \mathbb{R}$, then we denote by $\mathcal{D}_\Omega$ the subspace of $\mathcal{D}$ consisting of those functions $\varphi \in \mathcal{D}$ for which $\operatorname{supp}(\varphi) \subseteq \Omega$; $\mathcal{D}_0 \equiv \mathcal{D}_{[0,\infty)}$. If $\varphi$, $\psi : \mathbb{R} \to \mathbb{C}$ are measurable functions, the convolution products $\varphi * \psi$ and $\varphi *_0 \psi$ are defined by

$$\varphi * \psi(t) := \int_{-\infty}^{\infty} \varphi(t-s)\psi(s)\, ds \text{ and } \varphi *_0 \psi(t) := \int_0^t \varphi(t-s)\psi(s)\, ds,\ t \in \mathbb{R}.$$

If $\varphi \in \mathcal{D}$ and $f \in \mathcal{D}'$, or $\varphi \in \mathcal{E}$ and $f \in \mathcal{E}'$, then we define the convolution $f * \varphi$ by $(f * \varphi)(t) := f(\varphi(t - \cdot))$, $t \in \mathbb{R}$. For $f \in \mathcal{D}'$, or for $f \in \mathcal{E}'$, define $\check{f}$ by $\check{f}(\varphi) := f(\varphi(-\cdot))$, $\varphi \in \mathcal{D}$ $(\varphi \in \mathcal{E})$. In general, the convolution of two distributions $f$, $g \in \mathcal{D}'$, denoted by $f * g$, is defined by $(f * g)(\varphi) := g(\check{f} * \varphi)$, $\varphi \in \mathcal{D}$. It is well-known that $f * g \in \mathcal{D}'$ and $\operatorname{supp}(f * g) \subseteq \operatorname{supp}(f) + \operatorname{supp}(g)$. For every $t \in \mathbb{R}$, we define the Dirac distribution centered at point $t$, $\delta_t$ for short, by $\delta_t(\varphi) := \varphi(t)$, $\varphi \in \mathcal{D}$.

The space $\mathcal{D}'(E) := L(\mathcal{D}, E)$ is consisted of all continuous linear functions $\mathcal{D} \to E$; $\mathcal{D}'_\Omega(E)$ denotes the subspace of $\mathcal{D}'(E)$ containing $E$-valued distributions whose supports are contained in $\Omega$. Set $\mathcal{D}'_0(E) := \mathcal{D}'_{[0,\infty)}(E)$. If $E = \mathbb{C}$, then the above spaces are also denoted by $\mathcal{D}'$, $\mathcal{D}'_\Omega$ and $\mathcal{D}'_0$. For more details about vector-valued distributions, we refer the reader to L. Schwartz [40]-[41].

Let $\alpha \in (0, \infty) \setminus \mathbb{N}$, $f \in \mathcal{S}$ and $n = \lceil \alpha \rceil$. Let us recall that the Weyl fractional derivative $W_+^\alpha$ of order $\alpha$ is defined by

$$W_+^\alpha f(t) := \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^\infty (s-t)^{n-\alpha-1} f(s)\, ds,\ t \in \mathbb{R}.$$

If $\alpha = n \in \mathbb{N}_0$, then we set $W_+^n := (-1)^n \frac{d^n}{dt^n}$. It is well known that the following equality holds: $W_+^{\alpha+\beta} f = W_+^\alpha W_+^\beta f$, $\alpha$, $\beta > 0$, $f \in \mathcal{S}$.

Now we recall the definition of a $C$-distribution semigroup in Fréchet space (see [31]):

**Definition 1.1.** Let $\mathcal{G} \in \mathcal{D}'_0(L(E))$ satisfy $C\mathcal{G} = \mathcal{G}C$. Then it is said that $\mathcal{G}$ is a $C$-distribution semigroup, shortly (C-DS), if $\mathcal{G}$ satisfies the following conditions:

(i) $\mathcal{G}(\varphi *_0 \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi)$, for any $\varphi$, $\psi \in \mathcal{D}$.

(ii) $\mathcal{N}(\mathcal{G}) := \bigcap_{\varphi \in \mathcal{D}_0} N(\mathcal{G}(\varphi)) = \{0\}$.

A (C-DS) $\mathcal{G}$ is called dense if, in addition to the above,

(iii) $\mathcal{R}(\mathcal{G}) := \bigcup_{\varphi \in \mathcal{D}_0} R(\mathcal{G}(\varphi))$ is dense in $E$.

Let $\mathcal{G} \in \mathcal{D}_0'(L(E))$ be a (C-DS) and let $T \in \mathcal{E}_0'$, i.e., $T$ is a scalar-valued distribution with compact support contained in $[0, \infty)$. Define

$$G(T) := \Big\{(x, y) \in E \times E : \mathcal{G}(T * \varphi)x = \mathcal{G}(\varphi)y \ \text{ for all } \ \varphi \in \mathcal{D}_0\Big\}.$$

Then it can be easily seen that $G(T)$ is a closed linear operator. In general case, for every $\psi \in \mathcal{D}$, we have that $\psi_+ := \psi \mathbf{1}_{[0,\infty)} \in \mathcal{E}_0'$, where $\mathbf{1}_{[0,\infty)}$ stands for the characteristic function of $[0, \infty)$, so that the definition of $G(\psi_+)$ is clear. We define the (infinitesimal) generator $A$ of a pre-(C-DS) $\mathcal{G}$ by $A := G(-\delta')$. We know that $C^{-1}AC = A$ as well as that the following holds: Let $S, T \in \mathcal{E}_0'$, $\varphi \in \mathcal{D}_0$, $\psi \in \mathcal{D}$ and $x \in E$. Then we have:

A1. $G(S)G(T) \subseteq G(S * T)$ with $D(G(S)G(T)) = D(G(S * T)) \cap D(G(T))$, and $G(S) + G(T) \subseteq G(S + T)$.
A2. $(\mathcal{G}(\psi)x,\ \mathcal{G}(-\psi')x - \psi(0)Cx) \in A$.

We denote by $D(\mathcal{G})$ the set consisting of all elements $x \in E$ for which $x \in D(G(\delta_t))$ for all $t \geq 0$ and the mapping $t \mapsto G(\delta_t)x$, $t \geq 0$ is continuous. By A1., we have that

$$D\big(G(\delta_s)G(\delta_t)\big) = D\big(G(\delta_s * \delta_t)\big) \cap D\big(G(\delta_t)\big) = D\big(G(\delta_{t+s})\big) \cap D\big(G(\delta_t)\big),\ t,\ s \geq 0, \quad (1.1)$$

which clearly implies $G(\delta_t)(D(\mathcal{G})) \subseteq D(\mathcal{G})$, $t \geq 0$.

The notions of hypercyclicity, chaoticity, topological transitivity and topologically mixing property of $\mathcal{G}$ are introduced in the same way as in [26, Definition 3.1.29], where it has been assumed that the pivot space $E$ is one of Banach's:

(i) $\mathcal{G}$ is said to be hypercyclic iff there exists $x \in D(\mathcal{G})$ such that the set $\{G(\delta_t)x : t \geq 0\}$ is dense in $E$ (we call $x$ a hypercyclic vector of $\mathcal{G}$);
(ii) $\mathcal{G}$ is said to be chaotic iff $\mathcal{G}$ is hypercyclic and the set of periodic points of $\mathcal{G}$, $\mathcal{G}_{per}$ for short, defined by $\{x \in D(\mathcal{G}) : G(\delta_{t_0})x = x \text{ for some } t_0 > 0\}$, is dense in $E$;
(iii) $\mathcal{G}$ is said to be topologically transitive iff for every two open non-empty subsets $U$, $V$ of $E$, there exist $u \in D(\mathcal{G})$ and $t \geq 0$ such that $u \in U$ and $G(\delta_t)u \in V$;
(iv) $\mathcal{G}$ is said to be topologically mixing iff for every two open non-empty subsets $U$, $V$ of $E$, there exists $t_0 \geq 0$ such that, for every $t \geq t_0$, there exists $u_t \in D(\mathcal{G})$ such that $u_t \in U$ and $G(\delta_t)u_t \in V$, $t \geq t_0$.

In [26, Definition 3.1.29], we have introduced many other (subspace) topological dynamical properties of $C$-distribution semigroups, which will not be considered in the context of this paper.

Let us recall that the solution space for a closed linear operator $A$, denoted by $Z(A)$, is defined as the set of all $x \in E$ for which there exists a continuous mapping $u(\cdot, x) \in C([0, \infty) : E)$ satisfying $\int_0^t u(s, x)\, ds \in D(A)$ and $A \int_0^t u(s, x)\, ds = u(t, x) - x$, $t \geq 0$. It should be worth noting that $Z(A) = D(\mathcal{G})$, provided that $A$ generates a (C-DS) $\mathcal{G}$ :

**Proposition 1.2.** *Suppose that $A$ generates a* (C-DS) *$\mathcal{G}$. Then $Z(A) = D(\mathcal{G})$.*

*Proof.* Let $x \in D(\mathcal{G})$, and let $u(t, x) := G(\delta_t)x$, $t \geq 0$. Then $t \mapsto u(t, x)$, $t \geq 0$ is continuous and, to see that $D(\mathcal{G}) \subseteq Z(A)$, it suffices to show that $A \int_0^t u(s, x)\, ds =$

$u(t,x)-x$, $t\geq 0$, i.e.,

$$\int_0^t \mathcal{G}\bigl(-\varphi'\bigr)G(\delta_s)x\,ds=\mathcal{G}(\varphi)\bigl[G(\delta_t)x-x\bigr],\quad t\geq 0,\ \varphi\in\mathcal{D}_0.$$

By A1., we have $\mathcal{G}(\varphi)G(\delta_t)x=G(\delta_t)\mathcal{G}(\varphi)x=\mathcal{G}(\varphi(\cdot-t))x$, $t\geq 0$, $\varphi\in\mathcal{D}_0$. Hence, we need to prove that

$$\int_0^t \mathcal{G}\bigl(-\varphi'(\cdot-s)\bigr)x\,ds=\mathcal{G}(\varphi(\cdot-t))x-\mathcal{G}(\varphi)x,\quad t\geq 0,\ \varphi\in\mathcal{D}_0.$$

This simply follows from the continuity of $\mathcal{G}$ and the Newton-Leibniz formula, since $(\mathcal{G}(\varphi(\cdot-t))x)'=-\mathcal{G}(\varphi'(\cdot-t))x$ in variable $t\geq 0$, for $\varphi\in\mathcal{D}_0$. Suppose now that $x\in Z(A)$, $u(\cdot,x)\in C([0,\infty):E)$ satisfies $\int_0^t u(s,x)\,ds\in D(A)$ and $A\int_0^t u(s,x)\,ds=u(t,x)-x$, $t\geq 0$. It remains to be proved that $x\in D(\mathcal{G})$ and $u(t,x)=G(\delta_t)x$, $t\geq 0$. In other words, we know that

$$\mathcal{G}\bigl(-\varphi'\bigr)\int_0^t u(s,x)\,ds=\mathcal{G}(\varphi)[u(t,x)-x],\quad t\geq 0,\ \varphi\in\mathcal{D}_0 \tag{1.2}$$

and we need to prove that

$$\mathcal{G}\bigl(\varphi(\cdot-t)\bigr)x=\mathcal{G}(\varphi)u(t,x),\quad t\geq 0,\ \varphi\in\mathcal{D}_0. \tag{1.3}$$

Let $T>0$ and $\varphi\in\mathcal{D}_{[T,\infty)}$. Put $F(t):=\mathcal{G}(\varphi(\cdot+t))\int_0^t u(s,x)\,ds$, $0\leq t\leq T$. Then (1.2) in combination with the product rule and the continuity of $\mathcal{G}$ implies

$$F'(t)=\mathcal{G}(\varphi(\cdot+t))x,\quad 0\leq t\leq T.$$

Hence, $F(t)=F(t)-F(0)=\int_0^t F'(s)\,ds$, $0\leq t\leq T$, whence we may conclude that

$$\mathcal{G}(\varphi(\cdot+t))\int_0^t u(s,x)\,ds=\int_0^t \mathcal{G}(\varphi(\cdot+s))x\,ds,\quad 0\leq t\leq T.$$

Applying the operator $A$ on both sides of above equality, and using its closedness, the commutation with the operators $\mathcal{G}(\varphi(\cdot+t))$ for $0\leq t\leq T$, and the property A2., we get that

$$\mathcal{G}(\varphi(\cdot+t))[u(t,x)-x]=-\int_0^t \mathcal{G}(\varphi'(\cdot+s))x\,ds.$$

By the continuity of $\mathcal{G}$ and the Newton-Leibniz formula, we obtain from the above equality that:

$$\mathcal{G}(\varphi(\cdot+t))[u(t,x)-x]=-\mathcal{G}(\varphi(\cdot+t))x+\mathcal{G}(\varphi)x. \tag{1.4}$$

Suppose now that $\psi\in\mathcal{D}_0$. Then $\varphi=\psi(\cdot-T)\in\mathcal{D}_{[T,\infty)}$ and applying (1.4) with $t=T$, we immediately get that (1.3) holds with $\varphi$ and $t$ replaced respectively by $\psi$ and $T$ therein. Clearly, (1.3) holds for $t=0$ and this completes the proof of proposition. □

*Remark* 1.3. Since a Fréchet space valued distribution need not be of finite order (see [31] for the notion), we cannot give an alternative proof of Proposition 1.2 by using the theory of integrated $C$-semigroups, like it has been done in the Banach space case [30, Lemma 6(i)]. Observe also that the argumentation contained in the proof of inclusion $Z(A)\subseteq D(\mathcal{G})$ shows that for each $x\in Z(A)$ the function $u(\cdot,x)$ obeying the properties prescribed above must be unique.

