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GENERAL DECAY OF SOLUTION TO SOME NONLINEAR VECTOR EQUATION IN A FINITE DIMENSIONAL HILBERT SPACE

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ABSTRACT. The aim of this paper is to establish a general decay result for the vector equation: $u''+\phi(\|A^{\frac{1}{2}}u\|^2)Au+g(u')=0$, in a finite dimensional Hilbert space under suitable assumptions on g and ϕ. We can consider the cases where ϕ degenerate or non-degenerate and we use the multiplier method.

1. Introduction

Let H be a finite dimensional real Hilbert space, with norm denoted by $\|.\|$. We consider first the following nonlinear equation

$$u'' + \phi(\|A^{\frac{1}{2}}u\|^2)Au + g(u') = 0, \tag{1}$$

where A is a positive and symmetric linear operator on H. We denote by $(.,.)$ the inner product in H, A is coercive, which means :

$$\exists\lambda > 0, \ \forall u \in D(A), \ \ (Au, u) \geq \lambda\|u\|^2$$

We also define

$$\forall u \in H, \ \|A^{\frac{1}{2}}u\| := \|u\|_{D(A^{\frac{1}{2}})}$$

a norm equivalent to the norm in H. We assume that g and ϕ are locally Lipschitz continuous.

The consideration of the more complicated problem (1) is partially motivated by [5] in which a similar but harder (infinite dimensional) problem with general dissipation was studied with application to some PDE in a bounded domain. Under Neumann or Dirichlet boundary conditions, and for nonlinearities asymptotically homogeneous near 0 similar to the ones appearing in (1), they proved the existence of a global solution in Sobolev spaces to the initial boundary value problem of the (degenerate or non-degenerate) Kirchhoff equation with weak dissipation and they establish general stability estimates using the multiplier method and general weighted integral inequalities.

When $\phi(u) = |u|^\beta u$ and $g(u') = c|u'|^\alpha u'$, Haraux in [6] studied the decay rate of the energy of non trivial solutions to the scalar second order ODE with initial data

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$(u_0, u_1) \in R^2$. In addition, he showed that if $\alpha > \frac{\beta}{\beta+2}$ all non-trivial solutions are oscillatory and if $\alpha < \frac{\beta}{\beta+2}$ they are non-oscillatory.

We can also consider the equation

$$(\|u'\|^l u')' + \|A^{\frac{1}{2}} u\|^\beta Au + g(u') = 0, \tag{2}$$

where g is a locally Lipschitz continuous function. The equation (2) has been studied by Abdelli, Anguiano and Haraux [2], they proved the existence and uniqueness of a global solution $u \in \mathcal{C}^1(\mathbb{R}^+, H)$ with $\|u'\|^l u' \in \mathcal{C}^1(\mathbb{R}^+, H)$ for any initial data $(u_0, u_1) \in H \times H$ they used some techniques from Abdelli and Haraux [1]. They used some modified energy function to estimate the rate of decay and they used the method introduced by Haraux [6]. Finally, they discuss the optimality of these estimates when $g(s) = c\|s\|^\alpha s$ and $l < \alpha < \frac{\beta(1+l)+l}{\beta+2}$.

In this article, we use some technique from to establish an explicit and general decay result, depending on g and ϕ. The proof is based on the multiplier method and makes use of some properties of convex functions, the general Young inequality and Jensen's inequality.

The plan of this paper is as follows: In Section 2 we establish some basic preliminary inequalities, and in Section 3 we prove the energy estimates.

2. Assumptions and preliminary results

In order to state and prove our result, we require the following assumptions:

(A1) $g : H \to H$ and $\phi : H \to H$ are a locally Lipschitz continuous functions.

(A2) $\phi : R_+ \to R_+$ is of the Class $C^1(R_+)$ satisfying one of the following tow properties:

Degenerate case: $\phi(s) > 0$ on $]0, +\infty[$ and ϕ is non-decreasing.

