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OSCILLATION CRITERIA FOR DELAY DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. The present paper is dedicated to examine the oscillatory behavior of all solutions of first order delay dynamic equation

$$x^{\Delta}(t) + p(t)x(\tau(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{*}$$

We obtain a new oscillation criterion for this equation on time scale $\mathbb{T}$. In particular, we show that all solutions of $(*)$ oscillate under the condition

$$M > 2m + \frac{2}{\lambda_1} - 1$$

is satisfied when $M < 1$ and $0 < m \leq \frac{1}{e}$ such that the numbers m and M are defined as

$$m = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s$$

and

$$M = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s$$

where $\lambda_1 \in [1, e]$ is the unique root of the equation $\lambda = e^{m\lambda}$.

1. Introduction

In this paper, we study the oscillatory behavior of solutions of the first-order delay dynamic equation

$$x^{\Delta}(t) + p(t)x(\tau(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.1}$$

where $\mathbb{T}$ is a time scale that is unbounded above with $t_0 \in \mathbb{T}$, $p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$, $\tau \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ is nondecreasing on $\mathbb{T}$ and

$$\tau(t) \leq t, \lim_{t\to\infty} \tau(t) = \infty \quad \text{for } t \in \mathbb{T} \tag{1.2}$$

and $\sup \mathbb{T} = \infty$.

For a reader not familiar to the time scale calculus, it will be helpful to introduce the following introductory information. A time scale, which inherits the standard topology on $\mathbb{R}$, is a nonempty closed subset of reals. In this paper, a time scale

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will be denoted by the symbol $\mathbb{T}$, and the intervals with a subscript $\mathbb{T}$ are used to denote the intersection of the usual interval with $\mathbb{T}$. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma := \inf(t, \infty)_{\mathbb{T}}$ while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho := \sup(-\infty, t)_{\mathbb{T}}$, and the graininess function $\mu : \mathbb{T} \to \mathbb{R}_0^+$ is defined as $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is called right-dense if $\sigma(t) = t$ and/or equivalently $\mu(t) = 0$ holds; otherwise it is called right-scattered, and similarly left-dense and left scattered points are defined with respect to the backward jump operator. We also need the set $\mathbb{T}^\kappa$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be Δ-differentiable at the point $t \in \mathbb{T}^\kappa$ provided that there exists $f^\Delta(t)$ such that for every $\varepsilon > 0$ there exists a neighborhood U of t such that

$$\left|[f(\sigma(t) - f(s)] - f^\Delta(t)\,[\sigma(t) - s]\right| \leq \varepsilon\,|\sigma(t) - s| \text{ for all } s \in U.$$

We shall mean the Δ-derivative of a function when we only say derivative if it is not mentioned explicitly. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$, and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T},\mathbb{R})$.

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $C^1_{rd}(\mathbb{T},\mathbb{R})$. For $s, t \in \mathbb{T}$ and a function $f \in C_{rd}(\mathbb{T},\mathbb{R})$, the Δ-integral is defined by

$$\int_s^t f(\eta)\Delta(\eta) = F(t) - F(s)$$

where $F \in C^1_{rd}(\mathbb{T},\mathbb{R})$ is an anti-derivative of f, i.e., $F^\Delta = f$ on $\mathbb{T}^\kappa$. Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$ then F is defined by

$$F(t) = \int_{t_0}^t f(\eta)\Delta(\eta) \text{ for } t \in \mathbb{T}$$

which is an antiderivative of f. And, for $t \in \mathbb{T}^\kappa$

$$\int_t^{\sigma(t)} f(\eta)\Delta(\eta) = \mu(t)f(t).$$

It is obvious that if $f^\Delta \geq 0$, then f is nondecreasing.