Now we will recall the definition and basic properties of fractionally integrated $C$-semigroups in Fréchet spaces (cf. [26]-[27] for further information):

**Definition 1.4.** Suppose $A$ is a closed operator, $\alpha \geq 0$ and $0 < \tau \leq \infty$. If there exists a strongly continuous operator family $(S_\alpha(t))_{t\in[0,\tau)}$ ($S_\alpha(t) \in L(E)$, $t \in [0,\tau)$) such that:

(i) $S_\alpha(t)A \subseteq AS_\alpha(t)$, $t \in [0,\tau)$,
(ii) $S_\alpha(t)C = CS_\alpha(t)$, $t \in [0,\tau)$ and
(iii) for all $x \in E$ and $t \in [0,\tau)$: $\int_0^t S_\alpha(s)x\,ds \in D(A)$ and

$$A\int_0^t S_\alpha(s)x\,ds = S_\alpha(t)x - g_{\alpha+1}(t)Cx,$$

then it is said that $A$ is a subgenerator of a (local) $\alpha$-times integrated $C$-semigroup $(S_\alpha(t))_{t\in[0,\tau)}$. If $\tau = \infty$, then it is said that $(S_\alpha(t))_{t\geq 0}$ is an exponentially equicontinuous, $\alpha$-times integrated $C$-semigroup with a subgenerator $A$ iff, in addition to the above, there is a constant $\omega \in \mathbb{R}$ such that the operator family $\{e^{-\omega t}S_\alpha(t) : t \geq 0\} \subseteq L(E)$ is equicontinuous.

The integral generator $\hat{A}$ of $(S_\alpha(t))_{t\in[0,\tau)}$ is defined by

$$\hat{A} := \left\{(x,y) \in E \times E : S_\alpha(t)x - g_{\alpha+1}(t)Cx = \int_0^t S_\alpha(s)y\,ds,\ t \in [0,\tau)\right\}.$$

We know that the integral generator of $(S_\alpha(t))_{t\in[0,\tau)}$ is a closed linear operator which extends any subgenerator of $(S_\alpha(t))_{t\in[0,\tau)}$ and satisfies $C^{-1}\hat{A}C = \hat{A}$. In global case $\tau = \infty$, which will be only considered in the sequel, the integral generator $\hat{A}$ of $(S_\alpha(t))_{t\geq 0}$ is always its subgenerator.

Suppose that $\alpha \geq 0$, $n = \lceil \alpha \rceil$ and $\hat{A}$ is the integral generator of a global $\alpha$-times integrated $C$-semigroup $(S_\alpha(t))_{t\geq 0}$ on $E$. Then we have that:

$$\int_0^\infty W_+^\alpha \varphi(t)S_\alpha(t)x\,dt = (-1)^n\int_0^\infty \varphi^{(n)}(t)S_n(t)x\,dt, \quad x \in E,\ \varphi \in \mathcal{D}, \tag{1.5}$$

where $(S_n(t) \equiv (g_{n-\alpha} *_0 S_\alpha)(t))_{t\geq 0}$ is the global $n$-times integrated $C$-semigroup generated by $\hat{A}$. Furthermore, the following holds ([31]):

**Lemma 1.5.** *Assume that $\alpha \geq 0$ and $\hat{A}$ is the integral generator of a global $\alpha$-times integrated $C$-semigroup $(S_\alpha(t))_{t\geq 0}$ on $E$. Set*

$$\mathcal{G}_{S_\alpha}(\varphi)x := \int_0^\infty W_+^\alpha\varphi(t)S_\alpha(t)x\,dt, \quad x \in E,\ \varphi \in \mathcal{D}. \tag{1.6}$$

*Then $\mathcal{G}_{S_\alpha}$ is a* (C-DS) *whose integral generator is $\hat{A}$.*

Let us recall ([17], [27]) that an entire $C$-regularized group is an entire family of continuous linear operators $(T(z))_{z\in\mathbb{C}} \subseteq L(E)$ such that $T(0) = C$ and $T(z+\omega)C = T(z)T(\omega)$, $z,\ \omega \in \mathbb{C}$. The integral generator of $(T(z))_{z\in\mathbb{C}}$ is said to be the integral generator of $C$-regularized semigroup $(T(t))_{t\geq 0}$.

We refer the reader to [26]-[27] for the notion and basic properties of integrated $C$-cosine functions and exponentially equicontinuous, analytic integrated $C$-semigroups.

Let $k \in \mathbb{N}$, let $A$ be a linear operator on $E$, and let $\mathcal{G}$ be a $C$-distribution semigroup on $E$. Then we define the linear operator $\underbrace{A \oplus \cdots \oplus A}_{k}$ and $C$-distribution semigroup $\underbrace{\mathcal{G} \oplus \cdots \oplus \mathcal{G}}_{k}$ on $\underbrace{X \oplus \cdots \oplus X}_{k}$ by $D(\underbrace{A \oplus \cdots \oplus A}_{k}) := \underbrace{D(A) \oplus \cdots \oplus D(A)}_{k}$,

$$\underbrace{A \oplus \cdots \oplus A}_{k}\bigl(x_1, x_2, \cdots, x_k\bigr) := \bigl(Ax_1, Ax_2, \cdots, Ax_k\bigr),$$

for any $x_1, \cdots, \ x_k \in D(A)$, and

$$\underbrace{\mathcal{G} \oplus \cdots \oplus \mathcal{G}}_{k}(\varphi)\bigl(x_1, x_2, \cdots, x_k\bigr) := \bigl(\mathcal{G}(\varphi)x_1, \mathcal{G}(\varphi)x_2, \cdots, \mathcal{G}(\varphi)x_k\bigr),$$

for any $\varphi \in \mathcal{D}$ and $x_1, \cdots, \ x_k \in E$.

In a joint research study with J. Alberto Conejero and M. Murillo-Arcila [12], the first and third named author have recently extended various notions of hypercyclicity and disjoint hypercyclicity to (sequences of) multivalued linear operators (cf. [12, Definition 3.4] for the notion we will use henceforth). For our further work, it will be necessary to recall the following special case of $d$-Blow-up/Collapse Criterion for multivalued linear operators [12, Proposition 3.7]:

**Lemma 1.6.** *Let $N \in \mathbb{N}$, $N \geq 2$, and let $A_j$ be a linear operator in $E$ ($1 \leq j \leq N$). Suppose that $(a_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence of positive integers, as well as that the following holds:*

- *The set $E_0$, consisting of those elements $y \in D_\infty(A_1) \cap \cdots \cap D_\infty(A_N)$ satisfying that for each $j \in \mathbb{N}_N$ we have $\lim_{n\to\infty} A_j^{a_n} y = 0$, is dense in $E$.*
- *For each $j \in \mathbb{N}_N$ there exists a dense subset $E_{\infty,j}$ of $E$, consisting of those elements $z \in E$ for which there exist elements $\omega_{n,i}(z) \in D_\infty(A_j)$ ($n \in \mathbb{N}$, $1 \leq i \leq N$) such that $(\omega_{n,j}(z))_{n\in\mathbb{N}}$ is a null sequence in $E$, and $\lim_{n\to\infty} A_j^{a_n} \omega_{n,i}(z) = \delta_{i,j} z$ ($1 \leq i \leq N$).*

*Then the operators*

$$\underbrace{A_1 \oplus \cdots \oplus A_1}_{k}, \cdots, \underbrace{A_N \oplus \cdots \oplus A_N}_{k}$$

*are $d$-topologically transitive ($k \in \mathbb{N}$).*

## 2. Disjoint Hypercyclic and Disjoint Topologically Mixing $C$-Distribution Semigroups in Fréchet Spaces

We start this section by introducing the following definition:

**Definition 2.1.** Let $N \in \mathbb{N}$, $N \geq 2$ and $\mathcal{G}_i$ be a hypercyclic $C_i$-distribution semigroup in $E$, $i = 1, 2, \cdots, N$. We say that $(\mathcal{G}_i)_{1 \leq i \leq N}$ are:

(i) disjoint hypercyclic, $d$-hypercyclic in short, iff there exists $x \in \bigcap_{1 \leq i \leq N} D(\mathcal{G}_i)$ such that

$$\overline{\{(G_1(\delta_t)x, G_2(\delta_t)x, \cdots, G_N(\delta_t)x) \ : \ t \geq 0\}} = E^N. \tag{2.1}$$

An element $x \in \bigcap_{1 \leq i \leq N} D(\mathcal{G}_i)$ satisfying (2.1) is called a $d$-hypercyclic vector associated to $(\mathcal{G}_i)_{1 \leq i \leq N}$. If the set of all $d$-hypercyclic vectors is dense in $E$, then we say that $(\mathcal{G}_i)_{1 \leq i \leq N}$ are densely $d$-hypercyclic.

(ii) disjoint topologically transitive, $d$-topologically transitive in short, iff for any open non-empty subsets $V_0, V_1, V_2, \cdots, V_N$ of $E$, there exists $t \geq 0$ such that $V_0 \cap G_1(\delta_t)^{-1}(V_1) \cap G_2(\delta_t)^{-1}(V_2) \cap \cdots \cap G_N(\delta_t)^{-1}(V_N) \neq \emptyset$.

(iii) disjoint topologically mixing, $d$-topologically mixing in short, iff for any open non-empty subsets $V_0, V_1, V_2, \cdots, V_N$ of $E$, there exists $t_0 \geq 0$ such that for every $t \geq t_0$ we have that $V_0 \cap G_1(\delta_t)^{-1}(V_1) \cap G_2(\delta_t)^{-1}(V_2) \cap \cdots \cap G_N(\delta_t)^{-1}(V_N) \neq \emptyset$.

(iv) disjoint chaotic, $d$-chaotic in short, iff $(\mathcal{G}_i)_{1 \leq i \leq N}$ are $d$-transitive and the set of periodic elements of $(\mathcal{G}_i)_{1 \leq i \leq N}$, defined by $\mathcal{P}(\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_N) := \{(f_1, f_2, \cdots, f_N) \in D(\mathcal{G}_1) \times D(\mathcal{G}_2) \times \cdots \times D(\mathcal{G}_N) : \exists\, t > 0$ with $(G_1(\delta_t)f_1, G_2(\delta_t)f_2, \cdots, G_N(\delta_t)f_N) = (f_1, f_2, \cdots, f_N)\}$, is dense in $E^N$.

*Remark* 2.2. The notions introduced in Definition 2.1 can be considered in a more general framework. Speaking-matter-of-factly, let $N \in \mathbb{N}$, $N \geq 2$ and $(W_i(t))_{t \geq 0}$ be a family of linear operators in $E$, $i = 1, 2, \cdots, N$. Then we define the notion of a $d$-hypercyclic vector for $(W_i(\cdot))_{1 \leq i \leq N}$ similarly as above: $x \in \bigcap_{1 \leq i \leq N,\ t \geq 0} D(W_i(t))$ is said to be a $d$-hypercyclic vector for $(W_i(\cdot))_{1 \leq i \leq N}$ iff (2.1) holds with $G_i(\delta_t)$ replaced by $W_i(t)$ therein $(1 \leq i \leq N,\ t \geq 0)$. Further on, we say that $(f_1, f_2, \cdots, f_N) \in \bigcap_{t \geq 0} D(W_1(t)) \times \bigcap_{t \geq 0} D(W_2(t)) \times \cdots \times \bigcap_{t \geq 0} D(W_N(t))$ is a periodic element of $(W_i(\cdot))_{1 \leq i \leq N}$ iff there exists $t > 0$ such that $W_j(t)f_j = f_j$, $j \in \mathbb{N}_N^0$. After that, we can define the notions of (densely) $d$-hypercyclicity, $d$-topological transitivity, $d$-topologically mixing property and $d$-chaoticity of $(W_i(\cdot))_{1 \leq i \leq N}$ in the same way as above, with $G_i(\delta_t)$ replaced by $W_i(t)$ therein $(1 \leq i \leq N,\ t \geq 0)$.

Besides, we would like to point out the possibility to define the notion of disjoint chaos by considering the periodic points of the form $(f, f, \cdots, f)$ instead of $(f_1, f_2, \cdots, f_N)$, and to start the new paper about these peculiar phenomena of disjoint chaoticity for semigroups and fractional resolvent families.

The change of order in tuple $(\mathcal{G}_i)_{1 \leq i \leq N}$ does not have any influence on $d$-hypercyclicity of semigroups $(\mathcal{G}_i)_{1 \leq i \leq N}$. It can be almost trivially shown that the $d$-hypercyclicity of $(\mathcal{G}_i)_{1 \leq i \leq N}$ implies hypercyclicity of each component $\mathcal{G}_i$ for $1 \leq i \leq N$, and the relation $G_i \neq G_j$ for all $i,\ j \in \mathbb{N}_N$ with $i \neq j$.