Non-degenerate case: there exist $m_0,\ m_1$ such that $\phi(s) \geq m_0$ on R_+ and

$$s\phi(s) \geq m_1 \int_0^s \phi(\tau)\, d\tau \quad \text{on } R_+. \tag{3}$$

(A3) $g : R \to R$ is non decreasing function of class C^1 and $G : R_+ \to R_+$ is convex, increasing and of class $C^1(R_+) \cap C^2(]0, +\infty[)$ satisfying

$$\begin{cases} G(0) = 0 \text{ and } G \text{ is linear on } [0, r_0] \text{ or} \\ G'(0) = 0 \text{ and } G'' > 0 \text{ on }]0, r_0] \text{ such that} \\ c_2\|g(v)\|^2 \leq c_1\|v\|^2 \leq (g(v), v) \text{ if } \|v\| \geq r_0 \\ \|v\|^2 + \|g(v)\|^2 \leq G^{-1}(g(v), v) \text{ if } \|v\| \leq r_0 \end{cases} \tag{4}$$

where G^{-1} denotes the inverse function of G and r_0, c_1, c_2 are positive constants.

Remark 1

1. In both the degenerate and the non-degenerate cases, we have $\int_0^{+\infty} \phi(\tau)\, d\tau = +\infty$, and then $\widetilde{\phi}(s) = \frac{1}{2}\int_0^s \phi(\tau)\, d\tau$ is a bijection from $\mathbb{R}^+$ to $\mathbb{R}^+$. On the other hand, (3) is satisfied in the degenerate case (with $m_1 = 1$) as well.
2. In the degenerate case, it is enough to suppose that

$$\phi \in \mathcal{C}(\mathbb{R}^+) \cap \mathcal{C}^1(]0, +\infty[).$$

In this case, one can easily check that $\widetilde{\phi}(s) = \frac{1}{2}\int_0^s \phi(\tau)\, d\tau$ is a convex function. Indeed, let $x_1 \neq 0$ and $x_2 \neq 0$ such that $x_1 < x_2$. Because ϕ

is of the class $\mathcal{C}^1$ in $[x_1, x_2]$ and a non-decreasing function, $\widetilde{\phi}$ is a convex function. Now if $x_1 = 0$, we have, for all $0 \le \lambda \le 1$, that

$$\widetilde{\phi}(\lambda x_2) = \frac{1}{2}\int_0^{\lambda x_2} \phi(s)\, ds = \frac{1}{2}\lambda \int_0^{x_2} \phi(\lambda z)\, dz,$$

where we have made the change of variable $s = \lambda z$. As ϕ is a non-decreasing function and $\lambda x_2 \le x_2$ for all $\lambda \in [0, 1]$, it follows that

$$\widetilde{\phi}(\lambda x_2) \le \lambda \widetilde{\phi}(x_2).$$

Proposition 1 Let $(u_0, u_1) \in H \times H$ and suppose that g and ϕ satisfies **(A1)**. Then the problem (1) has a unique global solution

$$u \in \mathcal{C}(\mathbb{R}^+, H), \quad u' \in \mathcal{C}(\mathbb{R}^+, H) \quad \text{and} \quad u(0) = u_0, \quad u'(0) = u_1.$$

We introduce the energy associated to the solution of the problem (1) by

$$E(t) = \frac{1}{2}\|u'\|^2 + \frac{1}{2}\widetilde{\phi}(\|A^{\frac{1}{2}}u\|^2), \tag{5}$$

where

$$\widetilde{\phi}(s) = \int_0^s \phi(\tau)\, d\tau.$$

By multiplying equation (1) by u', we obtain easily

$$\frac{d}{dt}E(t) = -(g(u'), u') \le 0. \tag{6}$$

3. Asymptotic behavior

Lemma 1 Assume that **(A2)** and **(A3)** hold, then the functional

$$F(t) = ME(t) + (u, u'),$$

satisfies the following estimate, for some positive constants $M,\ c,\ m$:

$$F'(t) \le -mE(t) + c\|u'\|^2 + |(u, g(u'))|, \tag{7}$$

and $F(t) \sim E(t)$.