A function $f \in C_{rd}(\mathbb{T},\mathbb{C})$ is called regressive if $1 + f\mu \neq 0$ on $\mathbb{T}^\kappa$, and $f \in C_{rd}(\mathbb{T},\mathbb{C})$ is called positively regressive if $1 + f\mu > 0$ on $\mathbb{T}^\kappa$. The set of regressive functions and the set of positively regressive functions are denoted by $\mathcal{R}(\mathbb{T},\mathbb{C})$ and $\mathcal{R}^+(\mathbb{T},\mathbb{R})$ respectively. $\mathcal{R}^-(\mathbb{T},\mathbb{R})$ is defined similarly. For simplicity, we denote the set of regressive constants by $\mathcal{R}_c(\mathbb{T},\mathbb{C})$. Similarly, we define the sets $\mathcal{R}_c^+(\mathbb{T},\mathbb{R})$ and $\mathcal{R}_c^-(\mathbb{T},\mathbb{R})$.

A function $x : \mathbb{T} \to \mathbb{R}$ is called a solution of the equation (1.1), if $x(t)$ is delta differentiable for $t \in \mathbb{T}^\kappa$ and it satisfies the equation (1.1) for $t \in \mathbb{T}$. We say that a solution x of equation (1.1) has a generalized zero at t if $x(t) = 0$ or $\mu(t) > 0$ and $x(t)x(\sigma(t)) < 0$. Let $\sup \mathbb{T} = \infty$ and then a nontrivial solution x of equation (1.1) is called oscillatory on $[t, \infty)$ if it has arbitrarirly large generalized zeros in $[t, \infty)$.

In recent years, there has been an increasing interest in the oscillation of solutions of some dynamic equations. See [1-27] and the references cited therein. However, few papers ([3,25-27]) deal with only delay dynamic equations even in the case of first order linear equations.

Supposing $\mathbb{T} = \mathbb{R}$, then Eq. (1.1) is reduced to the first order delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0 \quad , \quad t \geq t_0. \tag{1.3}$$

Many authors studied the oscillatory behavior of Eq. (1.3), ([4, 8-11, 13-16, 18–20, 23-24]).

Similarly, in case that $\mathbb{T} = \mathbb{N}$, Eq. (1.1) turns into

$$\Delta x(n) + p(n)x(\tau(n)) = 0 \quad , \quad n = 0, 1, ... \tag{1.4}$$

Recently, many studies are performed on the oscillation of solutions of Eq. (1.4), [5-7, 21-22].

In 2002, Zhang and Deng [26], studied the oscillatory behavior of solutions of the following delay differential equation on time scales

$$x^{\Delta}(t) + p(t)x(\tau(t)) = 0 \quad , \quad t \geq t_0 \quad , \; t \in \mathbb{T}$$

where $p \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, $\tau \in C_{rd}(\mathbb{T}, \mathbb{T})$ and $\tau(t) < t$ for $t \in \mathbb{T}$, and $\sup \mathbb{T} = \infty$. They proved the following result by the help of cylinder transforms.

Theorem 1. *Define*

$$\alpha = \limsup_{t_0 \to \infty} \sup_{\lambda \in E} \left\{ \lambda \exp_{-\lambda p}(\tau(t), t) \right\} \tag{1.5}$$

where

$$\exp_{-\lambda p}(\tau(t), t) = \exp \int_{\tau(t)}^{t} \xi_{\mu(s)}(-\lambda p(s))\Delta s,$$

$E = \{\lambda : \lambda > 0, \; 1 - \lambda p(t)\mu(t) > 0\}$, *and*

$$\xi_h(z) = \begin{cases} \frac{Log(1+hz)}{h} & , \text{ if } h \neq 0 \\ z & , \text{ if } h = 0 \end{cases} .$$

If $\alpha < 1$, then all solutions of Eq.(1.1) are oscillatory.

In 2005, Bohner [3] gave the following result by using exponential functions notation for any time scale $\mathbb{T}$.

Theorem 2. *If Eq.(1.1) has an eventually positive solution, then α satisfies the condition $\alpha \geq 1$ defined by (1.5).*

Following these studies, Şahiner and Stavroulakis [25] gave the following result for Eq.(1.1).