Using Lemma 1.5, we can simply reformulate the above notions for fractionally integrated $C_i$-semigroups (cf. also [26, Theorem 3.1.32(i)]):

**Definition 2.3.** Let $N \in \mathbb{N}$, $N \geq 2$, and let $A_i$ be the integral generator of a global $\alpha_i$-times integrated $C_i$-semigroup $(S_{\alpha_i}(t))_{t \geq 0}$ on $E$, $i = 1, 2, \cdots, N$. We say that $(S_{\alpha_i}(\cdot))_{1 \leq i \leq N}$ are $d$-hypercyclic ($d$-topologically transitive, $d$-topologically mixing) iff $(\mathcal{G}_{S_{\alpha_i}})_{1 \leq i \leq N}$ are (cf. (1.6) with $\alpha = \alpha_i$, $C = C_i$ and $S_\alpha(\cdot) = S_{\alpha_i}(\cdot)$). Set $\mathcal{P}(S_{\alpha_1}, S_{\alpha_2}, \cdots, S_{\alpha_N}) := \mathcal{P}(\mathcal{G}_{S_{\alpha_1}}, \mathcal{G}_{S_{\alpha_2}}, \cdots, \mathcal{G}_{S_{\alpha_N}})$.

Before proceeding further, we would like to observe that we allow some regularizing operators $C_i$ to be mutually different as well as that (1.5) yields that $(S_{\alpha_i}(\cdot))_{1 \leq i \leq N}$ are $d$-hypercyclic ($d$-topologically transitive, $d$-topologically mixing) iff $(S_{\beta_i}(\cdot))_{1 \leq i \leq N}$ are $d$-hypercyclic ($d$-topologically transitive, $d$-topologically mixing), where $\beta_i \geq \alpha_i$ for all $i \in \mathbb{N}_N$ and $S_{\beta_i}(t)x = (g_{\beta_i - \alpha_i} *_0 S_{\alpha_i}(\cdot)x)(t)$, $t \geq 0$, $x \in E$.

Our first result reads as follows.

**Theorem 2.4.** *Suppose that $N \in \mathbb{N}$, $N \geq 2$, $T_1, \cdots, T_N$ are linear operators on $E$ $(1 \leq j \leq N)$ and $C \in L(E)$ is injective. Suppose that there exists a subset $E_0$ of*

*$D_\infty(T_1)\cap\cdots\cap D_\infty(T_N)$, dense in $E$, as well as dense subsets $E_1,\cdots,E_N$ of $E$ and mappings $S_{j,n}:E_j\to D_\infty(T_1)\cap\cdots\cap D_\infty(T_N)$ ($1\le j\le N$, $n\in\mathbb{N}$) such that:*

(i) $\lim_{n\to\infty}T_j^n x_0=0$, $x_0\in E_0$, $1\le j\le N$,
(ii) $\lim_{n\to\infty}S_{j,n}x_j=0$, $x_j\in E_j$, $1\le j\le N$,
(iii) $\lim_{n\to\infty}[T_i^n S_{j,n}x_j-\delta_{j,i}x_j]=0$, $x_j\in E_j$, $1\le i,\ j\le N$,
(iv) $R(C)\subseteq D_\infty(T_1)\cap\cdots\cap D_\infty(T_N)$ *and* $T_j^nC\in L(E)$, $1\le j\le N$, $n\in\mathbb{N}$,
(v) $CT_jx=T_jCx$, $x\in D_\infty(T_j)$, $1\le j\le N$,
(vi) $R(C)$ *is dense in* $E$.

*Then the operators $T_1,\cdots,T_N$ are densely d-hypercyclic.*

*Proof.* It is clear that $[R(C)]_\circledast$ is a separable Fréchet space. Define the operators $\mathfrak{T}_{j,n}\in L([R(C)]_\circledast,E)$ by $\mathfrak{T}_{j,n}(Cx):=T_j^nCx$, $x\in E$ (cf. (iv)). By (vi), it suffices to show that the sequences $(\mathfrak{T}_{1,j})_{j\in\mathbb{N}},\cdots,(\mathfrak{T}_{N,j})_{j\in\mathbb{N}}$ are densely d-hypercyclic. Since the final conclusions of [7, Remark 2.8] also hold for sequences of continuous linear operators acting between different Fréchet spaces, we need to prove the existence of a dense subset $E_0'$ of $[R(C)]_\circledast$, dense subsets $E_1',\cdots,E_N'$ of $E$ and mappings $S_{j,n}':E_j'\to[R(C)]$ ($1\le j\le N$, $n\in\mathbb{N}$) such that the following holds:

(a) $\lim_{n\to\infty}\mathfrak{T}_{j,n}x_0'=0$, $x_0'\in E_0'$,
(b) $\lim_{n\to\infty}S_{j,n}'x_j'=0$, $x_j'\in E_j'$, $1\le j\le N$, and
(c) $\lim_{n\to\infty}[\mathfrak{T}_{i,n}S_{j,n}'x_j'-\delta_{j,i}x_j']=0$, $x_j'\in E_j'$, $1\le i,\ j\le N$.

Set $E_j':=C(E_j)$, $0\le j\le N$ and $S_{j,n}':E_j'\to R(C)$ by $S_{j,n}'(Cx_j):=CS_{j,n}x$, $x_j\in E_j$ ($1\le j\le N$, $n\in\mathbb{N}$). By (vi) and the density of $E_j$ in $E$, we get that $E_0'$ is dense in $[R(C)]_\circledast$ and that $E_j'$ is dense in $E$ ($1\le j\le N$). The property (b) follows immediately from (ii) and definition of $S_{j,n}'$. The property (a) follows by making use the fact that $E_0$ belongs to $D_\infty(T_1)\cap\cdots\cap D_\infty(T_N)$, as well as from (v), (i) and definition of $\mathfrak{T}_{j,n}$. By (iii), (v) and inclusion $R(S_{j,n})\subseteq D_\infty(T_1)\cap\cdots\cap D_\infty(T_N)$, we have

$$\begin{aligned}\lim_{n\to\infty}&\big[\mathfrak{T}_i^nS_{j,n}'x_j'-\delta_{j,i}x_j'\big]\\&=\lim_{n\to\infty}\big[T_i^nCS_{j,n}x_j-C\delta_{j,i}x_j\big]\\&=\lim_{n\to\infty}C\big[T_i^nS_{j,n}x_j-\delta_{j,i}x_j\big]\\&=C\lim_{n\to\infty}\big[T_i^nS_{j,n}x_j-\delta_{j,i}x_j\big]=0,\end{aligned}$$

provided that $x_j'=Cx_j\in E_j'$, $1\le i,\ j\le N$. The proof of the theorem is thereby complete. □

Keeping in mind Theorem 2.4, it is very simple to prove the following $d$-Hypercyclicity Criterion for $C_i$-distribution semigroups in Fréchet spaces.

**Theorem 2.5.** *Suppose that $N\in\mathbb{N}$, $N\ge2$, $\mathcal{G}_i$ is a $C_i$-distribution semigroup in $E$ ($i=1,2,\cdots,N$), and $C\in L(E)$ is injective. Suppose that there exists a subset $E_0$ of $D(\mathcal{G}_1)\cap\cdots\cap D(\mathcal{G}_N)$, dense in $E$, as well as dense subsets $E_1,\cdots,E_N$ of $E$ and mappings $S_{j,n}:E_j\to D(\mathcal{G}_1)\cap\cdots\cap D(\mathcal{G}_N)$ ($1\le j\le N$, $n\in\mathbb{N}$) such that:*

(i) $\lim_{n\to\infty}G_j(\delta_n)x_0=0$, $x_0\in E_0$, $1\le j\le N$,
(ii) $\lim_{n\to\infty}S_{j,n}x_j=0$, $x_j\in E_j$, $1\le j\le N$,
(iii) $\lim_{n\to\infty}[G_i(\delta_n)S_{j,n}x_j-\delta_{j,i}x_j]=0$, $x_j\in E_j$, $1\le i,\ j\le N$,
(iv) $R(C)\subseteq D(\mathcal{G}_1)\cap\cdots\cap D(\mathcal{G}_N)$ *and* $G_j(\delta_n)C\in L(E)$, $1\le j\le N$, $n\in\mathbb{N}$,
(v) $CG_j(\delta_1)x=G_j(\delta_1)Cx$, $x\in D(\mathcal{G}_j)$, $1\le j\le N$,

(vi) *$R(C)$ is dense in $E$.*

*Then $(\mathcal{G}_i)_{1\leq i\leq N}$ are densely d-hypercyclic.*

*Proof.* Put $T_j := G_j(\delta_1)$, $1 \leq j \leq N$. Then the prescribed assumptions in combination with the property A1. and (1.1) imply by Theorem 2.4 that the operators $T_1, \cdots, T_N$ are densely $d$-hypercyclic. This immediately implies that $(\mathcal{G}_i)_{1\leq i\leq N}$ are densely $d$-hypercyclic, as well. □

Now we will prove the following continuous analogue of Lemma 1.6:

**Proposition 2.6.** *Suppose that $N \in \mathbb{N}$, $N \geq 2$, and $\mathcal{G}_i$ is a $C_i$-distribution semigroup in $E$ ($i = 1, 2, \cdots, N$). Suppose that $(a_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of positive integers, as well as that the following holds:*

- *The set $E_0$, consisting of those elements $y \in D(\mathcal{G}_1)\cap\cdots\cap D(\mathcal{G}_N)$ satisfying that for each $j \in \mathbb{N}_N$ we have $\lim_{n\to\infty} G_j(\delta_{a_n})y = 0$, is dense in $E$.*
- *For each $j \in \mathbb{N}_N$ there exists a dense subset $E_{\infty,j}$ of $E$, consisting of those elements $z \in E$ for which there exist elements $\omega_{n,i}(z) \in D(\mathcal{G}_j)$ ($n \in \mathbb{N}$, $1 \leq i \leq N$) such that $(\omega_{n,j}(z))_{n\in\mathbb{N}}$ is a null sequence in $E$, and $\lim_{n\to\infty} G_j(\delta_{a_n})\omega_{n,i}(z) = \delta_{i,j}z$ ($1 \leq i \leq N$).*

*Then*

$$\underbrace{\mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_1}_{k}, \cdots, \underbrace{\mathcal{G}_N \oplus \cdots \oplus \mathcal{G}_N}_{k}$$

*are d-topologically transitive ($k \in \mathbb{N}$).*

*Proof.* Keeping in mind A1., (1.1) and Lemma 1.6, it readily follows that the operators

$$\underbrace{G_1(\delta_1) \oplus \cdots \oplus G_1(\delta_1)}_{k}, \cdots, \underbrace{G_N(\delta_1) \oplus \cdots \oplus G_N(\delta_1)}_{k}$$

are d-topologically transitive ($k \in \mathbb{N}$). This proves the claimed assertion. □

The subsequent theorem is a continuous version of [6, Theorem 4.3], which has been proved by L. Bernal-González, and a disjoint version of the Desch-Schappacher-Webb criterion for $C$-distribution semigroups [26, Theorem 3.1.36(i)] (the case in which there exists an integer $p \in \mathbb{N}_N^0$ such that the set $D_p$ appearing below is not total in $E$ will be considered in Example 2.13).

**Theorem 2.7.** *Let $N \in \mathbb{N}$, $N \geq 2$, and let $A_j$ be the integral generator of a $C_j$-distribution semigroup $\mathcal{G}_j$ ($1 \leq j \leq N$). Suppose that for each $p \in \mathbb{N}_N^0$ there exists a total subset $D_p$ of $E$ such that the following holds:*

(i) *Any element of the set $D_p$ is an eigenvector of any operator $A_j$ ($p \in \mathbb{N}_N^0$, $j \in \mathbb{N}_N$); if $e \in D_p$, then there exists an eigenvalue $\lambda_{p,j}(e)$ of the operator $A_j$ for which $\lambda_{p,j}(e)e = A_je$ and* (ii)-(iii) *hold, where:*

(ii) $\lambda_{0,j}(e) \in \mathbb{C}_-$, $j \in \mathbb{N}_N$, $e \in D_0$ *and* $\lambda_{j,j}(e) \in \mathbb{C}_+$, $j \in \mathbb{N}_N$, $e \in D_j$;

(iii) *Suppose $i$, $j \in \mathbb{N}_N$ and $i \neq j$. Then, for every $e \in D_i$, we have $\Re(\lambda_{i,j}(e)) < \Re(\lambda_{i,i}(e))$.*

*Then $(\mathcal{G}_i)_{1\leq i\leq N}$ are d-topologically mixing.*

First of all, we will state and prove the following auxiliary lemma.