Proof. Using (1), (5) and (6), we obtain

$$\begin{aligned} F'(t) &= ME'(t) + \|u'\|^2 + (u, u'') \\ &\le \|u'\|^2 - (u, \phi(\|A^{\frac{1}{2}}u\|^2)Au) - (u, g(u')) \\ &\le \|u'\|^2 - \phi(\|A^{\frac{1}{2}}u\|^2)\|A^{\frac{1}{2}}u\|^2 - (u, g(u')). \end{aligned}$$

On the other hand, we have (in both the degenerate and the non-degenerate cases) $s\phi(s) \ge c\widetilde{\phi}(s)$. Then we deduce that

$$\begin{aligned} F'(t) &\le \|u'\|^2 - c\widetilde{\phi}(\|A^{\frac{1}{2}}u\|^2) + |(u, g(u'))| \\ &\le -mE(t) + c\|u'\|^2 + |(u, g(u'))|. \end{aligned}$$

To prove that $F(t) \sim E(t)$, we show that for some positive constants λ_1 and λ_2

$$\lambda_1 E(t) \le F(t) \le \lambda_2 E(t). \tag{8}$$

Using Young's inequality and the definition of E, we have (note also that $\widetilde{\phi}$ is a bijection from R_+ to R_+)

$$\begin{aligned}(u,u') &\le \frac{1}{2}\|u\|^2+\frac{1}{2}\|u'\|^2\\ &\le \frac{1}{2}\|A^{\frac{1}{2}}u\|^2+E(t)\\ &\le c\widetilde{\phi}^{-1}(E(t))+E(t).\end{aligned}$$

Using the fact that $s\mapsto \widetilde{\phi}^{-1}(s)$ is non-decreasing, we obtain

$$(u,u')\le c_1E(t),$$

and

$$\begin{aligned}(u,u') &\ge -\frac{1}{2}\|u\|^2-\frac{1}{2}\|u'\|^2\\ &\ge -\frac{1}{2}\|A^{\frac{1}{2}}u\|^2-E(t)\\ &\ge -c\widetilde{\phi}^{-1}(E(t))-E(t)\\ &\ge -c_2E(t).\end{aligned}$$

Then, for M large enough, we obtain (8). This completes the proof.

Theorem 1 Assume that **(A2)** and **(A3)** hold. Let $\widetilde{\phi}(t)=\int_0^t\phi(\tau)\,d\tau$. Then there exist $w,\ k,\ \varepsilon>0$ such that the energy E satisfies

A. The degenerate case:

$$E(t)\le\varphi_1\Big(\psi^{-1}(kt+\psi(E(0)))\Big),\ \forall t\ge 0, \tag{9}$$

where $\psi(t)=\int_t^1\frac{1}{w\varphi(\tau)}\,d\tau$ for $t>0$

$$\begin{cases}\varphi_1(s)=\sqrt{s},\ \ \varphi(s)=\widetilde{\phi}(s) & G \text{ is linear on }]0,r_0]\\ \varphi_1(s)=s,\ \ \varphi(s)=\frac{s^2}{\widetilde{\phi}^{-1}(s)}G'\Big(\varepsilon\frac{s^2}{\widetilde{\phi}^{-1}(s)}\Big) & \text{if } G'(0)=0 \text{ and } G''>0 \text{ on }]0,r_0],\end{cases} \tag{10}$$

B. The non-degenerate case:

$$E(t)\le\psi^{-1}(kt+\psi(E(0))),\ \forall t\ge 0, \tag{11}$$

where $\psi(t)=\int_t^1\frac{1}{w\varphi(\tau)}\,d\tau$ for $t>0$

$$\begin{cases}\varphi(s)=s & G \text{ is linear on }]0,r_0],\\ \varphi(s)=sG'(\varepsilon s) & \text{if } G'(0)=0 \text{ and } G''>0 \text{ on }]0,r_0].\end{cases} \tag{12}$$

Proof. We now estimate (7).