Theorem 3. *Assume that there exists a positive constant L such that*

$$\liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s)\Delta s > L \tag{1.6}$$

and

$$\limsup_{t \to \infty} \int_{\tau(t)}^{t} p(s)\Delta s > 1 - \frac{L^2}{4}. \tag{1.7}$$

Then Eq.(1.1) is oscillatory.

In 2005, the following criterias were given by Zhang et al. [27] for all solutions of Eq.(1.1) to be oscillatory.

Theorem 4. *Assume that (1.2) and the following inequality holds*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s > 1, \tag{1.8}$$

then all solutions of Eq.(1.1) are oscillatory.

Theorem 5. *Assume that (1.2) holds and* $m \in [0, \frac{1}{e}]$*. Furthermore,*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s > \frac{1+\ln\lambda_1}{\lambda_1} - \frac{1-m-\sqrt{1-2m-m^2}}{2}, \tag{1.9}$$

where $\lambda_1 \in [1, e]$ *is the unique root of the equation* $\lambda = e^{m\lambda}$*, then all solutions of Eq.(1.1) are oscillatory.*

This work is inspired by [27], [22] and [14]. In this paper, we use these studies to find a new criteria for all solutions of Eq.(1.1) to be oscillatory. The purpose of the present paper is essentially to extend these results to the dynamic equations on time scale $\mathbb{T}$. Finally, two examples are given for certain cases.

2. Main Results

In this section, we give an oscillatory criteria for all solutions of Eq.(1.1).
Here, we set

$$m = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s. \tag{2.1}$$

Lemma 1 ([27, Lemma 2.3])**.** *Let $x(t)$ be an eventually positive solution of Eq.(1.1) and* $m \in [0, (1/e)]$*. Then*

$$\liminf_{t\to\infty} \frac{x(\tau(t))}{x(t)} \geq \lambda_1, \tag{2.2}$$

where $\lambda_1 \in [1, e]$ *is the unique root of the equation* $\lambda = e^{m\lambda}$*.*

Lemma 2. *Let $x(t)$ be an eventually positive solution of Eq.(1.1) and* $m \in [0, (1/e)]$*. Assume that $\tau(t)$ is nondecreasing and there exists $\theta > 0$ such that*

$$\int_{\tau(u)}^{\tau(t)} p(s)\Delta s \geq \theta \int_{u}^{t} p(s)\Delta s \quad \textit{for all } \tau(t) \leq u \leq t. \tag{2.3}$$

Then

$$\liminf_{t\to\infty} \frac{x(\sigma(t))}{x(\tau(t))} \geq \frac{1-m-\sqrt{(1-m)^2-4A}}{2}, \tag{2.4}$$

where A is given by

$$A = \frac{e^{\lambda_1\theta m} - \lambda_1\theta m - 1}{(\lambda_1\theta)^2} \tag{2.5}$$

and $\lambda_1 \in [1, e]$ *is the unique root of the equation* $\lambda = e^{m\lambda}$*.*

Proof. If $m = 0$, then obviously inequality (2.4) holds.

If $m \neq 0$, then let $x(t)$ be eventually positive solution of Eq.(1.1). Define the functions $\overline{x}, \overline{p}, \overline{\tau}$ on $\mathbb{R}$ as follows

$$\overline{x}(t) = \begin{cases} x(t), & t \in \mathbb{T}, \\ x(s) + (x(\sigma(s)) - x(s))\frac{t-s}{\sigma(s)-s}, & s < t < \sigma(s),\ s \in \mathbb{T}, \end{cases}$$

$$\overline{p}(t) = \begin{cases} p(t), & t \in \mathbb{T}, \\ p(s), & s < t < \sigma(s),\ s \in \mathbb{T}, \end{cases}$$

$$\overline{\tau}(t) = \begin{cases} \tau(t), & t \in \mathbb{T}, \\ \tau(s), & s < t < \sigma(s),\ s \in \mathbb{T}. \end{cases}$$

Clearly, these functions are well defined under the assumption on $\mathbb{T}$. It is easy to see that the function $\overline{x}$ is continuous, nonincreasing and eventually positive on $\mathbb{R}$, and the function $\overline{\tau}$ is nondecreasing on $\mathbb{R}$ with $\lim_{t\to\infty} \overline{\tau}(t) = \infty$, $t \in \mathbb{R}$. And $\overline{p}(t) \geq 0$, for $t \geq t_0$, $t \in \mathbb{R}$.