**Lemma 2.8.** *Suppose $A$ generates a $C$-distribution semigroup $\mathcal{G}$, $\lambda \in \mathbb{C}$, $x \in E$ and $Ax = \lambda x$. Then $G(\delta_t)x = e^{\lambda t}x$, $t \geq 0$.*

*Proof.* By definition of $A$, we have that

$$\mathcal{G}\big(-\varphi'\big)x = \lambda\mathcal{G}(\varphi)x, \quad \varphi \in \mathcal{D}_0. \tag{2.2}$$

Fix a test function $\varphi \in \mathcal{D}_0$ and consider the function $F : [0,\infty) \to E$ defined by

$$F(t) := e^{-\lambda t}\mathcal{G}\big(\varphi(\cdot - t)\big)x, \quad t \geq 0.$$

Since $\mathcal{G}$ is continuous and (2.2) holds, it readily follows that $F'(t) = 0$, $t \geq 0$. This implies $F(t) = F(0)$ for all $t \geq 0$, i.e., $\mathcal{G}(e^{-\lambda t}\varphi(\cdot - t))x = \mathcal{G}(\varphi)x$. Hence, $\mathcal{G}(\delta_t * \varphi)x = e^{\lambda t}\mathcal{G}(\varphi)x$, $t \geq 0$ and, by definition of $G(\delta_t)$, $G(\delta_t)x = e^{\lambda t}x$, $t \geq 0$. □

*Proof.* Owing to Lemma 2.8, we have that the assumption $\lambda_{j,p}(e)e = A_je$ for some $p \in \mathbb{N}_N^0$, $j \in \mathbb{N}_N$ and $e \in D_p$ implies

$$G_j\big(\delta_t\big)e = e^{t\lambda_{j,p}(e)}e, \quad t \geq 0. \tag{2.3}$$

Having in mind this fact, the proof can be deduced by slightly modifying the arguments given in that of [6, Theorem 4.3]; we will include all relevant details for the sake of clarity. Let open non-empty subsets $V_0, V_1, V_2, \cdot\cdot\cdot, V_N$ of $E$ be given. We will have to prove that there exists $t_0 \geq 0$ such that, for every $t \geq t_0$, there exists a vector $x_t \in V_0 \cap D(\mathcal{G}_1) \cap \cdot\cdot\cdot \cap D(\mathcal{G}_N)$ such that $G_j(\delta_t)x_t \in V_j$ for all $j \in \mathbb{N}_N$. Since the linear span of $D_i$ is dense in $E$, $i = 0,1,2,\cdot\cdot\cdot,N$, it is enough to prove that for given $(N+1)$-vectors $u_i \in \operatorname{span}(D_i)$, $i = 0,1,2,\cdot\cdot\cdot,N$ there is a net $(x_t)_{t\geq 0}$ in $E$ such that $x_t \to u_0$ and $G_i(\delta_t)x_t \to u_i$, when $t \to \infty$ for all $i = 1,2,\cdot\cdot\cdot,N$. Let $u_i$, $i = 0,1,2,\cdots,N$ be fixed. Then there are finite sets $E_i = \{e_{i,1}, e_{i,2}, \cdot\cdot\cdot, e_{i,m(i)}\} \subseteq D_i$ and scalars $c_{i,1}, c_{i,2}, \cdot\cdot\cdot, c_{i,m(i)}$ such that $u_i = \sum_{l=1}^{m(i)} c_{i,l}e_{i,l}$. Making use of (2.3) and (ii), we obtain that, for every $j = 1,2,\cdot\cdot\cdot,N$,

$$G_j\big(\delta_t\big)e_{0,l} \to 0, \quad t \to \infty,\ l = 1,2,\cdot\cdot\cdot,m(0) \tag{2.4}$$

and

$$G_j\big(\delta_t\big)e_{j,l} \to \infty, \quad t \to \infty,\ l = 1,2,\cdot\cdot\cdot,m(j). \tag{2.5}$$

From (2.3) and (iii) we obtain that, for all $i,\ j \in \mathbb{N}_N$ with $i \neq j$,

$$\big\|G_j\big(\delta_t\big)e_{i,l}\big\| / \big\|G_i\big(\delta_t\big)e_{i,l}\big\| \to 0, \quad t \to \infty \ \ (l = 1,2,\cdot\cdot\cdot,m(i)). \tag{2.6}$$

Define

$$x_t := u_0 + \sum_{j=1}^{N}\sum_{l=1}^{m(j)} \frac{c_{j,l}}{e^{t\lambda_{j,j}(e_{j,l})}}e_{j,l}, \quad t \geq 0.$$

Using (2.3)-(2.6) and the arguments already seen in the proof of [6, Theorem 4.3], we get that $x_t \to u_0$ and $G_i(\delta_t)x_t \to u_i$, when $t \to \infty$ for all $i = 1,2,\cdot\cdot\cdot,N$, as claimed. The proof of the theorem is thereby complete. □

*Remark* 2.9. Let $N \in \mathbb{N}$, $N \geq 2$, and let $A_j$ be the integral generator of a $C_j$-distribution semigroup $\mathcal{G}_j$ $(1 \leq j \leq N)$. Concerning the existence of $d$-periodic points of semigroups $(\mathcal{G}_i)_{1\leq i\leq N}$, we have the following simple result: Suppose that $\Omega$ is an open connected subset of $\mathbb{C}$ satisfying $\Omega \cap i\mathbb{R} \neq \emptyset$. Let $f : \Omega \to E \setminus \{0\}$ be an analytic mapping such that $A_jf(\lambda) = \lambda f(\lambda)$ for all $\lambda \in \Omega$ and $j \in \mathbb{N}_N$. Set $\tilde{E} := \overline{span\{f(\lambda) : \lambda \in \Omega\}}$. Then $\tilde{E} = \overline{span\{f(\lambda) : \lambda \in \Omega \cap \exp(2\pi i\mathbb{Q})\}}$, which implies without any substantial difficulties that the set $\mathcal{P}(\mathcal{G}_1, \mathcal{G}_2, \cdot\cdot\cdot, \mathcal{G}_N)$ is dense in $\tilde{E}^N$.

*Remark* 2.10. Connections between the imaginary point spectrum and hypercyclicity of strongly continuous semigroups in Banach spaces have been analyzed by S. El Mourchid in [21] (see also [5] for related results). It is clear that the condition (iii) from the formulation of Theorem 2.7 seriously hinders our strivings to prove a disjoint analogue of [21, Theorem 2.1] (cf. also [26, Theorem 3.1.42(i)]).

*Remark* 2.11. The class of $C$-ultradistribution semigroups of $*$-class in Fréchet spaces has been recently introduced and analyzed in [31]; here, the asterisk $*$ stands for the Beurling case or for the Roumieu case. We define the notion of integral generator of a $C$-ultradistribution semigroup $\mathcal{G}$ of $*$-class, the notion of a closed linear operator $G(T)$ and the notions from Definition 2.1 similarly as above ($T$ is now a scalar-valued ultradistribution of $*$-class with compact support contained in $[0,\infty)$). Then A1.-A2., Proposition 1.2, Theorem 2.5, Proposition 2.6, Lemma 2.8 and Theorem 2.7 continue to hold for $C$-ultradistribution semigroups in Fréchet spaces.

Now we would like to illustrate Theorem 2.5 and Theorem 2.7 with two instructive examples which will be put into general form; a great number of concrete applications can be provided by using differential operators appearing in [2]-[3], [14]-[15], [19]-[20], [22], [24] and [35]-[36]. In the first example, we use $C$-regularized semigroups and, in the second one, we use integrated semigroups.

**Example 2.12.** (cf. also [26, Theorem 3.1.38(i)]) Suppose that $\theta \in (0,\pi/2)$, $-A$ generates an exponentially equicontinouus, analytic strongly continuous semigroup of angle $\theta$, $N \in \mathbb{N}$, $N \geq 2$, $P_j(z) = \sum_{i=0}^{n_j} a_{i,j} z^i$ is a non-zero complex polynomial with $a_{n_j,j} > 0$ and $n_j(\frac{\pi}{2}-\theta) < \frac{\pi}{2}$ ($j \in \mathbb{N}_N$). Set $A_j := P_j(A)$ ($j \in \mathbb{N}_N$) and assume further that there exist an open connected subset $\Omega$ of $\mathbb{C}$ and an analytic mapping $f : \Omega \to E \setminus \{0\}$ such that $\sigma_p(-A) \supseteq \Omega$, $f(\lambda) \in N(-A-\lambda) \setminus \{0\}$, $\lambda \in \Omega$ and that the supposition $(x^* \circ f)(\lambda) = 0$, $\lambda \in \Omega$, for some $x^* \in E^*$, implies $x^* = 0$.

Let $\alpha \in (1, \frac{\pi}{n_j\pi - 2n_j\theta})$ for all $j \in \mathbb{N}_N$. Then [27, Theorem 2.2.10] (see also [17, Theorem 8.2] for the concrete representation of operators $C_j$ below) implies that there exists $\omega \in \mathbb{R}$ such that, for every $j \in \mathbb{N}_N$, $A_j$ generates an entire $C_j$-regularized group $(T_j(z))_{z\in\mathbb{C}}$ with $C_j \equiv e^{-(P_j(A)-\omega)^\alpha}$; furthermore, $R(C_j)$ is dense in $E$ for all $j \in \mathbb{N}_N$. It can be easily checked that the set $\{f(\lambda) : \lambda \in \Omega'\}$ is total in $E$ for any non-empty subset $\Omega'$ of $\Omega$ which has a cluster point in $\Omega$, as well as that

$$P_j(-\Omega) \subseteq P_j\bigl(\sigma_p(A)\bigr) \subseteq \sigma_p\bigl(A_j\bigr) \text{ and } A_j f(\lambda) = P_j(-\lambda) f(\lambda),\ \lambda \in \Omega,\ j \in \mathbb{N}_N. \tag{2.7}$$

Suppose that, for every $p \in \mathbb{N}_N^0$, there exists a non-empty subset $\Omega_p$ of $\Omega$ which has a cluster point in $\Omega$, as well as the following holds:

$$P_j(-\lambda) \in \mathbb{C}_-,\ 1 \leq j \leq N,\ \lambda \in \Omega_0\ ;\ P_j(-\lambda) \in \mathbb{C}_+,\ 1 \leq j \leq N,\ \lambda \in \Omega_j \tag{2.8}$$

and

$$\Bigl(\forall i,\ j \in \mathbb{N}_N\Bigr)\ \Bigl(i \neq j \Rightarrow \Re\bigl(P_j(-\lambda)\bigr) < \Re\bigl(P_i(-\lambda)\bigr),\ \lambda \in \Omega_i\Bigr). \tag{2.9}$$

Keeping in mind (2.7)-(2.9), we can apply Theorem 2.7, with $D_p := \{f(\lambda) : \lambda \in \Omega_p\}$ ($p \in \mathbb{N}_N^0$), in order to see that $(T_i(\cdot))_{1\leq i\leq N}$ are $d$-topologically mixing; furthermore, we can apply Theorem 2.5, with the operator $C = C_1C_2\cdots C_N$, the set $E_j$ being the linear span of $D_j$ for $j \in \mathbb{N}_N^0$, and the mappings $S_{j,n} : E_j \to E$ defined by $S_{j,n}\sum_{l=1}^k \beta_l f(\lambda_l) := \sum_{l=1}^k \beta_l e^{-nP_j(-\lambda_l)} f(\lambda_l)$ for $j \in \mathbb{N}_N^0$, $k,\ n \in \mathbb{N}$ and $\lambda \in \Omega_j$, in order to see that $(T_i(\cdot))_{1\leq i\leq N}$ are densely $d$-hypercyclic, as well. The existence of real numbers $r \in \mathbb{R}$ and $\epsilon > 0$ such that $L(ir,\epsilon) \equiv \{z \in \mathbb{C} : |z - ir| < \epsilon\} \subseteq P_j(-\Omega)$, $j \in \mathbb{N}_N^0$ implies by the considerations given in Remark 2.9 that $(T_i(\cdot))_{1\leq i\leq N}$ are $d$-chaotic.

**Example 2.13.** Suppose that $\zeta \geq 0$, $-A \notin L(E)$, $-A$ generates an exponentially equicontinous $\zeta$-times integrated cosine function $(C_\zeta(t))_{t\geq 0}$, $N \in \mathbb{N}$, $N \geq 2$ and $P_j(z) = \sum_{i=0}^{n_j} a_{i,j} z^i$ is a non-zero complex polynomial with $a_{n_j,j} > 0$ $(j \in \mathbb{N}_N)$. Assume that there exist an open connected subset $\Omega$ of $\mathbb{C}$ and an analytic mapping $f : \Omega \to E \setminus \{0\}$ such that $\sigma_p(-A) \supseteq \Omega$ and $f(\lambda) \in N(-A-\lambda) \setminus \{0\}$, $\lambda \in \Omega$ (for example, let $a > 0$, let $\rho(x) := e^{-a|x|}$, $x \in \mathbb{R}$, $E := L^p_\rho(\mathbb{R})$, $D(B) := \{f \in X \mid f(\cdot) \text{ is loc. abs. continuous, } f' \in E\}$ and $Af := f'$, $f \in D(B)$, see [20]; then $A$ is the generator of a $C_0$-group on $E$ and the above requirements hold with $A = -B^2$, $\Omega = \{z^2 : |\Re z| < a\}$ and $f(z^2) = e^{z\cdot}$ for $|\Re z| < a$).

Let $\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}$, and let $\Omega'$ be a non-empty open connected subset of $\mathbb{C}$ such that $\lambda^2 \in \Omega$ for all $\lambda \in \Omega'$. Define $F : \Omega' \to (E\times E)\setminus\{(0,0)\}$ by $F(\lambda) := [f(\lambda^2)\ \lambda f(\lambda^2)]^T$, $\lambda \in \Omega'$. Then $F(\cdot)$ is analytic, $\sigma_p(\mathcal{A}) \supseteq \Omega'$ and $F(\lambda) \in N(\mathcal{A}-\lambda) \setminus \{(0,0)\}$, $\lambda\in\Omega'$; cf. [29, Lemma 32].