The degenerate case: we distinguish two cases.

1. G **is linear on** $[0,r_0]$

If $\|u'\|\ge r_0$, we use Young's inequality and (6), for any $\delta>0$, we have

$$\begin{aligned}|(u,g(u'))|+\|u'\|^2 &\le \delta\|u\|^2+C'_\delta\|g(u')\|^2+c(g(u'),u')\\ &\le \delta\|A^{\frac{1}{2}}u\|^2+C_\delta(g(u'),u')\\ &\le \delta\|A^{\frac{1}{2}}u\|^2+C_\delta(-E'(t))\\ &\le \delta\widetilde{\phi}^{-1}(E(t))+C_\delta(-E'(t)).\end{aligned} \tag{13}$$

If $\|u'\|<r_0$, we have

$$\|u'\|^2+|(u,g(u'))|\le\delta\widetilde{\phi}^{-1}(E(t))+C_\delta(-E'(t)). \tag{14}$$

We then use (13) and (14), to deduce from (7)

$$F'(t) \leq -\widetilde{\phi}(E(t))\Big(m\frac{E(t)}{\widetilde{\phi}(E(t))} - \delta\frac{\widetilde{\phi}^{-1}(E(t))}{\widetilde{\phi}(E(t))}\Big) + C_\delta(-E'(t)).$$

Using the fact that $\widetilde{\phi}$ is convex, increasing and choosing $\delta > 0$ small enough, we obtain

$$F'(t) \leq -d\widetilde{\phi}(E(t)) + C_\delta(-E'(t)). \tag{15}$$

By Lemma 1 and (15) the function $L(t) = F(t) + C_\delta E(t)$ satisfies

$$L'(t) \leq -d\varphi(L(t)), \tag{16}$$

where $\varphi(s) = \widetilde{\phi}(s)$, and

$$L(t) \sim E(t). \tag{17}$$

We choose $\varphi(t) = -\frac{w}{\psi'(t)}$, where $\psi(t)$ is defined in Theorem 1.
Using (16), we arrive at

$$(\psi(L(t))' = L'(t)\psi'(L(t)) \leq c.$$

A simple integration leads to

$$\psi(L(t)) \leq ct + \psi(L(0)),$$

consequently,

$$L(t) \leq \psi^{-1}(kt + \psi(L(0))).$$

Using (20), we obtain (9).
2. $G'(0) = 0$ **and** $G'' > 0$ **on** $]0, r_0]$**.**
If $\|u'\| \geq r_0$. Using Young's inequality, we have, for any $\delta > 0$,

$$\begin{aligned} |(u, g(u'))| + \|u'\|^2 &\leq \delta\|A^{\frac{1}{2}}u\|^2 + C_\delta\|g(u')\|^2 + \|u'\|^2 \\ &\leq \delta\widetilde{\phi}^{-1}(E(t)) + C_\delta(\|g(u')\|^2 + \|u'\|^2) \\ &\leq \delta\widetilde{\phi}^{-1}(E(t)) + C_\delta(-E'(t)), \end{aligned} \tag{18}$$

and if $\|u'\| < r_0$, we have

$$\begin{aligned} |(u, g(u'))| + \|u'\|^2 &\leq \delta\|u\|^2 + C_\delta\|g(u')\|^2 + \|u'\|^2 \\ &\leq \delta\widetilde{\phi}^{-1}(E(t)) + C_\delta G^{-1}(g(u'), u'). \end{aligned} \tag{19}$$