From the proof of Lemma 2.4 in [27] we know that $\overline{x}$ is a solution of the following differential equation

$$\overline{x}'_+(t) + \overline{p}(t)\overline{x}(\overline{\tau}(t)) = 0, \quad t \geq t_0,\ t \in \mathbb{R}, \tag{2.6}$$

where $\overline{x}'_+(t)$ means the right derivative of $\overline{x}$ at t.

On the other hand, from (2.3),we get

$$\int_{\overline{\tau}(u)}^{\overline{\tau}(t)} \overline{p}(s)\Delta s \geq \theta \int_u^t \overline{p}(s)\Delta s \qquad \text{for all } \overline{\tau}(t) \leq u \leq t.$$

Therefore, from Lemma 2 in [14] we have

$$\overline{x}(t) \geq \frac{1}{2}\left[1 - m - \sqrt{(1-m)^2 - 4A}\right] \overline{x}(\overline{\tau}(t)).$$

for $t \in \mathbb{R}$.

If $s \leq t < \sigma(s)$, $s \in \mathbb{T}$, then we have $\overline{x}(\overline{\tau}(t)) = \overline{x}(\tau(s)) = x(\tau(s))$. So, we get

$$\overline{x}(t) \geq \frac{1}{2}\left[1 - m - \sqrt{(1-m)^2 - 4A}\right] x(\tau(s)).$$

Let $t \to \sigma(s) - 0$ and from the continuity of $\overline{x}$,

$$\overline{x}(\sigma(s)) \geq \frac{1}{2}\left[1 - m - \sqrt{(1-m)^2 - 4A}\right] x(\tau(s)).$$

It should be noted that $\lim\limits_{t\to\sigma(s)-0} \overline{x}(t) = \overline{x}(\sigma(s)) = x(\sigma(s))$. Thus, we prove that for all $s \leq t < \sigma(s)$, $s \in \mathbb{T}$,

$$x(\sigma(s)) \geq \frac{1}{2}\left[1 - m - \sqrt{(1-m)^2 - 4A}\right] x(\tau(s)).$$

Finally, we obtain (2.4). □

Theorem 6. *Consider the Eq.(1.1) and let $M < 1$, $m \in \left[0, \frac{1}{e}\right]$. Assume that (1.2) holds and there exists $\theta > 0$ such that (2.3) holds. If $\tau(t)$ is nondecreasing and*

$$M > \frac{1 + \ln \lambda_1}{\lambda_1} - \frac{1 - m - \sqrt{(1-m)^2 - 4A}}{2}, \tag{2.7}$$

where $\lambda_1 \in [1, e]$ is the unique root of the eqution $\lambda = e^{k\lambda}$ and A is given by (2.5), then all solutions of Eq.(1.1) oscillate.

Proof. If $m = 0$, then the inequality (2.7) reduces to (1.8). Thus with the help of Theorem 1.4, we get the conclusion.

Let $0 < m \leq \frac{1}{e}$. Assume for the sake of contradiction that x is an eventually positive solution of Eq.(1.1). Then there exists $t_0 \leq t_1 \in \mathbb{T}$ such that $x(\tau(t)) > 0$ for $t > t_1$. We define $\overline{x}$, $\overline{\tau}$, $\overline{p}$ as in Lemma 2.2, then $\overline{x}$ satisfies the delay differential equation (2.6). From Lemma 2.1 and Lemma 2.2, it follows that

$$\liminf_{t\to\infty}\frac{x(\tau(t))}{x(t)} \geq \lambda_1, \quad \liminf_{t\to\infty}\frac{x(\sigma(t))}{x(\tau(t))} \geq \frac{1-m-\sqrt{(1-m)^2-4A}}{2} := \beta.$$