It is well known that the operator $\mathcal{A}$ generates an exponentially equicontinuous $(\zeta+1)$-times integrated semigroup $(S_{\zeta+1}(t))_{t\geq 0}$ in $E \times E$, which is given by

$$S_{\zeta+1}(t) := \begin{pmatrix} \int_0^t C_\zeta(s)\,ds & \int_0^t (t-s)C_\zeta(s)\,ds \\ C_\zeta(t) - g_{\zeta+1}(t)C & \int_0^t C_\zeta(s)\,ds \end{pmatrix}, \quad t \geq 0.$$

Since $\mathcal{A}^2 = \begin{pmatrix} -A & 0 \\ 0 & -A \end{pmatrix}$ generates an exponentially equicontinous $\zeta$-times integrated cosine function $(C_\zeta(t) \oplus C_\zeta(t))_{t\geq 0}$ (see also [1, Example 3.16.10]), the abstract Weierstrass formula [27, Theorem 2.2.18(ii)] yields that the operator $\mathcal{A}^2$ generates an exponentially equicontinuous, analytic $(\zeta/2)$-times integrated semigroup of angle $\pi/2$. Set $Q_1(z) := z$ and $Q_j(z) := -P_j(-z^2)$ $(z \in \mathbb{C},\ 2 \leq j \leq N)$, as well as $A_j := Q_j(\mathcal{A})$. Arguing as in the proof of [18, Theorem 9], we get that for each number $\eta > \zeta/2$, the operator $A_j$ generates an exponentially equicontinuous, analytic $\eta$-times integrated semigroup $(S^j_\eta(t))_{t\geq 0}$ of angle $\pi/2$, for $2 \leq j \leq N$ (observe that our choice of operator $A_1 = \mathcal{A}$ is motivated by the fact that, in the Banach space case, the operator $\mathcal{A}$ cannot generate a strongly continuous semigroup in $E \times E$ by [1, Corollary 3.14.9]). Set

$$\hat{E} := \overline{span(\{F(\lambda) : \lambda \in \Omega'\})}.$$

Suppose that, for every $p \in \mathbb{N}_N^0$, there exists a non-empty subset $\Omega'_p$ of $\Omega'$ which has a cluster point in $\Omega'$ (this obviously implies by the analyticity of $F(\cdot)$ that $\hat{E} = \overline{span(\{F(\lambda) : \lambda \in \Omega'_p\})}$), as well as that (2.8)-(2.9) holds with the polynomials $P_j(-\cdot)$, $P_i(-\cdot)$ and sets $\Omega_0$, $\Omega_j$, $\Omega_i$ replaced therein by the polynomials $Q_j(\cdot)$, $Q_i(\cdot)$ and sets $\Omega'_0$, $\Omega'_j$, $\Omega'_i$, respectively. Then the proof of Theorem 2.7 shows that the integrated semigroups $(S_{\zeta+1}(\cdot), (S^j_\eta(\cdot))_{2\leq j\leq N})$ are $\hat{E}$-disjoint topologically mixing in the following sense: For any open non-empty subsets $V_0, V_1, V_2, \cdots, V_N$ of $E$, there exists $t_0 \geq 0$ such that for every $t \geq t_0$ we have that $(V_0 \cap \hat{E}) \cap G_1(\delta_t)^{-1}(V_1 \cap \hat{E}) \cap G_2(\delta_t)^{-1}(V_2 \cap \hat{E}) \cap \cdots \cap G_N(\delta_t)^{-1}(V_N \cap \hat{E}) \neq \emptyset$; here, $G_j$ denotes the induced distribution semigroup generated by $A_j$, for $1 \leq j \leq N$ (notice that a similar statement can be established in a general situation of Theorem 2.7, provided that the set $D_p$ is not total in $E$ for all $p \in \mathbb{N}_N^0$, and $\hat{E} = \overline{span(D_p)} = \overline{span(D_{p'})}$ for all

$p,\ p' \in \mathbb{N}_N^0$). The existence of real numbers $r \in \mathbb{R}$ and $\epsilon > 0$ such that $L(ir, \epsilon) \subseteq Q_j(\Omega')$, $j \in \mathbb{N}_N^0$ implies that the set of all periodic points of $(S_{\zeta+1}(\cdot), (S_\eta^j(\cdot))_{2\le j\le N})$ is dense in $\hat{E}^N$. Finally, it should be noted that the case in which $\hat{E} = E$ is very difficult to be satisfied if we consider disjoint topologically mixing properties of ill-posed abstract Cauchy problems of first order whose solutions are governed by integrated semigroups.

In [26, Theorem 3.1.32(ii)], we have reconsidered the assertion of [19, Theorem 4.6] for $C$-distribution semigroups in Banach spaces and characterized (subspace) hypercyclicity, topological transitivity and chaoticity of a $C$-distribution semigroup $\mathcal{G}$ in terms of bounded operators $\mathcal{G}(\varphi)$, for $\varphi \in \mathcal{D}$; the extension of [26, Theorem 3.1.32(ii)] to $C$-(ultra)distribution semigroups in Fréchet spaces is obvious. It is also possible to characterize disjoint topological dynamical properties introduced above in a similar way; we leave this question to the interested reader to pursue.

## 3. Disjoint Transitivity and Disjoint Chaoticity of Strongly Continuous Semigroups of Composition Operators

Let $X$ be a locally compact, Hausdorff, and $\sigma$-compact topological space. Let $\mu$ be a positive, locally finite, Borel measure on $X$. In particular, $\mu$ is $\sigma$-finite by the fact $X$ is $\sigma$-compact. For $1 \le p < \infty$, let $L^p(\mu)$ be the Lebesgue space with respect to $\mu$, with the norm $\|f\| = (\int_X |f|^p d\mu)^{1/p}$ ([25]).

Let $I$ be a non-empty set, and let $\varphi : I \times X \to X$ be a mapping such that $\varphi(t, \cdot)$ is injective and continuous for all $t \in I$. For given $N \ge 2$, we define $N$-families of composition operators on $L^p(\mu)$ by $T_{\varphi_k}(t)f := f \circ \varphi_k(t, \cdot) := f(\varphi_k(t, \cdot))$ for $k = 1, 2, \cdots, N$. We assume $\varphi_k$ satisfies the condition in [25, Theorem 2.1] so that $T_{\varphi_k}(t)$ is well defined and continuous for all $t \in I$.

Let $\nu_{k,t} := \mu^{\varphi_k(t,\cdot)}$ be the image measure of $\mu$ under $\varphi_k(t, \cdot)$. Also, for a Borel set $B \subseteq X$, let

$$\nu_{k,-t}(B) := \mu(\varphi_k(t, B)) = \left(\mu\mid_{\varphi_k(t,X)}\right)^{\varphi_k(-t,\cdot)}(B),$$

where $\varphi_k(-t, \cdot)$ is the inverse mapping from $\varphi_k(t, X)$ to $X$ and $\mu\mid_{\varphi_k(t,X)} = \mu(\cdot \cap \varphi_k(t, X))$ (cf. [25, Remark 2.3]). In order to study d-transitivity of $T_{\varphi_1}, T_{\varphi_2}, \cdots, T_{\varphi_N}$, we put

$$\lambda_{m,k,t}(B) := \int \chi_B\big(\varphi_m(-t, \varphi_k(t, \cdot))\big)\, d\mu$$

for a Borel set $B$ of $X$ with $m \ne k$. We recall that $T_{\varphi_1}, T_{\varphi_2}, \cdots, T_{\varphi_N}$ are disjoint transitive iff for any non-empty subsets $U, V_1, V_2, \cdots, V_N$ of $X$, there exists $t \in I$ such that

$$U \cap T_{\varphi_1}(t)^{-1}(V_1) \cap T_{\varphi_2}(t)^{-1}(V_2) \cap \cdots \cap T_{\varphi_N}(t)^{-1}(V_N) \ne \emptyset.$$

Before proceeding to analyze the results, we would like to mention that a sufficient condition for weighted composition operators on general $L^p$ spaces to be disjoint topologically mixing was obtained in [10, Theorem 1.3, Proposition 2.5] where the mapping $\varphi$, however, is defined in a different way.

**Theorem 3.1.** *Under the general assumptions, we have that* (ii) *implies* (i)*, where:*

(i) $T_{\varphi_1}, T_{\varphi_2}, \cdots, T_{\varphi_N}$ *are disjoint transitive.*

(ii) *For each compact subset $K$ of $X$, there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ in $K$ and a sequence $(t_n)_{n\in\mathbb{N}}$ in $I$ such that*

$$\lim_{n\to\infty}\mu\big(K\setminus L_n\big)=\lim_{n\to\infty}\lambda_{m,k,t_n}\big(L_n\big)=0,$$

*and*

$$\lim_{n\to\infty}\nu_{k,t_n}\big(L_n\big)=\lim_{n\to\infty}\nu_{k,-t_n}\big(L_n\big)=0$$

*for $k=1,2,\cdot\cdot\cdot,N$ with $m\neq k$.*

*Proof.* For $1\leq k\leq N$, let $U$ and $V_k$ be non-empty open subsets of $L^p(\mu)$. Since the space $C_c(X)$ of continuous functions on $X$ with compact support is dense in $L^p(\mu)$, we can pick $f,g_k\in C_c(X)$ with $f\in U$ and $g_k\in V_k$ for $k=1,2,\cdot\cdot\cdot,N$. Let $K$ be the union of the compact supports of $f$ and all $g_k$. Let $(L_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ satisfy condition (ii) for $K$.

Let

$$\begin{aligned}v_n &= f\chi_{L_n}+g_1\big(\varphi_1(-t_n,\cdot)\big)\chi_{\varphi_1(t_n,L_n)}+g_2\big(\varphi_2(-t_n,\cdot)\big)\chi_{\varphi_2(t_n,L_n)}+\cdot\cdot\cdot\\ &+ g_N\big(\varphi_N(-t_n,\cdot)\big)\chi_{\varphi_N(t_n,L_n)}.\end{aligned}$$

Since $\varphi_k(-t_n,\cdot)(\varphi_k(t_n,L_n))=L_n$ and $\mu|_{\varphi_k(t_n,X)}=(\nu_{k,-t_n})^{\varphi_k(t_n,\cdot)}$, we have

$$\begin{aligned}\big\|g_k(\varphi_k\big(-t_n,\cdot)\big)\chi_{\varphi_k(t_n,L_n)}\big\|^p &= \int_{\varphi_k(t_n,L_n)}\big|g_k\big(\varphi_k(-t_n,\cdot)\big)\big|^p\,d\mu\\ &\leq \|g_k\|_\infty^p\int_{\varphi_k(t_n,L_n)}d\mu\\ &= \|g_k\|_\infty^p\int_{\varphi_k(t_n,L_n)}d\big(\mu|_{\varphi_k(t_n,X)}\big)\\ &= \|g_k\|_\infty^p\nu_{k,-t_n}\big(L_n\big)\end{aligned}$$

for each $k=1,2,\cdot\cdot\cdot,N$. By the inequality $\|f+g\|^p\leq 2^p\|f\|^p+2^p\|g\|^p$, we arrive at

$$\begin{aligned}2^{-Np}\big\|v_n-f\big\|_p^p &\leq \|f\|_\infty^p\mu(K\setminus L_n)+\|g_1\|_\infty^p\nu_{1,-t_n}(L_n)\\ &+ \|g_2\|_\infty^p\nu_{2,-t_n}(L_n)+\cdot\cdot\cdot+\|g_N\|_\infty^p\nu_{N,-t_n}(L_n)\end{aligned}$$

which implies $v_n\to f$ as $n\to\infty$. Moreover, one has the following estimates:

$$\begin{aligned}\big\|T_{\varphi_k}(t_n)(f\chi_{L_n})\big\|^p &= \int\big|f\big(\varphi_k(t_n,\cdot)\big)\big|^p\chi_{L_n}\big(\varphi_k(t_n,\cdot)\big)\,d\mu\\ &\leq \|f\|_\infty^p\int_{\varphi_k(-t_n,\cdot)(L_n)}d\mu\\ &= \|f\|_\infty^p\int_{L_n}d\mu^{\varphi_k(t_n,\cdot)}\\ &= \|f\|_\infty^p\nu_{k,t_n}(L_n),\end{aligned}$$

and

$$\Big\|T_{\varphi_k}(t_n)\big(g_m(\varphi_m(-t_n,\cdot))\chi_{\varphi_m(t_n,L_n)}\big)\Big\|^p$$
$$= \int \big|g_m\big(\varphi_k(t_n,\varphi_m(-t_n,\cdot))\big)\big|^p \chi_{\varphi_m(t_n,L_n)}(\varphi_k(t_n,\cdot))\,d\mu$$
$$\leq \|g_m\|_\infty^p \int \chi_{\varphi_m(t_n,L_n)}(\varphi_k(t_n,\cdot))\,d\mu$$
$$= \|g_m\|_\infty^p \int \chi_{L_n}\big(\varphi_m(-t_n,\varphi_k(t_n,\cdot))\big)\,d\mu$$
$$= \|g_m\|_\infty^p \lambda_{m,k,t_n}\big(L_n\big)$$

for $m \neq k$. Hence, together with this equality

$$T_{\varphi_k}(t_n)\big(g_k(\varphi_k(-t_n,\cdot))\chi_{\varphi_k(t_n,L_n)}\big) = g_k\chi_{L_n},$$

we obtain $T_{\varphi_k}(t_n)v_n \to g_k$ as $n \to \infty$, which follows from

$$2^{-Np}\big\|T_{\varphi_k}(t_n)v_n - g_k\big\|_p^p$$
$$\leq \|f\|_\infty^p \nu_{k,t_n}(L_n) + \|g_1\|_\infty^p \lambda_{1,k,t_n}(L_n) + \cdots + \|g_{k-1}\|_\infty^p \lambda_{k-1,k,t_n}(L_n)$$
$$+ \|g_k\|_\infty^p \mu(K \setminus L_n) + \|g_{k+1}\|_\infty^p \lambda_{k+1,k,t_n}(L_n) + \cdots + \|g_N\|_\infty^p \lambda_{N,k,t_n}(L_n).$$

Therefore $T_{\varphi_1}, T_{\varphi_2}, \cdots, T_{\varphi_N}$ are disjoint transitive. □

**Theorem 3.2.** *Under the general assumptions, we have that* (i) *implies* (ii)*, where:*

(i) *$T_{\varphi_1}, T_{\varphi_2}, \cdots, T_{\varphi_N}$ are disjoint transitive.*

(ii) *For each compact subset $K$ of $X$, there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ in $K$ and a sequence $(t_n)_{n\in\mathbb{N}}$ in $I$ such that*

$$\lim_{n\to\infty} \mu\big(K \setminus L_n\big) = 0,$$

*and*

$$\lim_{n\to\infty} \nu_{k,t_n}\big(L_n\big) = \lim_{n\to\infty} \nu_{k,-t_n}\big(L_n\big) = 0$$

*for $k = 1, 2, \cdots, N$.*

*Proof.* The proof is similar to that of implication (i) ⇒ (ii) in [25, Theorem 2.4] and therefore omitted. □

Let $\Omega$ be an open non-empty subset of $\mathbb{R}^d$. A continuous function $\varphi : [0,\infty)\times\Omega \to \Omega$ is called a semiflow iff $\varphi(0,\cdot) = id_\Omega$ and $\varphi(t,\cdot)\circ\varphi(s,\cdot) = \varphi(t+s,\cdot)$ for all $t, s \geq 0$, and iff $\varphi(t,\cdot)$ is injective for all $t \geq 0$. In particular, $\varphi$ is the solution semiflow of some initial value problem (see [25]). Moreover, if $\mu$ is an $(L^p)$-admissible Borel measure on $\Omega$, then $(T_\varphi(t))_{t\geq 0}$ is a well defined $C_0$-semigroup (see [25, Definition 3.3, Theorem 3.16]).