By Lemma 1, (18) and (19), for δ small enough, the function $L(t) = F(t) + C_\delta E(t)$ satisfies

$$L'(t) \leq -\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big(m\frac{\widetilde{\phi}^{-1}(E(t))}{E(t)} - \delta\Big(\frac{\widetilde{\phi}^{-1}(E(t))}{E(t)}\Big)^2\Big), + C_\delta G^{-1}(g(u'), u'),$$

and

$$L(t) \sim E(t). \tag{20}$$

Using the fact that $s \to \frac{s}{\widetilde{\phi}^{-1}(s)}$ is non-decreasing and choosing $\delta > 0$ small enough, we obtain

$$L'(t) \leq -d\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))} + C_\delta G^{-1}(g(u'), u'). \tag{21}$$

For $c_0 > 0$, we define $\widetilde{E}$ by

$$\widetilde{E}(t) = G'\Big(\varepsilon\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big)L(t) + c_0E(t).$$

Then, we see easily that, for $a_1,\ a_2 > 0$

$$a_1\widetilde{E}(t) \le E(t) \le a_2\widetilde{E}(t). \tag{22}$$

By recalling that $E' \le 0,\ G' > 0,\ G'' > 0$ on $(0, r_0]$ and using the fact that $s^2 \mapsto \frac{s}{\widetilde{\phi}^{-1}(s)}$ is non-decreasing, we obtain making use of (5) and (21), we obtain

$$\widetilde{E}'(t) = \varepsilon\Big(\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big)' G''\Big(\varepsilon\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big)L(t) + G'\Big(\varepsilon\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big)L('t) + c_0E'(t), \tag{23}$$

making use of (5) and (21), we obtain from (23) that

$$\widetilde{E}'(t) \le -d\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}G'\Big(\varepsilon\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big) + C_\delta G^{-1}(g(u'), u')G'\Big(\varepsilon\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big) + c_0E'(t). \tag{24}$$

On the other hand, let G^* denote the dual function of the convex function G (in the sense of Young, see Arnold [4], p. 46 , for the definition, and Lasiecka [7]. Because $G > 0$ on $]0, 1]$ and $G(0) = 0$, we can assume, without loss generality, that G defines a bijection from R^+ to R^+. Then G^* is the Legendre transform of G, which is given by (see Arnold [4], p. 61-62, Lasiecka [7], Liu and Zuazua [8], Alabau-Boussouira [3] and others).

$$G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)],$$

and G satisfies the generalized Young's inequality

$$AB \le G^*(A) + G(B)$$

with $A = G'\Big(\varepsilon\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big)$ and $B = G^{-1}(g(u'), u')$

$$\begin{aligned} G'\Big(\varepsilon\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big)G^{-1}(g(u'), u') &\le G^*\Big(G'\Big(\varepsilon\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big)\Big) + (g(u'), u') \\ &\le \varepsilon\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}G'\Big(\varepsilon\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big) + (g(u'), u'), \end{aligned} \tag{25}$$

Choosing $c_0 > C_\delta$ and ε small enough, we obtain and

$$\widetilde{E}'(t) \le -k_1\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}G'\Big(\varepsilon\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big) = -k_1\varphi\Big(\varepsilon\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big), \tag{26}$$

where $\varphi(t) = tG'(\varepsilon t)$. Since

$$\varphi'(t) = G'(\varepsilon t) + t\varepsilon G''(\varepsilon t).$$

and G is convex on $(0, \varepsilon]$, we find that $\varphi'(t) > 0$ and $\varphi(t) > 0$ on $(0, 1]$. By setting $H(t) = \frac{a_1^2\widetilde{E}^2(t)}{\widetilde{\phi}^{-1}(E(0))}$ (a_1 is given in (22)). we easily see that, by (22), we have

$$H(t) \sim \widetilde{E}^2(t).$$

using (26), we arrive at

$$H'(t) \le -k_2\varphi(H(t)),$$

where $\varphi(t) = -\frac{w}{\psi'(t)}$ and $\psi(t) = \int_t^1 \frac{1}{w\varphi(\tau)}\,d\tau$, hence

$$(\psi(H(t))' = H'(t)\psi'(H(t)) \le k.$$

By integrating over $(0, t)$, we get

$$\psi(H(t)) \le kt + \psi(H(0)).$$

Consequently,

$$H(t) \leq \psi^{-1}(kt + \psi(H(0))). \tag{27}$$

Using (22) and (27), we obtain (9).