Hence, for $\forall\varepsilon > 0$ such that $\varepsilon < \min\left\{\lambda_1, \frac{1-m-\sqrt{(1-m)^2-4A}}{2}\right\}$, we have

$$\frac{x(\tau(t))}{x(t)} \geq \lambda_1 - \varepsilon, \quad \frac{x(\sigma(t))}{x(\tau(t))} \geq \beta - \varepsilon, \quad \text{for } t > t_2 \geq t_1, \quad t \in \mathbb{T}.$$

By the definitions of $\overline{x}$, $\overline{\tau}$, $\overline{p}$ in Lemma 2.2, we also have

$$\frac{\overline{x}(\overline{\tau}(t))}{\overline{x}(t)} \geq \lambda_1 - \varepsilon, \quad \frac{\overline{x}(\sigma(t))}{\overline{x}(\overline{\tau}(t))} \geq \beta - \varepsilon, \quad \text{for } t > t_2, \quad t \in \mathbb{R}.$$

Hence, for a fixed $t > t_2,\ t \in \mathbb{R}$, there exists $t^* \in (\overline{\tau}(t), t)$, $t^* \in \mathbb{R}$ such that

$$\frac{\overline{x}(\overline{\tau}(t))}{\overline{x}(t^*)} = \lambda_1 - \varepsilon.$$

Integrating Eq.(2.6) from t^* to $\sigma(t)$ and using the monotonicity of $\overline{x}$ and $\overline{\tau}$, we have

$$\begin{aligned} 0 &= \overline{x}(\sigma(t)) - \overline{x}(t^*) + \int_{t^*}^{\sigma(t)} \overline{x}(\overline{\tau}(s))\overline{p}(s)ds \\ &= \overline{x}(\sigma(t)) - \overline{x}(t^*) + \int_{t^*}^{t} \overline{x}(\overline{\tau}(s))\overline{p}(s)ds + \int_{t}^{\sigma(t)} \overline{x}(\overline{\tau}(s))\overline{p}(s)ds \\ &\geq \overline{x}(\sigma(t)) - \overline{x}(t^*) + \overline{x}(\overline{\tau}(t)) \int_{t^*}^{\sigma(t)} \overline{p}(s)ds \end{aligned}$$

and then,

$$\int_{t^*}^{\sigma(t)} \overline{p}(s)ds \leq \frac{\overline{x}(t^*)}{\overline{x}(\overline{\tau}(t))} - \frac{\overline{x}(\sigma(t))}{\overline{x}(\overline{\tau}(t))} < \frac{1}{\lambda_1 - \varepsilon} - (\beta - \varepsilon). \tag{2.8}$$

Dividing Eq.(2.6) by $\overline{x}(t)$ and integrating it from $\tau(t)$ to t^*, we have

$$\int_{\tau(t)}^{t^*} \frac{\overline{x}_+'(s)}{\overline{x}(s)}ds = -\int_{\tau(t)}^{t^*} \overline{p}(s)\frac{\overline{x}(\overline{\tau}(t))}{\overline{x}(s)}ds \leq -\left(\lambda_1 - \varepsilon\right)\int_{\tau(t)}^{t^*} \overline{p}(s)ds$$

and then

$$\int_{\tau(t)}^{t^*} \overline{p}(s)ds \leq -\frac{1}{\lambda_1 - \varepsilon}\int_{\tau(t)}^{t^*} \frac{\overline{x}_+'(s)}{\overline{x}(s)}ds = \frac{\ln\left(\lambda_1 - \varepsilon\right)}{\lambda_1 - \varepsilon}. \tag{2.9}$$

On the other hand, from [27] we get

$$\int_{t_1}^{t_2} \overline{p}(s)ds = \int_{t_1}^{t_2} p(s)\Delta s \quad , \quad \forall t_1 \leq t_2 \quad , \quad t_1, t_2 \in \mathbb{T}.$$