**Corollary 3.3.** *Given some $N \geq 2$, let $\varphi_k$ be a semiflow for $1 \leq k \leq N$, and let $\mu$ be an $(L^p)$-admissible Borel measure on $\Omega$. Then we have that* (ii) *implies* (i)*, where:*

(i) *The $C_0$-semigroups $(T_{\varphi_i}(\cdot))_{1\leq i\leq N}$ are disjoint transitive.*

(ii) *For each compact subset $K$ of $\Omega$, there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ in $K$ and a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \mu\big(K \setminus L_n\big) = \lim_{n\to\infty} \lambda_{m,k,t_n}\big(L_n\big) = 0,$$

*and*

$$\lim_{n\to\infty} \nu_{k,t_n}(L_n) = \lim_{n\to\infty} \nu_{k,-t_n}(L_n) = 0$$

*for* $k = 1, 2, \cdots, N$ *with* $m \neq k$.

**Corollary 3.4.** *Given some* $N \geq 2$*, let* $\varphi_k$ *be a semiflow for* $1 \leq k \leq N$*, and let* $\mu$ *be an* $(L^p)$*-admissible Borel measure on* $\Omega$*. Then we have that* (i) *implies* (ii)*, where:*

(i) *The* $C_0$*-semigroups* $(T_{\varphi_i}(\cdot))_{1\leq i\leq N}$ *are disjoint transitive.*
(ii) *For each compact subset* $K$ *of* $\Omega$*, there are a sequence of measurable subsets* $(L_n)_{n\in\mathbb{N}}$ *in* $K$ *and a sequence of positive numbers* $(t_n)_{n\in\mathbb{N}}$ *such that*

$$\lim_{n\to\infty} \mu(K \setminus L_n) = 0,$$

*and*

$$\lim_{n\to\infty} \nu_{k,t_n}(L_n) = \lim_{n\to\infty} \nu_{k,-t_n}(L_n) = 0$$

*for* $k = 1, 2, \cdots, N$.

In the following theorem, we turn our attention to give a sufficient condition for $C_0$-semigroups $(T_{\varphi_1}(t))_{t\geq 0}$, $(T_{\varphi_2}(t))_{t\geq 0}, \cdots, (T_{\varphi_N}(t))_{t\geq 0}$ to be disjoint chaotic.

**Theorem 3.5.** *Given some* $N \geq 2$*, let* $\varphi_k$ *be a semiflow such that for every compact subset* $K$ *of* $\Omega$*, there is a number* $t_K > 0$ *satisfying* $\varphi_k(t, K) \cap K = \emptyset$ *for all* $t > t_K$ *and* $k = 1, 2, \cdots, N$*. Let* $\mu$ *be an* $(L^p)$*-admissible Borel measure on* $\Omega$*. Then we have that* (ii) *implies* (i)*, where:*

(i) *The* $C_0$*-semigroups* $(T_{\varphi_i}(\cdot))_{1\leq i\leq N}$ *are disjoint chaotic.*
(ii) *For each compact subset* $K$ *of* $\Omega$*, there are a sequence of measurable subsets* $(L_n)_{n\in\mathbb{N}}$ *in* $K$ *and a strictly increasing sequence of positive numbers* $(t_n)_{n\in\mathbb{N}}$ *tending to infinity such that, for* $k = 1, 2, \cdots, N$ *with* $m \neq k$*, we have*

$$\lim_{n\to\infty} \mu(K \setminus L_n) = \lim_{n\to\infty} \lambda_{m,k,t_n}(L_n) = 0 \quad \text{and} \qquad \lim_{n\to\infty} s_{k,n} = 0,$$

*where*

$$s_{k,n} := \sum_{l=1}^{\infty} \nu_{k,lt_n}(L_n) + \sum_{l=1}^{\infty} \nu_{k,-lt_n}(L_n).$$

*Proof.* By Corollary 3.3, the $C_0$-semigroups $(T_{\varphi_1}(t))_{t\geq 0}$, $(T_{\varphi_2}(t))_{t\geq 0}, \cdots, (T_{\varphi_N}(t))_{t\geq 0}$ are disjoint transitive. We will show that $\mathcal{P}(T_{\varphi_1}, T_{\varphi_2}, \cdots, T_{\varphi_N})$ is dense in $(L^p(\mu))^N$.

Choose $f_1, f_2, \cdots, f_N \in C_c(X)$ and a compact set $K$ of $\Omega$ containing the union support of $f_1, f_2, \cdots, f_N$. Let $(L_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be as in (ii) for $K$, where we may assume w.l.o.g. that $t_1 > t_K$. For $k = 1, 2, \cdots, N$, set

$$v_{k,n} := f_k \chi_{L_n} + \sum_{l=1}^{\infty} f_k\big(\varphi_k(lt_n, \cdot)\big)\chi_{\varphi_k(-lt_n, L_n)} + \sum_{l=1}^{\infty} f_k\big(\varphi_k(-lt_n, \cdot)\big)\chi_{\varphi_k(lt_n, L_n)}.$$

Using the equality $\varphi_k(t_n, K) \cap K = \emptyset$, we get that:

$$\big\|v_{k,n} - f_k\big\|_p^p \leq \|f_k\|_\infty^p \mu(K \setminus L_n) + \sum_{l=1}^{\infty} \|f_k\|_\infty^p \nu_{k,lt_n}(L_n) + \sum_{l=1}^{\infty} \|f_k\|_\infty^p \nu_{k,-lt_n}(L_n).$$

Hence $v_{k,n} \to f_k$ as $n \to \infty$. Moreover, $(v_{1,n}, v_{2,n}, \cdots, v_{N,n}) \in \mathcal{P}(T_{\varphi_1}, T_{\varphi_2}, \cdots, T_{\varphi_N})$ by the facts that $\varphi_k(t,\cdot) \circ \varphi_k(s,\cdot) = \varphi_k(t+s,\cdot)$ and

$$\begin{aligned}
&T_{\varphi_k}(t_n)v_{k,n} \\
&\quad = f_k\big(\varphi_k(t_n,\cdot)\big)\chi_{\varphi_k(-t_n,L_n)} + \sum_{l=1}^{\infty} f_k\big(\varphi_k((l+1)t_n,\cdot)\big)\chi_{\varphi_k(-(l+1)t_n,L_n)} \\
&\quad + \sum_{l=1}^{\infty} f_k\big(\varphi_k(-(l-1)t_n,\cdot)\big)\chi_{\varphi_k((l-1)t_n,L_n)} \\
&\quad = \sum_{l=1}^{\infty} f_k\big(\varphi_k(lt_n,\cdot)\big)\chi_{\varphi_k(-lt_n,L_n)} + f_k\chi_{L_n} + \sum_{l=1}^{\infty} f_k\big(\varphi_k(-lt_n,\cdot)\big)\chi_{\varphi_k(lt_n,L_n)} \\
&\quad = v_{k,n}.
\end{aligned}$$

□

Arguing similarly as in the proof of implication (ii) ⇒ (iii) in [25, Theorem 5.3], we can deduce the following result.

**Theorem 3.6.** *Given some $N \geq 2$, let $\varphi_k$ be a semiflow such that for every compact subset $K$ of $\Omega$, there is a number $t_K > 0$ satisfying $\varphi_k(t,K) \cap K = \emptyset$ for all $t > t_K$ and $k = 1, 2, \cdots, N$. Let $\mu$ be an $(L^p)$-admissible Borel measure on $\Omega$. Then we have that* (i) *implies* (ii)*, where:*

(i) *The $C_0$-semigroups $(T_{\varphi_i}(\cdot))_{1\leq i\leq N}$ are disjoint chaotic.*

(ii) *For each compact subset $K$ of $\Omega$, there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ in $K$ and a strictly increasing sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ tending to infinity such that for $k = 1, 2, \cdots, N$,*

$$\lim_{n\to\infty} \mu\big(K \setminus L_n\big) = 0 \qquad \textit{and} \qquad \lim_{n\to\infty} s_{k,n} = 0$$

*where*

$$s_{k,n} := \sum_{l=1}^{\infty} \nu_{k,lt_n}\big(L_n\big) + \sum_{l=1}^{\infty} \nu_{k,-lt_n}\big(L_n\big).$$

Next, we consider the space of continuous functions $C_{0,\rho}(X)$, where $C_{0,\rho}(X) := \{f : X \to \mathbb{C} \text{ continuous }; \forall\varepsilon > 0 \text{ the set } \{x \in X : |f(x)|\rho(x) \geq \varepsilon\} \text{ is compact}\}$. Equipped with the norm $\|f\|_{\infty,\rho} := \sup_{x\in X} |f(x)|\rho(x)$, $C_{0,\rho}(X)$ becomes a Banach space; here, $\rho : X \to (0,\infty)$ is upper semicontinuous ([25]).

Now we give a sufficient condition for composition operators on $C_{0,\rho}(X)$ to be disjoint transitive.

**Theorem 3.7.** *Additionally to the general hypotheses, given some $N \geq 2$, for $k = 1, 2, \cdots N$, we assume that $\varphi_k(t,\cdot) : X \to X$ is an open mapping for all $t \in I$, and $\inf_{x\in K} \rho(x) > 0$ for all compact subsets $K$ of $X$. Then* (ii) *implies* (i)*, where:*

(i) *$T_{\varphi_1}, T_{\varphi_2}, \cdots, T_{\varphi_N}$ are disjoint transitive.*

(ii) *For each compact subset $K$ of $X$, there is a sequence $(t_n)_{n\in\mathbb{N}}$ in $I$ such that for $k = 1, 2, \cdots, N$ with $m \neq k$, we have*

$$\lim_{n\to\infty} \sup_{x\in\varphi_m(t_n,\varphi_k(-t_n,K))} \rho(x) = 0,$$

*and*

$$\lim_{n\to\infty} \sup_{x\in\varphi_k(t_n,K)} \rho(x) = \lim_{n\to\infty} \sup_{x\in\varphi_k(t_n,\cdot)^{-1}(K)} \rho(x) = 0.$$

*Proof.* For $1 \leq k \leq N$, let $U$ and $V_k$ be non-empty open subsets of $C_{0,\rho}(X)$. Choose $f, g_k \in C_c(X)$ such that $f \in U$ and $g_k \in V_k$ for $k = 1, 2, \cdots, N$. Let $K$ be the union of the compact supports of $f$ and all $g_k$. Then the mapping $g_k \circ \varphi_k(-t, \cdot) : \varphi(t, X) \to \mathbb{C}$ is continuous and its support is contained in the compact set $\varphi(t, K)$ so that $g_k \circ \varphi_k(-t, \cdot) \in C_c(\varphi(t, X))$. Further on, we extend $g_k \circ \varphi_k(-t, \cdot)$ to a compactly supported continuous function $\widetilde{g}_{k,t}$ by putting it to be equal to 0 outside $\varphi(t, X)$. Clearly, $T_{\varphi_k}(t)\widetilde{g}_{k,t} = g_k$.