The non-degenerate case: we distinguish two cases.

1.G is linear on $[0, r_0]$

For $\|u'\| \geq r_0$, we have, thanks to Young's inequality, for any $\delta > 0$

$$\begin{aligned} |(u, g(u'))| &\leq \delta\|u\|^2 + C_\delta\|g(u')\|^2 \\ &\leq \delta\|A^{\frac{1}{2}}u\|^2 + C_\delta(g(u'), u') \\ &\leq \delta\|A^{\frac{1}{2}}u\|^2 + C_\delta(-E'(t)) \\ &\leq \delta\widetilde{\phi}^{-1}(E(t)) + C_\delta(-E'(t)) \\ &\leq \delta\frac{\widetilde{\phi}^{-1}(E(t))}{E(t)}E(t) + C_\delta(-E'(t)). \end{aligned}$$

Using the fact that $\widetilde{\phi}^{-1}(s) < cs$ and choosing $\delta > 0$ small enough. we have

$$|(u, g(u'))| \leq c\delta E(t) + C_\delta(-E'(t)),$$

and

$$\|u'\|^2 \leq c(g(u'), u') \leq c(-E'(t)),$$

then

$$\|u'\|^2 + |(u, g(u'))| \leq c\delta E(t) + C_\delta(-E'(t)), \tag{28}$$

and for $\|u'\| < r_0$, we have

$$\|u'\|^2 + |(u, g(u'))| \leq c\delta E(t) + C_\delta(-E'(t)) \tag{29}$$

By Lemma 1, (28) and (29), we obtain

$$\begin{aligned} F'(t) &\leq -(m - c\delta)E(t) + C_\delta(-E'(t)) \\ &\leq -dE(t) + C_\delta(-E'(t)), \end{aligned}$$

we take $L(t) = F(t) + C_\delta E(t)$ and $L \sim E$, we have

$$E'(t) \leq -dE(t).$$

A simple integration leads to

$$E(t) \leq c'e^{-c''t} = c\psi^{-1}(c''t),$$

where $\varphi(s) = s$.

2.G is non-linear on $[0, r_0]$

For $\|u'\| \geq r_0$, we have, thanks to Young's inequality, for any $\delta > 0$

$$\begin{aligned} |(u, g(u'))| &\leq \delta\|u\|^2 + C_\delta\|g(u')\|^2 \\ &\leq \delta\widetilde{\phi}^{-1}(E(t)) + C_\delta(-E'(t)). \end{aligned}$$

Using fact that $\widetilde{\phi}^{-1}(s) < cs$ and choosing $\delta > 0$ small enough. we have

$$|(u, g(u'))| \leq c\delta E(t) + C_\delta(-E'(t)),$$

and

$$\|u'\|^2 \leq c(g(u'), u') \leq c(-E'(t)),$$

then

$$\|u'\|^2 + |(u, g(u'))| \leq c\delta E(t) + C_\delta(-E'(t)),$$

and for $\|u'\| < r_0$, we have

$$\begin{aligned}\|u'\|^2 + |(u, g(u'))| &\leq c\delta E(t) + \|u'\|^2 + C(\delta)\|g(u')\|^2 \\ &\leq c\delta E(t) + c(\|u'\|^2 + \|g(u')\|^2) \\ &\leq c\delta E(t) + cG^{-1}(g(u'), u')\end{aligned}$$

$$\begin{aligned}F'(t) &\leq -(m - c\delta)E(t) + cG^{-1}(g(u'), u') + C_\delta(-E'(t)) \\ &\leq -dE(t) + cG^{-1}(g(u'), u') + C_\delta(-E'(t))\end{aligned}$$