Combining inequalities (2.8) and (2.9), we have

$$\int_{\tau(t)}^{\sigma(t)} p(s)\Delta s = \int_{\tau(t)}^{\sigma(t)} \overline{p}(s)ds \leq \frac{1 + \ln(\lambda_1 - \varepsilon)}{\lambda_1 - \varepsilon} - (\beta - \varepsilon).$$

Letting $t \to \infty$ and $\varepsilon \to 0$, we have

$$\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s \leq \frac{1 + \ln \lambda_1}{\lambda_1} - \frac{1 - m - \sqrt{(1-m)^2 - 4A}}{2},$$

which contradicts to (2.7). Thus, the proof is completed. □

Remark 1. *Observe that when $\theta = 1$, then*

$$A = \frac{e^{\lambda_1 m} - \lambda_1 m - 1}{(\lambda_1)^2}$$

and (2.7) reduces to

$$M > 2m + \frac{2}{\lambda_1} - 1.$$

Now, we give two examples in cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$.

Example 1. *For $\mathbb{T} = \mathbb{R}$, consider the delay differential equation*

$$x'(t) + \frac{1}{e}x(t - \sin^2 \sqrt{t} - 1) = 0, \tag{2.10}$$

where $p = \frac{1}{e}$, $a = 1$ and $pa = \frac{1}{e}$. Then

$$m = \liminf_{t\to\infty} \int_{\tau(t)}^{t} \frac{1}{e} ds = \liminf_{t\to\infty} \frac{1}{e}(\sin^2 \sqrt{t} + 1) = \frac{1}{e}$$

and

$$M = \limsup_{t\to\infty} \int_{\tau(t)}^{t} \frac{1}{e} ds = \limsup_{t\to\infty} \frac{1}{e}(\sin^2 \sqrt{t} + 1) = \frac{1}{e} + \frac{1}{e} = \frac{2}{e}.$$

Thus, according to Theorem 2.3, all solutions of Eq.(2.10) oscillate.

Example 2. *For $\mathbb{T} = \mathbb{N}$, consider the following delay difference equation*

$$\Delta x(n) + p(n)x(n-5) = 0 \quad , \quad n = 0, 1, \ldots, \tag{2.11}$$

where

$$\begin{aligned} p(6n) &= p(6n+1) = \ldots = p(6n+4) = \frac{1}{5e}, \\ p(6n+5) &= \frac{1}{5e} + 0.113 \quad , \quad n = 0, 1, \ldots. \end{aligned}$$

Then

$$m = \liminf_{n\to\infty} \sum_{s=n-5}^{n-1} p(s) = \frac{1}{e} \cong 0.3678 \quad \textit{and } \lambda_1 = e$$

and

$$M = \limsup_{n\to\infty} \sum_{s=n-5}^{n} p(s) = \frac{6}{5e} + 0.113 \cong 0.55446 < 1$$

hold. Since

$$M = 0.55446 > 2m + \frac{2}{\lambda_1} - 1 \cong 0.47135,$$

all solutions of Eq.(2.11) oscillate by Theorem 2.3.

In [22], the authors gave some incorrect results. Finally, we give a correction to the [22].

Correction.

i) In Lemma 2.1 [22], the condition (2.1) was given by

$$p\left(\tau(n)\right)\Delta\left(\tau(n)\right) \geq \theta p(n).$$

This condition should be changed as follows

$$\sum_{j=\tau(u)}^{\tau(n)-1} p(j) \geq \theta \sum_{j=u}^{n-1} p(j) \quad \text{for all } \tau(n) \leq u \leq n.$$

ii) In the proof of Lemma 2.1 [22], the $\sigma(t)$ is defined as follows

$$\sigma(t) = \tau(n) + \left(\Delta\tau(n)\right)(t-n) \text{ for } n \leq t < n+1, \quad n = 0, 1, \ldots.$$

This definition should be changed as follows

$$\sigma(t) = \tau(n) \text{ for } n \leq t < n+1, \quad n = 0, 1, \ldots.$$

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