Assume $(t_n)_{n\in\mathbb{N}}$ is as in the condition (ii) for compact set $K$, and let

$$v_n = f + \widetilde{g}_{1,t_n} + \widetilde{g}_{2,t_n} + \cdots + \widetilde{g}_{N,t_n}.$$

Then

$$\begin{aligned} \big\|v_n - f\big\|_{\infty,\rho} \quad \leq \quad & \sup_{x\in\varphi_1(t_n,K)} \big|g_1\big(\varphi_1(-t_n, x)\big)\big|\rho(x) + \sup_{x\in\varphi_2(t_n,K)} \big|g_2\big(\varphi_1(-t_n, x)\big)\big|\rho(x) \\ & + \cdots + \sup_{x\in\varphi_N(t_n,K)} \big|g_N\big(\varphi_N(-t_n, x)\big)\big|\rho(x) \\ \leq \quad & \|g_1\|_\infty \sup_{x\in\varphi_1(t_n,K)} \rho(x) + \|g_2\|_\infty \sup_{x\in\varphi_2(t_n,K)} \rho(x) \\ & + \cdots + \|g_N\|_\infty \sup_{x\in\varphi_N(t_n,K)} \rho(x) \end{aligned}$$

which says that $v_n \to f$ as $n \to \infty$. Moreover,

$$\big\|T_{\varphi_k}(t_n)f\big\|_{\infty,\rho} = \sup_{x\in\varphi_k(t_n,\cdot)^{-1}(K)} \big|f\big(\varphi_k(t_n, x)\big)\big|\rho(x) \leq \|f\|_\infty \sup_{x\in\varphi_k(t_n,\cdot)^{-1}(K)} \rho(x),$$

and

$$\begin{aligned} & \big\|T_{\varphi_k}(t_n)\widetilde{g}_{m,t_n}\big\|_{\infty,\rho} \\ & = \big\|T_{\varphi_k}(t_n)(g_m(\varphi_m(-t_n, \cdot))\big\|_{\infty,\rho} \\ & = \big\|g_m\big(\varphi_k(t_n, \varphi_m(-t_n, \cdot))\big)\big\|_{\infty,\rho} \\ & \leq \sup_{x\in\varphi_m(t_n,\varphi_k(-t_n,K))} \big|g_m\big(\varphi_k(t_n, \varphi_m(-t_n, \cdot))\big)\big|\rho(x) \\ & \leq \|g_m\|_\infty \sup_{x\in\varphi_m(t_n,\varphi_k(-t_n,K))} \rho(x). \end{aligned}$$

for $k = 1, 2, \cdots, N$ with $m \neq k$. Together with the equality

$$T_{\varphi_k}(t_n)\big(g_k(\varphi_k(-t_n, \cdot)\big) = g_k,$$

we obtain

$$\begin{aligned} & \big\|T_{\varphi_k}(t_n)v_n - g_k\big\|_{\infty,\rho} \leq \|f\|_\infty \sup_{x\in\varphi_k(t_n,\cdot)^{-1}(K)} \rho(x) \\ + \quad & \|g_1\|_\infty \sup_{x\in\varphi_1(t_n,\varphi_k(-t_n,K))} \rho(x) + \cdots + \|g_{k-1}\|_\infty \sup_{x\in\varphi_{k-1}(t_n,\varphi_k(-t_n,K))} \rho(x) \\ + \quad & \|g_{k+1}\|_\infty \sup_{x\in\varphi_{k+1}(t_n,\varphi_k(-t_n,K))} \rho(x) + \cdots + \|g_N\|_\infty \sup_{x\in\varphi_N(t_n,\varphi_N(-t_n,K))} \rho(x), \end{aligned}$$

proving that $T_{\varphi_k}(t_n)v_n \to g_k$ as $n \to \infty$. Therefore $T_{\varphi_1}, T_{\varphi_2}, \cdots, T_{\varphi_N}$ are disjoint transitive. □

Applying a similar argument as in the proof of (i) ⇒ (ii) of [25, Theorem 2.9], we obtain the following result immediately.

**Theorem 3.8.** *Additionally to the general hypotheses, given some $N \geq 2$, for $k = 1, 2, \cdots, N$, we assume that $\inf_{x\in K} \rho(x) > 0$ for all compact subsets $K$ of $X$. Then* (i) *implies* (ii)*, where:*

- (i) $T_{\varphi_1}, T_{\varphi_2}, \cdots, T_{\varphi_N}$ *are disjoint transitive.*
- (ii) *For each compact subset $K$ of $X$, there is a sequence $(t_n)_{n\in\mathbb{N}}$ in $I$ such that for $k = 1, 2, \cdots, N$, we have*
$$\lim_{n\to\infty} \sup_{x\in\varphi_k(t_n,K)} \rho(x) = \lim_{n\to\infty} \sup_{x\in\varphi_k(t_n,\cdot)^{-1}(K)} \rho(x) = 0.$$

As in the case of $L^p(\mu)$-space, if $\rho$ is a $C_0$-admissible weight function on an open set $\Omega \subseteq \mathbb{R}^d$ for the semiflow $\varphi_k$, then $(T_{\varphi_k}(t))_{t\geq 0}$ is a well-defined $C_0$-semigroup (see [25, Theorem 3.4, Definition 3.5]). In this case, one has the results below by Theorem 3.7 and Theorem 3.8.

**Corollary 3.9.** *Given some $N \geq 2$, let $\varphi_k$ be a semiflow for $1 \leq k \leq N$, and let $\rho$ be a $C_0$-admissible weight function for the semiflow $\varphi_k$. Then* (ii) *implies* (i)*, where:*

- (i) *The $C_0$-semigroups $(T_{\varphi_i}(\cdot))_{1\leq i\leq N}$ are disjoint transitive.*
- (ii) *For each compact subset $K$ of $X$, there is a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that for $k = 1, 2, \cdots, N$ with $m \neq k$, we have*
$$\lim_{n\to\infty} \sup_{x\in\varphi_m(t_n,\varphi_k(-t_n,K))} \rho(x) = 0,$$
*and*
$$\lim_{n\to\infty} \sup_{x\in\varphi_k(t_n,K)} \rho(x) = \lim_{n\to\infty} \sup_{x\in\varphi_k(t_n,\cdot)^{-1}(K)} \rho(x) = 0.$$

**Corollary 3.10.** *Given some $N \geq 2$, let $\varphi_k$ be a semiflow for $1 \leq k \leq N$, and let $\rho$ be a $C_0$-admissible weight function for the semiflow $\varphi_k$. Then* (i) *implies* (ii)*, where:*

- (i) *The $C_0$-semigroups $(T_{\varphi_i}(\cdot))_{1\leq i\leq N}$ are disjoint transitive.*
- (ii) *For each compact subset $K$ of $X$, there is a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that for $k = 1, 2, \cdots, N$ with $m \neq k$, we have*
$$\lim_{n\to\infty} \sup_{x\in\varphi_k(t_n,K)} \rho(x) = \lim_{n\to\infty} \sup_{x\in\varphi_k(t_n,\cdot)^{-1}(K)} \rho(x) = 0.$$

Based on the above results, we give a sufficient condition for the $C_0$-semigroups $(T_{\varphi_1}(t))_{t\geq 0}$, $(T_{\varphi_2}(t))_{t\geq 0}, \cdots$, $(T_{\varphi_N}(t))_{t\geq 0}$ to be disjoint chaotic on $C_{0,\rho}(X)$.

**Theorem 3.11.** *Given some $N \geq 2$, let $\varphi_k$ be a semiflow such that for every compact subset $K$ of $\Omega$, there is a number $t_K > 0$ satisfying $\varphi_k(t, K) \cap K = \emptyset$ for all $t > t_K$ and $k = 1, 2, \cdots, N$. Let $\rho$ be a $C_0$-admissible weight function such that $\inf_{x\in K} \rho(x) > 0$ for all compact subsets $K$ of $\Omega$. Then* (ii) *implies* (i)*, where:*

- (i) *The $C_0$-semigroups $(T_{\varphi_i}(\cdot))_{1\leq i\leq N}$ are disjoint chaotic.*
- (ii) *For each compact subset $K$ of $X$, there is an integer $p \in \mathbb{N}$ such that for $k = 1, 2, \cdots, N$ with $m \neq k$, we have*
$$\lim_{n\to\infty} \sup_{x\in\varphi_m(np,\varphi_k(-np,K))} \rho(x) = 0,$$
*and*
$$\lim_{n\to\infty} \sup_{x\in\varphi_k(np,K)} \rho(x) = \lim_{n\to\infty} \sup_{x\in\varphi_k(np,\cdot)^{-1}(K)} \rho(x) = 0.$$

*Proof.* By Corollary 3.9, the $C_0$-semigroups $(T_{\varphi_1}(t))_{t\geq 0}, (T_{\varphi_2}(t))_{t\geq 0}, \cdots, (T_{\varphi_N}(t))_{t\geq 0}$ are disjoint transitive. We will show that $\mathcal{P}(T_{\varphi_1}, T_{\varphi_2}, \cdots, T_{\varphi_N})$ is dense in $(C_{0,\rho}(X))^N$.

Choose $f_1, f_2, \cdots, f_N \in C_c(X)$ and a compact set $K$ of $\Omega$ containing the union support of $f_1, f_2, \cdots, f_N$. Let $p$ be as in (ii) for $K$, we may assume that $p > t_K$.

For $k = 1, 2, \cdots, N$, let

$$v_{k,n} = f_k + \sum_{l=1}^{\infty} f_k\big(\varphi_k(lnp, \cdot)\big) + \sum_{l=1}^{\infty} f_k\big(\varphi_k(-lnp, \cdot)\big).$$

Then by $\varphi_k(np, K) \cap K = \emptyset$, we have

$$\begin{aligned} &\big\|v_{k,n} - f_k\big\|_{\infty,\rho} = \sup_{x\in\Omega}\big|v_{k,n}(x) - f_k(x)\big|\rho(x) \\ &= \max\left\{\sup_{l\in\mathbb{N}} \sup_{x\in\varphi_k(lnp,\cdot)^{-1}(K)}\big|f_k\big(\varphi_k(lnp, x)\big)\big|\rho(x),\ \sup_{l\in\mathbb{N}} \sup_{x\in\varphi_k(lnp,K)}\big|f_k\big(\varphi_k(-lnp, x)\big)\big|\rho(x)\right\} \\ &\leq \|f_k\|_\infty \left\{\sup_{l\in\mathbb{N}} \sup_{x\in\varphi_k(lnp,\cdot)^{-1}(K)} \rho(x) + \sup_{l\in\mathbb{N}} \sup_{x\in\varphi_k(lnp,K)} \rho(x)\right\}, \end{aligned}$$

which implies $v_{k,n} \to f_k$ as $n \to \infty$. Moreover, by a simple computation, one has $T_{\varphi_k}(np)v_{k,n} = v_{k,n}$. Therefore $\mathcal{P}(T_{\varphi_1}, T_{\varphi_2}, \cdots, T_{\varphi_N})$ is dense in $(C_{0,\rho}(X))^N$. □

Again, we have the result below by using Corollary 3.10, and a similar argument as in the proof of implication (ii) ⇒ (iii) in [25, Theorem 5.7].

**Theorem 3.12.** *Given some $N \geq 2$, let $\varphi_k$ be a semiflow such that for every compact subset $K$ of $\Omega$, there is a number $t_K > 0$ satisfying $\varphi_k(t, K) \cap K = \emptyset$ for all $t > t_K$ and $k = 1, 2, \cdots, N$. Let $\rho$ be a $C_0$-admissible weight function such that $\inf_{x\in K} \rho(x) > 0$ for all compact subsets $K$ of $\Omega$. Then we have that* (i) *implies* (ii), *where:*

(i) *The $C_0$-semigroups $(T_{\varphi_i}(\cdot))_{1\leq i\leq N}$ are disjoint chaotic.*
(ii) *For each compact subset $K$ of $X$, there is an integer $p \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} \sup_{x\in\varphi_k(np,K)} \rho(x) = \lim_{n\to\infty} \sup_{x\in\varphi_k(np,\cdot)^{-1}(K)} \rho(x) = 0.$$

Using [27, Theorem 3.1.40] and the proof of Theorem 3.11, we can simply clarify some sufficient conditions for $d$-chaoticity of strongly continuous semigroups induced by semiflows on the Fréchet space $C(\Omega)$.

**Theorem 3.13.** *Suppose that $N \in \mathbb{N}$, $N \geq 2$, $\varphi_i : [0, \infty) \times \Omega \to \Omega$ is a semiflow for all $i = 1, 2, \cdots, N$ and, for every compact set $K \subseteq \Omega$, there exists an integer $p \in \mathbb{N}$ satisfying the following condition: For every compact set $K' \subseteq \Omega$ there exists $k_0(K') \in \mathbb{N}$ such that:*

- *$\varphi_i(kp, \cdot)^{-1}(\varphi_j(kp, K)) \ \cap \ K' = \emptyset,\ i,\ j \in \mathbb{N}_N,\ i \neq j,\ k \geq k_0(K')$ and*
- *$\varphi_i(kp, K) \ \cap \ K' = \varphi_i(kp, \cdot)^{-1}(K) \ \cap \ K' = \emptyset,\ i \in \mathbb{N}_N,\ k \geq k_0(K')$.*

*Then $(T_{\varphi_i}(t))_{t\geq 0}$ is a strongly continuous semigroup in $C(\Omega)$ for every $i \in \mathbb{N}_N$ and $(T_{\varphi_i})_{1\leq i\leq N}$ are d-chaotic in $C(\Omega)$.*

We would like to recommend for the readers problem of finding some necessary (sufficient) conditions for disjoint topologically mixing of strongly continuous semigroups induced by semiflows. As pointed out in [32], it is a very non-trivial problem to clarify a condition which would be both necessary and sufficient for strongly continuous semigroups induced by semiflows to be disjoint topologically mixing (disjoint hypercyclic, disjoint topologically transitive). See also [10] for the study on disjoint topologically mixing and stronger notions of other general semigroups.

We close the paper by verifying that strongly continuous semigroups appearing in [27, Example 3.1.41, Example 3.1.42] are also $d$-chaotic.