we take $L(t) = F(t) + C_\delta E(t)$ and $L \sim E$

$$L'(t) \leq -dE(t) + cG^{-1}(g(u'), u'), \tag{30}$$

we define H by

$$H(t) = G'\Big(\varepsilon\frac{E(t)}{E(0}\Big)L(t) + c_0E(t).$$

Then, we see easily that, for $\lambda_1, \lambda_2 > 0$

$$\lambda_1 H(t) \leq E(t) \leq \lambda_2 H(t) \tag{31}$$

By recalling that $E' \leq 0, G' > 0,\ G'' > 0$ on $(0, r_0]$ and making use of (5) and (30), we obtain

$$\begin{aligned}H'(t) &= \varepsilon\frac{E'(t)}{E(0)}G''\Big(\varepsilon\frac{E(t)}{E(0)}\Big)L(t) + G'\Big(\varepsilon\frac{E(t)}{E(0)}\Big)L'(t) + c_0E'(t) \\ &\leq -dE(t)G'\Big(\varepsilon\frac{E(t)}{E(0)}\Big) + cG'\Big(\varepsilon\frac{E(t)}{E(0)}\Big)G^{-1}(g(u'), u') + c_0E'(t).\end{aligned} \tag{32}$$

Let G^* be the convex conjugate of G in the sense of Young (see Arnold [4], p. 61-62), then

$$G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)], \quad \text{if } \ s \in (0, G'(r_0)], \tag{33}$$

and G satisfies the generalized Young's inequality

$$AB \leq G^*(A) + G(B) \ \text{ if } \ A \in (0, G'(r_0)], \ \ B \in (0, r_0], \tag{34}$$

with $A = G'(\varepsilon E(t)/E(0))$ and $B = G^{-1}(g(u'), u')$, using (6) and (32)-(34)

$$\begin{aligned}H'(t) &\leq -dE(t)G'\Big(\varepsilon\frac{E(t)}{E(0)}\Big) + cG^*\Big(\Big(\varepsilon\frac{E(t)}{E(0)}\Big)\Big) + (g(u'), u') + c_0E'(t) \\ &\leq -dE(t)G'\Big(\varepsilon\frac{E(t)}{E(0)}\Big) + c\varepsilon\frac{E(t)}{E(0)}G'\Big(\varepsilon\frac{E(t)}{E(0)}\Big) - cE'(t) + c_0E'(t).\end{aligned}$$

Choosing $c_0 > c$ and ε small enough, we obtain

$$H'(t) \leq -k\frac{E(t)}{E(0)}G'\Big(\varepsilon\frac{E(t)}{E(0)}\Big) = -k\varphi\Big(\frac{E(t)}{E(0)}\Big), \tag{35}$$

where $\varphi(s) = sG'(\varepsilon s)$ and $\widetilde{E_0}(t) = \frac{\lambda_1 H(t)}{E(0)}$, ($\lambda_1$ is given in (31)), we easily see that, by (31), we have

$$\widetilde{E_0}(t) \sim E(t). \tag{36}$$

Using (35), we arrive at

$$\widetilde{E_0'}(t) \leq -k\varphi(\widetilde{E_0}(t)),$$

where $\varphi(t) = -\frac{w}{\psi'(t)}$ and $\psi(t) = \int_t^1 \frac{1}{w\varphi(\tau)}\,d\tau$, hence

$$(\psi(\widetilde{E_0}(t))' = \widetilde{E_0}'(t)\psi'(t) \leq k.$$

A simple integration leads to

$$\psi(\widetilde{E_0}(t)) \leq kt + \psi(\widetilde{E_0}(0)).$$

Consequently,

$$\widetilde{E_0}(t) \leq \psi^{-1}(kt + \psi(\widetilde{E_0}(0))). \tag{37}$$

Using (36) and (37) we obtain (11). This completes the proof of Theorem.

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