**Example 3.14.** (i) Suppose $p \in [1,\infty)$, $\Omega = [1,\infty)$, $N \in \mathbb{N}$, $N \geq 2$ and $0 < \alpha_1 < \cdots < \alpha_N \leq 1$. Define $\varphi_i : [0,\infty) \times \Omega \to \Omega$, $i = 1, 2, \cdots, N$ and $\rho = \rho_1 : \Omega \to (0,\infty)$ by:

$$\varphi_i(t,x) := \big(t + x^{\alpha_i}\big)^{1/\alpha_i} \text{ and } \rho_1(x) := e^{-x^{\alpha_1}},\ t \geq 0,\ x \in \Omega.$$

We know that the strongly continuous semigroups $(T_{\varphi_i}(\cdot))_{1\leq i\leq N}$ are d-topologically transitive in $L^p_{\rho_1}(\Omega)$ and $C_{0,\rho}(\Omega)$. Let a compact set $K = [a,b] \subseteq \Omega$ be given. Let $L_k = K,\ k \in \mathbb{N}$ and let $(t_k)_{k\in\mathbb{N}}$ be any increasing sequence of positive real numbers satisfying $\lim_{k\to\infty} t_k = \infty$ and $t_1 \geq \max(b^{\alpha_1}, \cdots, b^{\alpha_n})$. In order to prove that $(T_{\varphi_i}(\cdot))_{1\leq i\leq N}$ are $d$-chaotic in $L^p_{\rho_1}(\Omega)$, it suffices to show by Theorem 3.5 (here $\mu = \rho_1(x)\,dx$ with $dx$ being the Lebesgue measure) that, for every $k \in \mathbb{N}_N$, we have:

$$\lim_{n\to\infty} \sum_{l=1}^{\infty} \left[ \int_{\varphi_k(lt_n,K)} \rho_1(x)\,dx + \int_{\varphi_k(-lt_n,K)} \rho_1(x)\,dx \right] = 0, \tag{3.1}$$

i.e., that

$$\lim_{n\to\infty} \sum_{l=1}^{\infty} \int_{\varphi_k(lt_n,K)} \rho_1(x)\,dx = 0 \tag{3.2}$$

since $\varphi_k(-lt_n, K) = \emptyset$ for all $l \in \mathbb{N}$ and $k \in \mathbb{N}_N$. To show (3.2), it suffices to observe that, for every $k \in \mathbb{N}_N$, there exists a finite constant $c_k > 0$ such that:

$$\begin{aligned}
&\sum_{l=1}^{\infty} \int_{\varphi_k(lt_n,K)} \rho_1(x)\,dx \\
&\quad \leq \sum_{l=1}^{\infty} \big(lt_n + b^{\alpha_k}\big)^{1/\alpha_k} e^{-(lt_n + b^{\alpha_k})^{1/\alpha_k}} \\
&\quad \leq c_k \sum_{l=1}^{\infty} \Big[\big(lt_n\big)^{1/\alpha_k} + b\Big] e^{-(lt_n)^{1/\alpha_k}} \\
&\quad \leq c_k t_n^{(m-1)/\alpha_k} \sum_{l=1}^{\infty} \left[ \frac{m!}{l^{(m-1)/\alpha_k}} + \frac{bm!}{l^{m/\alpha_k}} \right],
\end{aligned}$$

where $m \in \mathbb{N}$ is chosen so that $m - 1 > \alpha_k$. One can similarly prove by Theorem 3.11 that $(T_{\varphi_i}(\cdot))_{1\leq i\leq N}$ are $d$-chaotic in $C_{0,\rho}(\Omega)$.

(ii) Let $p \in [1,\infty)$, $N \in \mathbb{N}$, $N \geq 2$, $a_i > 0$ for $i \in \mathbb{N}_N$ and $a_i \neq a_j$ for $i \neq j$, $q > 1/2$ and $\Omega = (0,\infty)$. Define semiflows $\varphi_i : [0,\infty) \times \Omega \to \Omega$, $i = 1, 2, \cdots, N$ and the weight function $\rho_1 : \Omega \to (0,\infty)$ by

$$\varphi_i(t,x) := e^{a_i t} \text{ and } \rho_1(x) := \frac{1}{(1+x^2)^q}, \quad t \geq 0,\ x > 0.$$

Then we know that $(T_{\varphi_i}(\cdot))_{1\leq i\leq N}$ are $d$-topologically transitive strongly continuous semigroups in $L^p_{\rho_1}(\Omega)$ ($L_k = K,\ k \in \mathbb{N}$); in order to prove that $(T_{\varphi_i}(\cdot))_{1\leq i\leq N}$ are also $d$-chaotic, it suffices to show that, for every $a>0,\ b>0$ with $a<b$ and for every strictly increasing sequence of positive reals $(t_n)_{n\in\mathbb{N}}$ tending to infinity, one has (cf. (3.1)):

$$\lim_{n\to\infty}\sum_{l=1}^{\infty}\Biggl[\int_{e^{lt_na_k}a}^{e^{lt_na_k}b}\frac{dx}{(1+x^2)^q}+\int_{e^{-lt_na_k}a}^{e^{-lt_na_k}b}\frac{dx}{(1+x^2)^q}\Biggr]=0,\quad 1\leq k\leq N.$$

This simply follows from the following elementary computation:

$$\sum_{l=1}^{\infty}\Biggl[\int_{e^{lt_na_k}a}^{e^{lt_na_k}b}\frac{dx}{(1+x^2)^q}+\int_{e^{-lt_na_k}a}^{e^{-lt_na_k}b}\frac{dx}{(1+x^2)^q}\Biggr]$$
$$\leq(b-a)\sum_{l=1}^{\infty}\Biggl[\frac{e^{a_klt_n}}{(1+a^2e^{2a_klt_n})^q}+\frac{e^{-a_klt_n}}{(1+a^2e^{-2a_klt_n})^q}\Biggr]$$
$$\leq(b-a)\sum_{l=1}^{\infty}\Biggl[a^{-2q}e^{(1-2q)a_klt_n}+e^{-a_klt_n}\Biggr]\to 0,\quad n\to\infty,\ 1\leq k\leq N.$$

The same conclusion holds in the case that $\Omega=\mathbb{R}$ but then we must use an appropriate sequence $(L_k)$ of measurable subsets of $K$ satisfying $0\notin L_k^{\circ},\ k\in\mathbb{N}$. Let us recall once more that, for every $i\in\mathbb{N}_N$, $(T_{\varphi_i}(t))_{t\geq 0}$ is a non-hypercyclic strongly continuous semigroup in $C_{0,\rho_1}(\Omega)$ $(C_{0,\rho_1}(\mathbb{R}))$.

(iii) Suppose that $\Omega=\{(x,y)\in\mathbb{R}^2 : x^2+y^2>1\}$, $N\in\mathbb{N}$, $N\geq 2$, $0<p_1<\cdots<p_N<\infty$, $q_i\in\mathbb{R}$ for $i\in\mathbb{N}_N$, and:

$$\varphi_i(t,x,y)=e^{p_it}\bigl(x\cos q_it-y\sin q_it, x\sin q_it+y\cos q_it\bigr),$$

for any $t\geq 0$, $(x,y)\in\Omega$ and $i\in\mathbb{N}_N$. Applying Theorem 3.12 and the arguments already seen in [27, Example 3.1.42], we get that the strongly continuous semigroups $(T_{\varphi_i}(\cdot))_{1\leq i\leq N}$ are $d$-chaotic in $C(\Omega)$.

## References


[1] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems,* Birkhäuser/Springer Basel AG, Basel, 2001.

[2] F. Astengo and B. di Blasio, Dynamics of the heat semigroup in Jacobi analysis, *J. Math. Anal. Appl.,* **391(1)** (2012), 48–56.

[3] J. Banasiak and M. Lachowicz, Topological chaos for birth-and-death-type-models with proliferation, *Math. Models Methods Appl. Sci.,* **12(06)** (2002), 755–775.

[4] F. Bayart and E. Matheron, *Dynamics of Linear Operators,* Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, UK, **179(1)**, 2009.

[5] F. Bayart and E. Matheron, Mixing operators and small subsets of the circle, *J. Reine Angew. Math.,* **715** (2016), 75–123.

[6] L. Bernal-González, Disjoint hypercyclic operators, *Studia Math.,* **182(2)** (2007), 113–131.

[7] J. Bés and A. Peris, Disjointness in hypercyclicity, *J. Math. Anal. Appl.,* **336(1)** (2007), 297–315.

[8] J. Bès, Ö. Martin and R. Sanders, Weighted shifts and disjoint hypercyclicity, *J. Operator Theory,* **72(1)** (2014), 15–40.

[9] J. Bés, Ö. Martin and A. Peris, Disjoint hypercyclic linear fractional composition operators, *J. Math. Anal. Appl.,* **381(2)** (2011), 843–856.

[10] J. Bès, Ö. Martin, A. Peris and S. Shkarin, Disjoint mixing operators, *J. Funct. Anal.,* **263(5)** (2013), 1283–1322.

[11] C.-C. Chen, Disjoint topological transitivity for cosine operator functions on groups, *Filomat,* **31(8)** (2017), 2413–2423.

[12] C.-C. Chen, J. A. Conejero, M. Kostić and M. Murillo-Arcila, Dynamics of multivalued linear operators, *Open Math.,* **15(1)** (2017), 948–958.

[13] C.-C. Chen, J. A. Conejero, M. Kostić and M. Murillo-Arcila, Hypercyclicity and disjoint hypercyclicity of binary relations on topological spaces, preprint.
[14] J. A. Conejero and E. Mangino, Hypercyclic semigroups generated by Ornstein-Uhlenbeck operators, *Mediterr. J. Math.,* **7(1)** (2010), 101–109.
[15] J. A. Conejero and F. Martinez-Gimenez, Chaotic differential operators, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM,* **105(2)** (2011), 423–431.
[16] J. A. Conejero, V. Müller and A. Peris, Hypercyclic behaviour of operators in a hypercyclic $C_0$-semigroup, *J. Funct. Anal.,* **244(1)** (2007), 342–348.
[17] R. deLaubenfels, *Existence Families, Functional Calculi and Evolution Equations,* Lect. Notes Math., Vol. **1570**, Springer-Verlag, New York, 1994.
[18] R. deLaubenfels, Polynomials of generators of integrated semigroups, *Proc. Amer. Math. Soc.,* **107(1)** (1989), 197–204.
[19] R. deLaubenfels, H. Emamirad and K.-G. Grosse-Erdmann, Chaos for semigroups of unbounded operators, *Math. Nachr.,* **261/262(1)** (2003), 47–59.
[20] W. Desch, W. Schappacher and G. F. Webb, Hypercyclic and chaotic semigroups of linear operators, *Ergodic Theory Dynam. Systems,* **17(4)** (1997), 1–27.
[21] S. El Mourchid, The imaginary point spectrum and hypercyclicity, *Semigroup Forum,* **73(2)** (2006), 313–316.
[22] H. Emamirad, G. R. Goldstein and J. A. Goldstein, Chaotic solution for the Black-Scholes equation, *Proc. Amer. Math. Soc.,* **140(6)** (2012), 2043–2052.
[23] K.-G. Grosse-Erdmann and A. Peris, *Linear Chaos*, Springer-Verlag, London, 2011.
[24] L. Ji and A. Weber, Dynamics of the heat semigroup on symmetric spaces, *Ergodic Theory Dynam. Systems,* **30(2)** (2010), 457–468.
[25] T. Kalmes, Hypercyclic, mixing, and chaotic $C_0$-semigroups induced by semiflows, *Ergodic Theory Dynam. Systems,* **27(5)** (2007), 1599-1631.
[26] M. Kostić, *Generalized Semigroups and Cosine Functions*, Mathematical Institute SANU, Belgrade, 2011.
[27] M. Kostić, *Abstract Volterra Integro-Differential Equations*, CRC Press, Boca Raton, Fl, 2015.
[28] M. Kostić, *Abstract Degenerate Volterra Integro-Differential Equations: Linear Theory and Applications*, Book Manuscript, 2016.
[29] M. Kostić, Hypercyclic and chaotic integrated $C$-cosine functions, *Filomat,* **26(1)** (2012), 1–44.
[30] M. Kostić, Chaotic $C$-distribution semigoups, *Filomat,* **23(3)** (2009), 51–65.
[31] M. Kostić, S. Pilipović and D. Velinov, $C$-distribution semigroups and $C$-distribution ultradistribution semigroups in locally convex spaces, *Siberian Math J.,* **58(3)** (2017), 476–492.
[32] M. Kostić, On hypercyclicity and supercyclicity of strongly continuous semigroups induced by semiflows. Disjoint hypercyclic semigroups, *Discrete Contin. Dyn. Syst.,* in press.
[33] Y.-X. Liang and Z.-H. Zhou, Disjoint supercyclic powers of weighted shifts on weighted sequence spaces, *Turkish J. Math.,* **38(6)** (2014), 1007–1022.
[34] Ö. Martin, *Disjoint Hypercyclic and Supercyclic Composition Operators*, PhD Thesis, Bowling Green State University, 2010.
[35] G. Metafune, $L^p$-Spectrum of Ornstein-Uhlenbeck operators, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.,* **30(1)** (2001), 97–124.
[36] M. Pramanik and R. P. Sarkar, Chaotic dynamics of the heat semigroup on Riemannian symmetric spaces, *J. Funct. Anal.,* **266(5)** (2014), 2867–2909.
[37] Y. Puig, Disjoint Hypercyclicity along filters, preprint, arXiv:1411.7721.
[38] S. Rolewicz, On orbits of elements, *Studia Math.,* **32(1)** (1969), 17–22.
[39] H. N. Salas, Dual disjoint hypercyclic operators, *J. Math. Anal. Appl.,* **374(1)** (2011), 106–117.
[40] L. Schwartz, Théorie des distributions à valeurs vectorielles. I, Ann. Inst. Fourier (Grenoble) **7** (1957), 1–141.
[41] L. Schwartz, Théorie des distributions à valeurs vectorielles. II, Ann. Inst. Fourier (Grenoble) **8** (1958), 1–209.
[42] S. Shkarin, A short proof of existence of disjoint hypercyclic operators, *J. Math. Anal. Appl.,* **367(2)** (2010), 713-715.

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